Linear programming, integer linear programming, mixed integer linear programming

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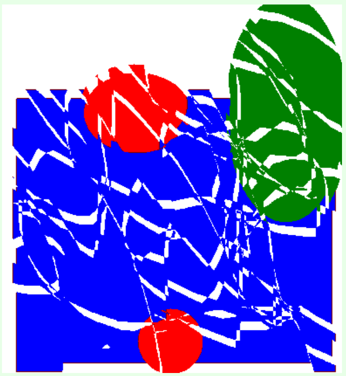
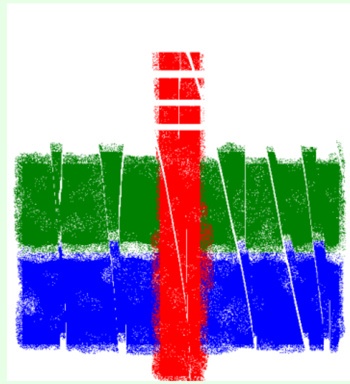
Example linear program

- We make reproductions of two paintings





- Painting 1 sells for \$30, painting 2 sells for \$20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red

maximize $3x + 2y$

subject to

$$4x + 2y \leq 16$$

$$x + 2y \leq 8$$

$$x + y \leq 5$$

$$x \geq 0$$

$$y \geq 0$$

Solving the linear program graphically

maximize $3x + 2y$

subject to

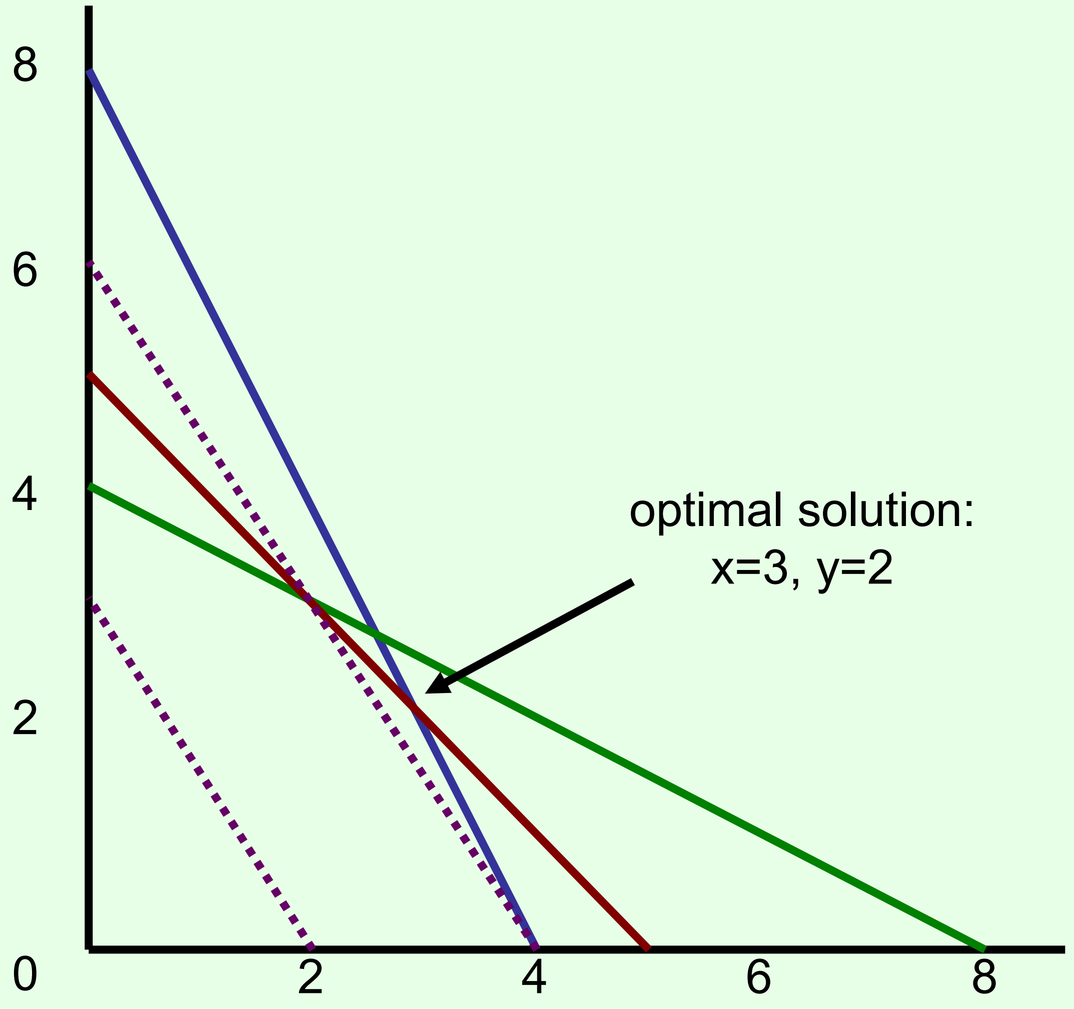
$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$



Proving optimality

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Recall: optimal solution: $x=3$, $y=2$

Solution value = 9+4 = 13

How do we prove this is optimal (without the picture)?

Proving optimality…

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

We can rewrite the blue constraint as

$2x + y \leq 8$

If we add the red constraint

$x + y \leq 5$

we get

$3x + 2y \leq 13$

Matching upper bound!

(Really, we added .5 times the blue constraint to 1 times the red constraint)

Linear combinations of constraints

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

$b(4x + 2y \leq 16) +$
$g(x + 2y \leq 8) +$
$r(x + y \leq 5) =$

$(4b + g + r)x +$
$(2b + 2g + r)y \leq$
$16b + 8g + 5r$

$4b + g + r$ must be at least 3

$2b + 2g + r$ must be at least 2

Given this, minimize $16b + 8g + 5r$

Using LP for getting the best upper bound on an LP

maximize $3x + 2y$

subject to

$4x + 2y \leq 16$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

minimize $16b + 8g + 5r$

subject to

$4b + g + r \geq 3$

$2b + 2g + r \geq 2$

$b \geq 0$

$g \geq 0$

$r \geq 0$

the dual of the original program

- Duality theorem: any linear program has the same optimal value as its dual!

Modified LP

maximize $3x + 2y$

subject to

$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$

$y \geq 0$

Optimal solution: $x = 2.5$, $y = 2.5$

Solution value = $7.5 + 5 = 12.5$

Half paintings?

Integer (linear) program

maximize $3x + 2y$

subject to

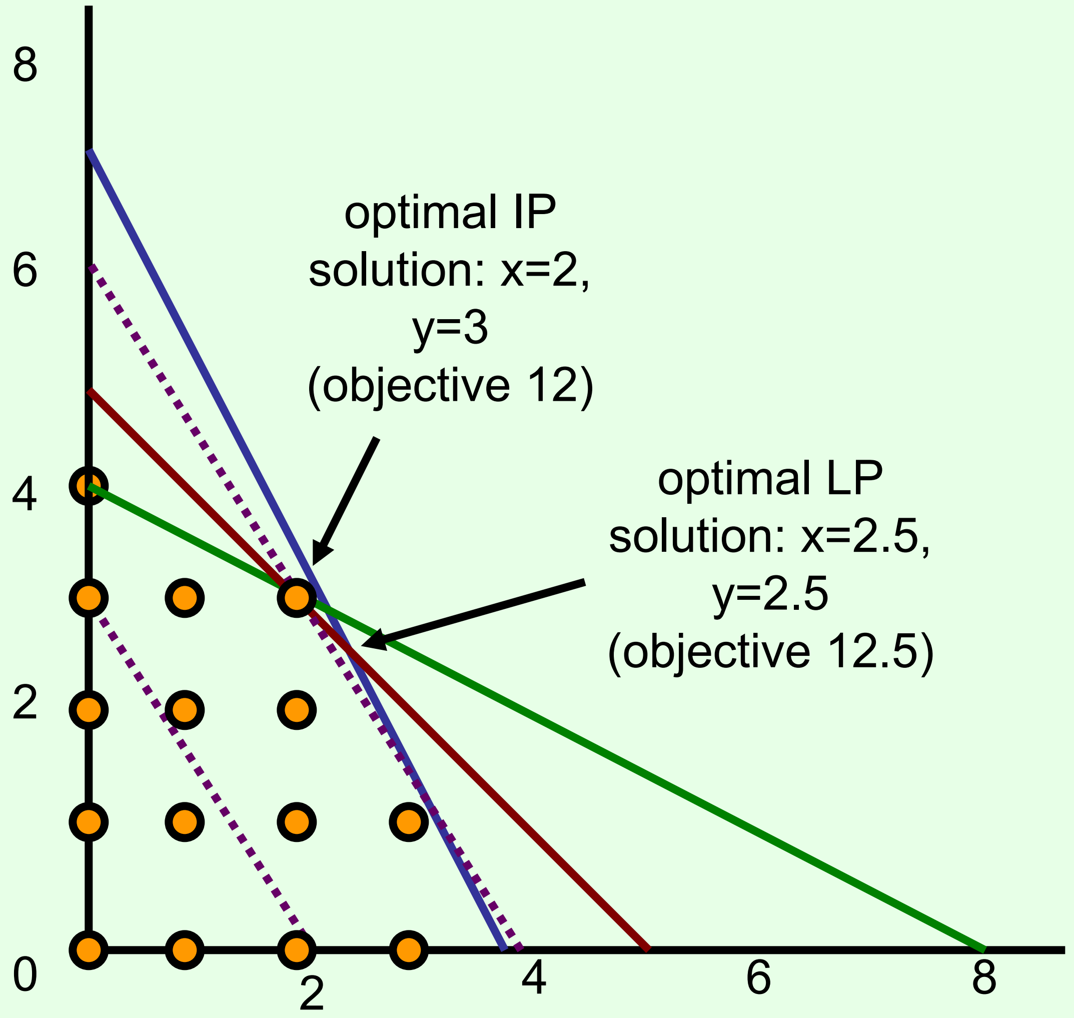
$4x + 2y \leq 15$

$x + 2y \leq 8$

$x + y \leq 5$

$x \geq 0$, integer

$y \geq 0$, integer



Mixed integer (linear) program

maximize 3x + 2y

subject to

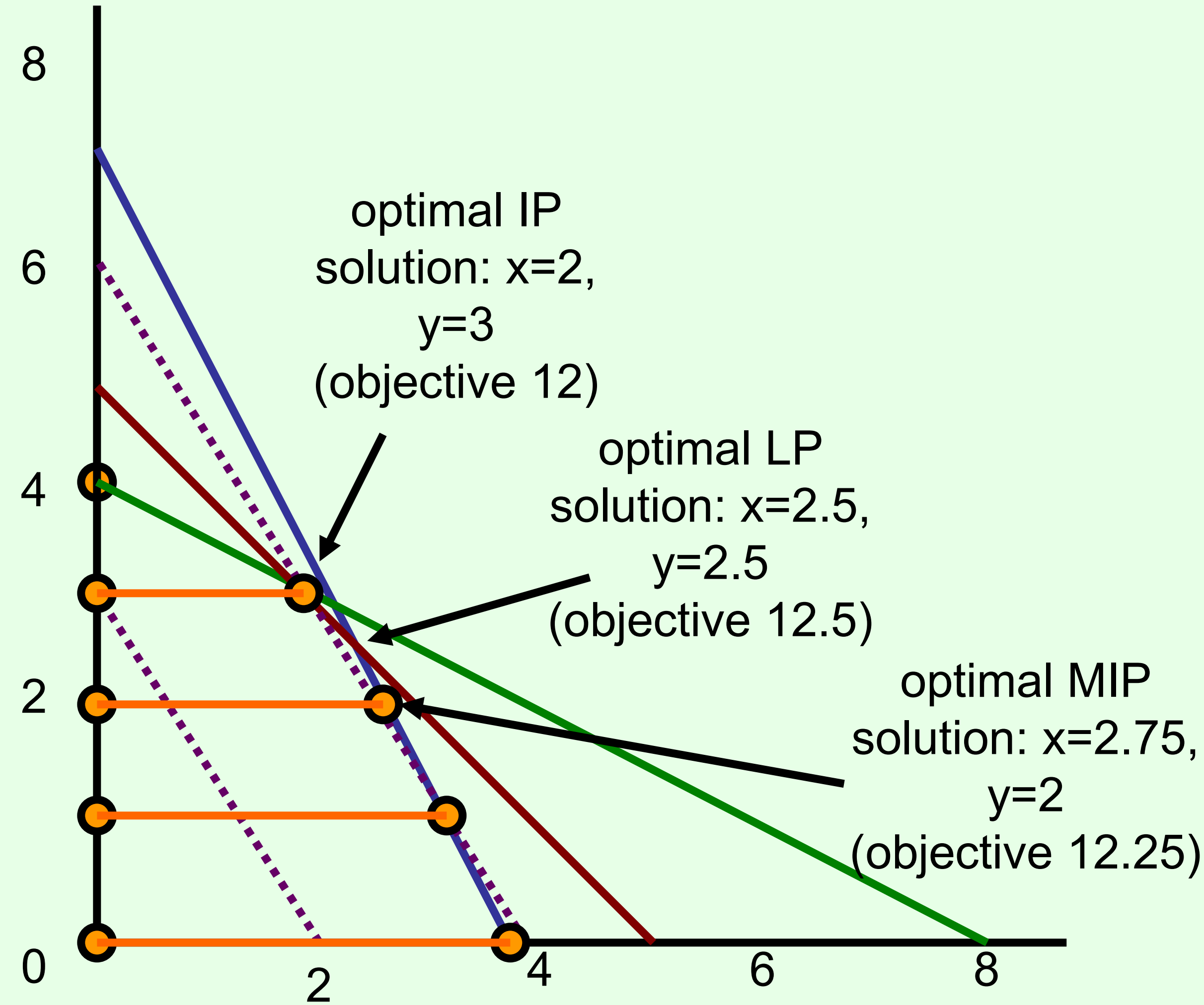
4x + 2y ≤ 15

x + 2y ≤ 8

x + y ≤ 5

x ≥ 0

y ≥ 0, integer



Solving linear/integer programs

- Linear programs can be solved efficiently
  - Simplex, ellipsoid, interior point methods…
- (Mixed) integer programs are NP-hard to solve
  - Quite easy to model many standard NP-complete problems as integer programs (try it!)
  - Search type algorithms such as branch and bound
- Standard packages for solving these
  - GNU Linear Programming Kit, CPLEX, Gurobi, …
- LP relaxation of (M)IP: remove integrality constraints
  - Gives upper bound on MIP (~admissible heuristic)

Exercise in modeling: knapsack-type problem

- We arrive in a room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for $11
  - There are 3 units available
- Unit of object B: 4kg, 4 liters, sells for $4
  - There are 4 units available
- Unit of object C: 6kg, 3 liters, sells for $9
  - Only 1 unit available
- What should we take?

The *general* version of this knapsack IP

maximize $\Sigma_j \, p_j \, x_j$

subject to

$\Sigma_j \, w_j \, x_j \leq W$

$\Sigma_j \, v_j \, x_j \leq V$

(for all j) $x_j \leq a_j$

(for all j) $x_j \geq 0$, x_j integer

Exercise in modeling: cell phones (set cover)

- We want to have a working phone in every continent (besides Antarctica)
- … but we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E

Exercise in modeling: hot-dog stands

- We have two hot-dog stands to be placed in somewhere along the beach
- We know where the people that like hot dogs are, how far they are willing to walk
- Where do we put our stands to maximize #hot dogs sold? (price is fixed)