1 Bayes' Nets Warm Up

1. Are the following statements true or false? If it is false, briefly justify why not.

   (a) In a Bayes Net, a directed acyclic graph, each node can correspond to multiple random variables.

   False - each node will correspond to exactly one random variable.

   (b) Bayes Nets allow us to represent conditional independence relationships in a concise way.

   True

   (c) The number of entries in a conditional probability table in a Bayes Net is exponentially larger than in a joint probability table.

   False - The number of entries in a conditional probability table in a Bayes Net is exponentially smaller than in a joint probability table. A joint probability table will need to list out all combinations of possible domain values for the variables, which will be an exponential number of entries. In a Bayes Net, it only needs the different conditional probability relationships.

   (d) In a Bayes Net with no edges, the variables are all dependent on each other.

   False - the variables will all be independent of each other since there are no edges connecting them.

2 Bayes' Nets

1. Now let's look at some common types of queries we might be interested in answering using a Bayes' Net.

   Let C = some variable we care about, D = some variable we don't care about (and we also don't know the value of), E = some variable we've observed to have the value e (i.e., current knowledge/evidence). (Suppose for now C, D, E are the only nodes in our BN.)

   Express each of the following generic queries using probability notation, then write how we would compute it using the full joint distribution, $P(C, D, E)$. The first expression has been written for you as an example to get started.

   (a) What's the probability of the outcome $C = c$ given we have observed $E = e$?

   $$P(c \mid e) = \frac{P(c, e)}{P(e)} =$$

   $$P(c \mid e) = \frac{P(c, e)}{P(e)} = \frac{\sum_d P(c, d, e)}{\sum_{c'} \sum_d P(c', d, e)}$$

   (b) What are the probabilities of all possible outcomes of C given we have observed $E = e$?

   $$P(C \mid e) = \frac{P(C, e)}{P(e)} =$$

   $$P(C \mid e) = \frac{P(C, e)}{P(e)} = \frac{\sum_d P(C, d, e)}{\sum_c \sum_d P(c, d, e)}$$

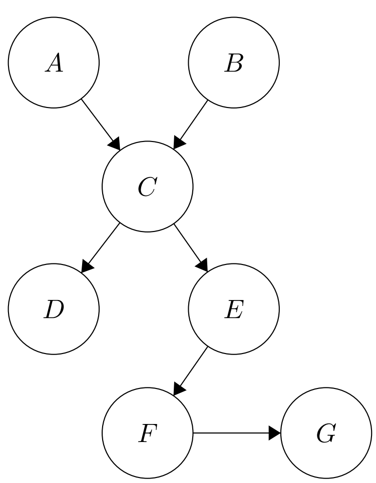
   (c) Which value of C is least likely given we have observed $E = e$?

   $$\underset{c}{\operatorname{argmin}} P(c \mid e) = \underset{c}{\operatorname{argmin}} \frac{P(c, e)}{P(e)} = \underset{c}{\operatorname{argmin}} \frac{\sum_d P(c, d, e)}{\sum_{c'} \sum_d P(c', d, e)}$$

   Note that for parts (a) and (c) above, we use c' in the denominator as the variable being summed over to differentiate it from c itself, which is already in the expression.

2. Bayesian networks represent our beliefs about the conditional independence relationships between the variables in a system. The joint distribution is factored into conditional distributions, which are represented by the connections between nodes.

   Consider the following Bayes net.



(a) Suppose all variables have domains of size 2. How many entries are in the full joint probability table $P(A, B, C, D, E, F, G)$?

$2^7 = 128$ (assuming none have been implicitly omitted)

(b) How many entries are in the conditional probability table at each of the individual nodes?

Variable	A	B	C	D	E	F	G
# Entries							

Variable	A	B	C	D	E	F	G
# Rows	2	2	8	4	4	4	4

(c) How many CPT entries are needed to represent this Bayes net in total?

Summing the number of entries in the table above, we get 28.

(d) How do we represent the full joint probability using the conditional probability tables of each variable?

Joint probability = product of all conditional probability distributions.
$P(A, B, C, D, E, F, G) = P(A)P(B)P(C \mid A, B)P(D \mid C)P(E \mid C)P(F \mid E)P(G \mid F)$

(e) Give an expression for the probability $P(D, E \mid C)$. Use as many conditional independence relations as possible from the Bayes Net.

$P(D, E \mid C) = P(D, E)$ (since D and E are conditionally independent given C)
$= \sum_{a,b,c,f,g} P(a)P(b)P(c \mid a, b)P(D \mid c)P(E \mid c)P(f \mid E)P(g \mid f)$

(f) Now give an expression for the probability $P(A, B \mid C)$. Still use as many conditional independence relations as possible from the Bayes Net.

Since A and B are not conditionally independent given C, we need to express the query in terms of the joint probability and sum out the variables we're not interested in.

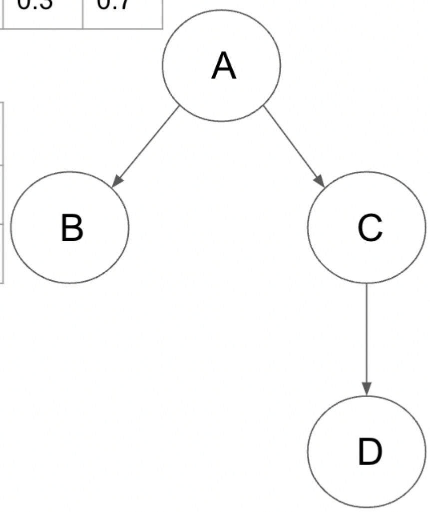
$P(A, B \mid C) = \frac{P(A,B,C)}{P(C)} = \frac{\sum_{d,e,f,g} P(A)P(B)P(C|A,B)P(d|C)P(e|C)P(f|e)P(g|f)}{\sum_{a,b,d,e,f,g} P(a)P(b)P(C|a,b)P(d|C)P(e|C)P(f|e)P(g|f)}$

3 Bayes Net Calculations

Consider the following Bayes Net:

	+a	-a
P(A)	0.3	0.7

	P(+b|A)	P(-b|A)
+a	0.5	0.5
-a	0.4	0.6



	P(+c|A)	P(-c|A)
+a	0.8	0.2
-a	0.7	0.3

	P(+d|C)	P(-d|C)
+c	0.9	0.1
-c	0.2	0.8

Calculate the following probabilities based on the given Bayes Net:

1. $P(+a, -b, -c, +d)$

   $P(+a, -b, -c, +d) = P(+a)P(-b \mid +a)P(-c \mid +a)P(+d \mid -c) = 0.3 * 0.5 * 0.2 * 0.2 = 0.006$

2. $P(+c)$

   $P(+c) = \sum_a P(a)P(+c \mid a) = 0.73$

3. $P(-d \mid -c)$

   $P(-d \mid -c) = 0.8$ (directly from the probability table)

4. $P(-b, -c, +d)$

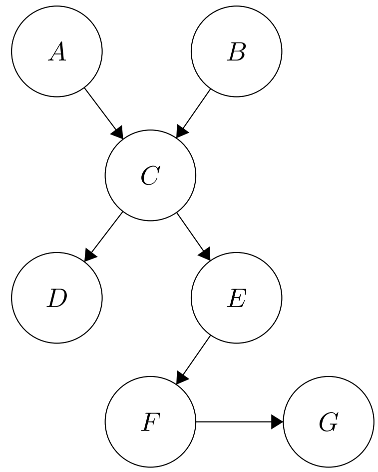
   $P(-b, -c, +d) = \sum_a P(a)P(-b \mid a)P(-c \mid a)P(+d \mid -c) = 0.0312$

5. $P(+a \mid -b, -c, +d)$

   $P(+a \mid -b, -c, +d) = \frac{P(+a,-b,-c,+d)}{P(-b,-c,+d)} = 0.1923$

4 Bayes Net Independence

Consider the Bayes net from the Question 2 (included below again for convenience). We will be considering whether or not certain combinations of variables will be conditionally independent given other variables.



1. Are A and B conditionally independent given C?

   No. Using the "Bayes Ball" method, we can determine that the (A, B, C) triple is active given C (common effect) and thus the ball can reach B starting at A.

2. Are A and B conditionally independent given D?

   No. Using the "Bayes Ball" method, we can determine that the (A, B, C) triple is active given D (extended common effect) and thus the ball can reach B starting at A.

3. Are A and D conditionally independent given C?

   Yes. The (A, C, D) triple is inactive (causal chain) and thus the ball cannot move through C.

4. Are D and G conditionally independent given E?

   Yes. The (C, E, F) triple is inactive (causal chain) and thus the ball cannot move through E.

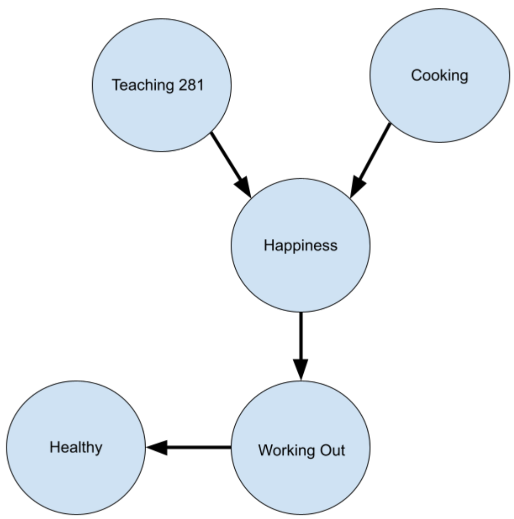
5. Are D and G conditionally independent given C?

   Yes. The (D, C, E) triple is inactive (common cause) and thus the ball cannot move through C.

5 Pinky's Bayes' Net

Pinky the Penguin loves spending time at CMU. Some of her favorite activities include working out (W), cooking (C), and teaching 281 (T). She has seen that her happiness level (H) is directly affected by whether or not she teaches 281 (T) and whether or not she is able to cook her meals (C). If Pinky is happy (H), she has the energy to be able to work out (W). Overall, Pinky feels that working out has allowed her to live a healthy lifestyle (Y).

1. Using the variables defined above W, C, H, T, and Y, draw the Bayes net representing Pinky's life from the information given in the description above.



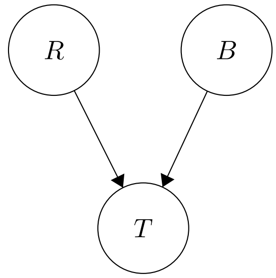
2. Can you find a conditional independence relationship between variables? How about a group of three variables such that two variables are NOT conditionally independent given the third?

   Example of Conditionally Independent Variables: $H \perp\!\!\!\perp Y \mid W$

   Example of Not Conditionally Independent Variables: $T \not\perp\!\!\!\perp C \mid H$

6 Common Effect

Recall the common effect example from lecture: both raining (R) and a ballgame (B) can cause traffic (T).



In this example $P(R = 1) = 0.7, P(B = 1) = 0.2$, and the probability table of T given R and B is below:

R	B	$P(T = 1\|R, B)$	$P(T = 0\|R, B)$
1	1	0.9	0.1
1	0	0.5	0.5
0	1	0.5	0.5
0	0	0.1	0.9

We will now answer a few queries on this Bayes Net to determine the probability of a ballgame given different sets of knowledge.

1. What is the factorization of the joint distribution? What is the probability that it is not raining and there is no ballgame but there is still traffic?

   Based on the Bayes net, the joint distribution $P(R, B, T) = P(R)P(B)P(T|R, B)$.
   So $P(R = 0, B = 0, T = 1) = P(R = 0)P(B = 0)P(T = 1|R = 0, B = 0) = .3 * .8 * .1 = .024$.

2. What is the probability of a ballgame if we have not observed any of the variables?

   Since we have not observed any variables, the probability is simply $P(B = 1) = 0.2$.

3. What is the probability of a ballgame given that there is traffic? How does it compare to the probability from the previous question? Is ballgame independent of traffic?

   $P(B = 1|T = 1) = \frac{P(B=1,T=1)}{P(T=1)}$.
   To find $P(B = 1, T = 1)$, we sum the joint distribution over the domain of R:
   $\sum_r P(R = r, B = 1, T = 1) = P(R = 0, B = 1, T = 1) + P(R = 1, B = 1, T = 1) = .3*.2*.5+.7*.2*.9 = .156$.
   To find $P(T = 1)$, we sum the joint distribution over the domains of R and B:
   $\sum_{r,b} P(R = r, B = b, T = 1) = P(R = 0, B = 0, T = 1) + P(R = 0, B = 1, T = 1) + P(R = 1, B = 0, T = 1) + P(R = 1, B = 1, T = 1) = .3 * .8 * .1 + .3 * .2 * .5 + .7 * .8 * .5 + .7 * .2 * .9 = .46$.
   So therefore $P(B = 1|T = 1) = \frac{P(B=1,T=1)}{P(T=1)} = \frac{.156}{.46} = .339$.
   The probability of ballgame given traffic is greater than the probability of a ballgame without observations, so ballgame is not independent of traffic.

4. What is the probability of a ballgame given that there is traffic but no rain? How does it compare to the probability from the previous questions? Is ballgame conditionally independent of rain given traffic?

$P(B=1|R=0,T=1) = \frac{P(R=0,B=1,T=1)}{P(R=0,T=1)}$.
$P(R=0,B=1,T=1) = .3 * .2 * .5 = .03$.
To find $P(R=0,T=1)$, we sum the joint distribution over the domain of B:
$\sum_b P(R=0,B=b,T=1) = P(R=0,B=0,T=1) + P(R=0,B=1,T=1) = .3*.8*.1 + .3*.2*.5 = .054$.
So therefore $P(B=1|R=0,T=1) = \frac{P(R=0,B=1,T=1)}{P(R=0,T=1)} = \frac{.03}{.054} = .556$.
The probability of a ballgame given traffic and no rain is even greater than the probability of a ballgame given just traffic. The absence of rain increases the probability of a ballgame as the cause of traffic. Because the probability changed, ballgame is not conditionally independent of rain given traffic.