

1 Definitions

1. Random Variable: A variable (usually denoted with a capital letter) whose value is the outcome of a non-deterministic system (e.g., the weather of tomorrow, etc.).
2. Domain: The set of all possible values for a random variable (e.g., $\{0, 1\}$ for binary variables, $\{v_1, v_2, \dots\}$ for a discrete variable V).
3. Conditional Probability: $P(A | B) =$

$$P(A | B) = \frac{P(A, B)}{P(B)}$$

4. Product Rule: $P(A, B) =$

$$P(A, B) = P(A|B)P(B)$$

5. Bayes' Theorem: $P(A | B) =$

$$P(A | B) = \frac{P(B|A)P(A)}{P(B)}$$

6. Normalization: $P(A | B) =$

$$P(A | B) = \frac{P(A, B)}{P(B)} = \frac{P(A, B)}{\sum_a P(a, B)}$$

7. Chain Rule: $P(A, B, C) =$

$$P(A, B, C) = P(A | B, C)P(B | C)P(C)$$

8. Law of Total Probability: [using only $P(B)$ and $P(A | B)$] $P(A) =$

$$P(A) = \sum_{b \in \text{Domain}(B)} P(A | b)P(b)$$

$$P(A) = P(A | b_1)P(b_1) + P(A | b_2)P(b_2) \quad (\text{For binary } B)$$

9. Independence: A, B independent, $P(A, B) =$

$$\text{If } A \text{ and } B \text{ are independent, then } P(A, B) = P(A)P(B)$$

10. Conditional Independence: If A and B are conditionally independent given C , then $P(A, B | C) =$

$$\text{If } A \text{ and } B \text{ are conditionally independent given } C, \text{ then } P(A, B | C) = P(A | C)P(B | C)$$

11. Expectation: A weighted sum of all the values in the domain.

$$\mathbb{E}(X) = \sum_{x \in \text{Domain}(X)} P(X = x) \cdot x$$

2 Warm Up

- (a) State the two ways to write the chain rule (conditional probability decomposition) for $P(A, B)$

$$P(A)P(B | A) = P(B)P(A | B)$$

- (b) Rearrange the above equation to find $P(A | B)$

$$P(A | B) = \frac{P(A)P(B|A)}{P(B)}$$

- (c) Find $P(a)$ in terms of the joint $P(a, b)$ for any $a \in A, b \in B$ (Hint: these are specific values, answer should include a sum)

$$P(a) = \sum_{b \in B} P(a, b)$$

- (d) Find $P(b | a)$ in terms of the joint $P(a, b)$ for any $a \in A, b \in B$

$$P(b | a) = \frac{P(a, b)}{\sum_{b' \in B} P(a, b')}$$

- (e) Find $P(b | a)$ in terms of the distributions $P(b)$, $P(a | b)$, for any $a \in A, b \in B$

$$P(b | a) = \frac{P(a|b)P(b)}{\sum_{b' \in B} P(a|b')P(b')}$$

- (f) Assume we had some fixed a and wanted to find each element of $P(b | a)$ (i.e. wanted to find $P(B | a)$). Would the numerator of the fraction in the previous question change for each value of b ? What about the denominator? How could you use this to do the calculation with less steps?

The numerator changes because the value of b changes. The denominator is constant because $P(a)$ will be the same for every value of b that we change. We can calculate all the numerators first, then normalize/equivalently compute the denominator at the end.

- (g) Assume A is a random variable that can take 3 values, B is a random variable that can take 2 values, and C is a random variable that can take 1 value. What do the following probability tables sum to?

- (a) $P(A | b)$
- (b) $P(A | C)$
- (c) $P(C | B)$
- (d) $P(B | a)$
- (e) $P(B | A)$

$P(A | b), P(A | C)$, and $P(B | a)$ sum to 1. $P(C | B)$ sums to 2 because B can take 2 values (b_1 and b_2). $P(C | b_1)$ and $P(C | b_2)$ each sum to 1, so if we add them, we get 2. $P(B | A)$ sums to 3 because A can take 3 values (a_1, a_2 , and a_3). Each of $P(B | a_1), P(B | a_2)$, and $P(B | a_3)$ sum to 1, so the total sums to 3.

3 Cake



Consider the above cake with 12 slices. Let s_1 indicate a slice with no sprinkles and s_2 be a slice with sprinkles. Let c_1 indicate a slice with no candles and c_2 be a slice with candles. Let S be a random variable indicating sprinkles and C be a random variable indicating candles. Calculate the following probabilities.

1. $P(C = c_1)$

By counting the number of slices that don't have candles, we can see that the probability of getting a slice with no candles is $4/12$.

2. $P(S = s_1, C = c_2)$

By counting the number of slices that don't have sprinkles but have candles, we can see that the probability of getting a slice with candles and no sprinkles is $4/12$.

3. $P(C = c_2 | S = s_1)$

We first constrain our world to only include the slices that contain no sprinkles which is 6 slices. 4 of those slices contain candles, so this probability becomes $4/6$. Another way to calculate this probability is by using the definition of conditional probability, $P(C = c_2 | S = s_1) = \frac{P(C=c_2, S=s_1)}{P(S=s_1)} = \frac{4/12}{6/12} = \frac{4}{6}$

4. $\sum_{s \in \{s_1, s_2\}} \sum_{c \in \{c_1, c_2\}} P(s, c)$

$P(s_1, c_1) + P(s_1, c_2) + P(s_2, c_1) + P(s_2, c_2) = 1$. Because we are summing up all possible disjoint combinations of the given sample space, the answer is 1.

5. $\sum_{c \in \{c_1, c_2\}} \sum_{s \in \{s_1, s_2\}} P(s | c)$

$P(s_1 | c_1) + P(s_2 | c_1) + P(s_1 | c_2) + P(s_2 | c_2) = 2/4 + 2/4 + 4/8 + 4/8 = 2$. Intuitively, we are summing up two different complete probability distributions, $P(S | c_1)$ and $P(S | c_2)$: one world where there are no candles, and another world where there are definitely candles.

6. $\sum_{s \in \{s_1, s_2\}} \sum_{c \in \{c_1, c_2\}} P(c | s)$

$P(c_1 | s_1) + P(c_2 | s_1) + P(c_1 | s_2) + P(c_2 | s_2) = 2/6 + 4/6 + 2/6 + 4/6 = 2$. Intuitively, we are summing up two different complete probability distributions, $P(C | s_1)$ and $P(C | s_2)$: one world where there is no sprinkles, and another world where there is definitely sprinkles.

4 Queries on a Large Joint Distribution

Consider binary (two outcomes) random variables A, B, C, D , and the following joint distribution table of all four variables.

A	B	C	D	$P(A, B, C, D)$
$+a$	$+b$	$+c$	$+d$	$12/64$
$+a$	$+b$	$+c$	$-d$	$4/64$
$+a$	$+b$	$-c$	$+d$	$2/64$
$+a$	$+b$	$-c$	$-d$	$2/64$
$+a$	$-b$	$+c$	$+d$	$8/64$
$+a$	$-b$	$+c$	$-d$	$4/64$
$+a$	$-b$	$-c$	$+d$	$2/64$
$+a$	$-b$	$-c$	$-d$	$4/64$
$-a$	$+b$	$+c$	$+d$	$6/64$
$-a$	$+b$	$+c$	$-d$	$3/64$
$-a$	$+b$	$-c$	$+d$	$4/64$
$-a$	$+b$	$-c$	$-d$	$6/64$
$-a$	$-b$	$+c$	$+d$	$2/64$
$-a$	$-b$	$+c$	$-d$	$1/64$
$-a$	$-b$	$-c$	$+d$	$3/64$
$-a$	$-b$	$-c$	$-d$	$1/64$

1. Calculate the following probabilities:

(a) $P(+c)$

Sum all the entries that contain $+c$ to get: $P(+c) = \sum_a \sum_b \sum_d P(a, b, +c, d) = 40/64$

(b) $P(+a, -b)$

Sum all the entries that contain both $+a$ and $-b$ to get: $P(+a, -b) = \sum_c \sum_d P(+a, -b, c, d) = 18/64$

(c) $P(-b \mid +a)$

1) Sum all entries with both $+a$ and $-b$ to get: $P(+a, -b) = \sum_c \sum_d P(+a, -b, c, d) = 18/64$

2) Sum all entries with $+a$ to get: $P(+a) = \sum_b \sum_c \sum_d P(+a, b, c, d) = 38/64$

3) Use definition of conditional probability to compute: $P(-b \mid +a) = \frac{P(+a, -b)}{P(+a)} = 18/38$

(d) $P(-a, +b, -d)$

Sum all entries with $-a, +b$, and $-d$ to get: $P(-a, +b, -d) = \sum_c P(-a, +b, c, -d) = P(-a, +b, +c, -d) + P(-a, +b, -c, -d) = 9/64$

(e) $P(+c \mid -a, +b, -d)$

1) Find entry with $-a, +b, +c, -d$ to get: $P(-a, +b, +c, -d) = 3/64$

2) Sum all entries with $-a, +b, -d$ to get: $P(-a, +b, -d) = \sum_c P(-a, +b, c, -d) = 9/64$

3) Use definition of conditional probability to compute: $P(+c \mid -a, +b, -d) = \frac{P(-a, +b, +c, -d)}{P(-a, +b, -d)} = 3/9$

(f) $P(+c, +d \mid +a, +b)$

1) Find entry with $+a, +b, +c, +d$ to get: $P(+a, +b, +c, +d) = 12/64$

2) Sum all entries with $+a, +b$ to get: $P(+a, +b) = \sum_c \sum_d P(+a, +b, c, d) = 20/64$

3) Use definition of conditional probability to compute: $P(+c, +d \mid +a, +b) = \frac{P(+a, +b, +c, +d)}{P(+a, +b)} = 12/20$

2. What value do the following probability tables sum to?

(a) $P(B)$

The short answer is that we are considering the entries for all the possible values of B , so this should sum to 1. You could calculate both entries in this table to convince yourself, $P(+b)$ and $P(-b)$.

(b) $P(+b \mid C, +d)$

Sadly, there is no shortcut here. The two entries in this table come from two different worlds that are unrelated: one world where $+c$ and $+d$ are given; and another world where $-c$ and $+d$ are given.

Important note: There is no real reason to add these numbers together in this strange probability table. This is primarily a counterexample to show that these do *not* sum to one.

We compute these two values similar to the methods in the previous questions, and then add them together.

$$P(+b \mid +c, +d) = \frac{P(+b, +c, +d)}{P(+c, +d)} = 9/14$$

$$P(+b \mid -c, +d) = \frac{P(+b, -c, +d)}{P(-c, +d)} = 6/11$$

When we add these two values together, we get 1.1883.

(c) $P(C, D \mid +a, +b)$

The short answer is that we are considering all possible entries for a single world where $+a$ and $+b$ are given, so this should sum to 1. You could calculate all four entries in this table to convince yourself, $P(+c, +d \mid +a, +b)$, $P(+c, -d \mid +a, +b)$, $P(-c, +d \mid +a, +b)$, $P(-c, -d \mid +a, +b)$.