Securities & Expressive Securities Markets

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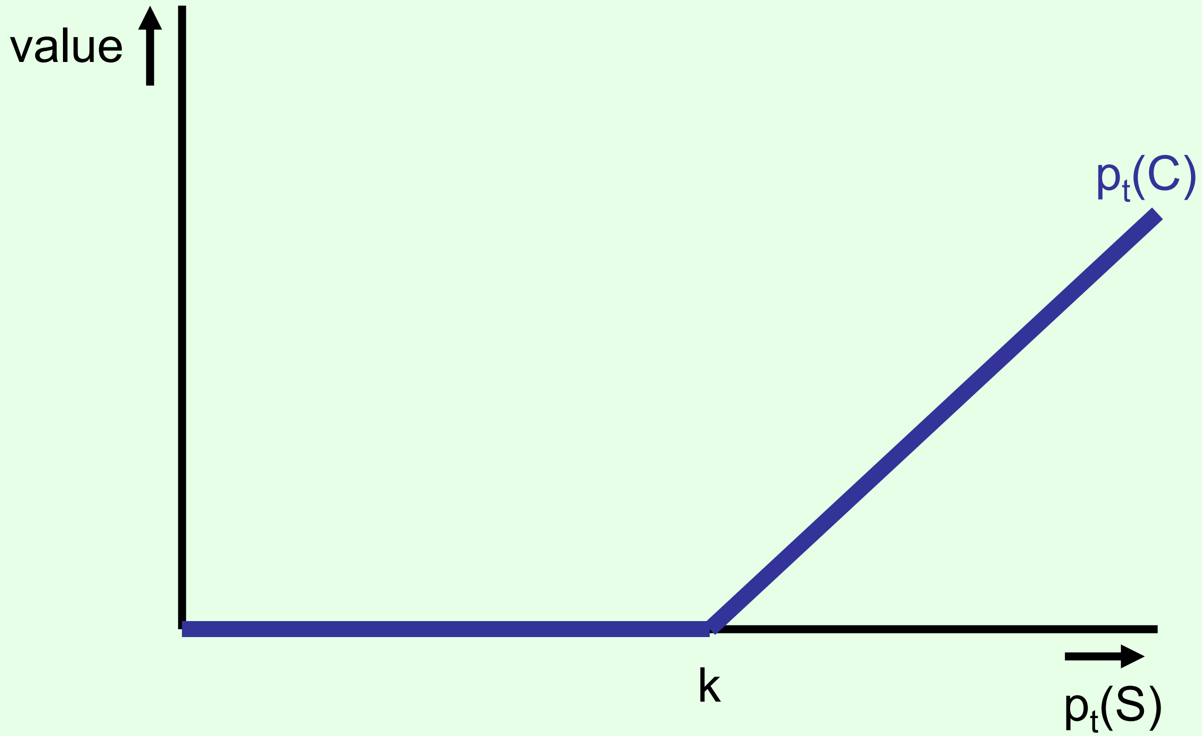
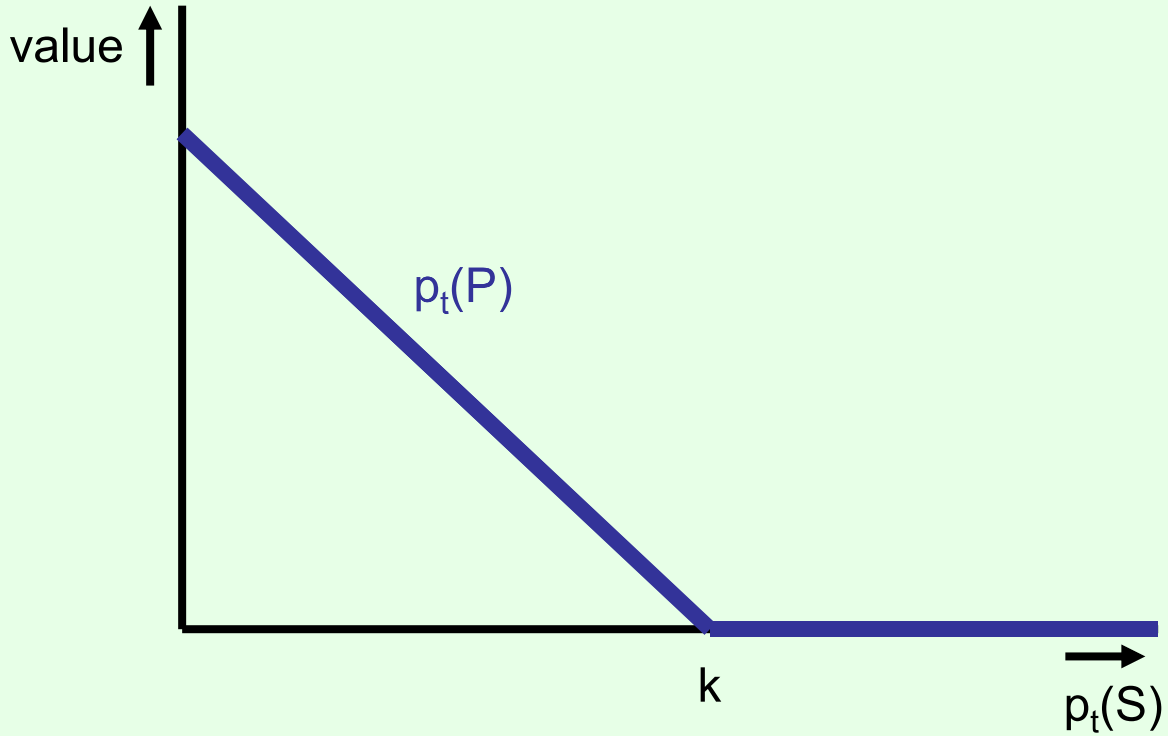
Call options

- A (European) call option $C(S, k, t)$ gives you the right to buy stock S at (strike) price k on (expiry) date t
  - American call option can be exercised early
  - European one easier to analyze
- How much is a call option worth at time t (as a function of the price of the stock)?



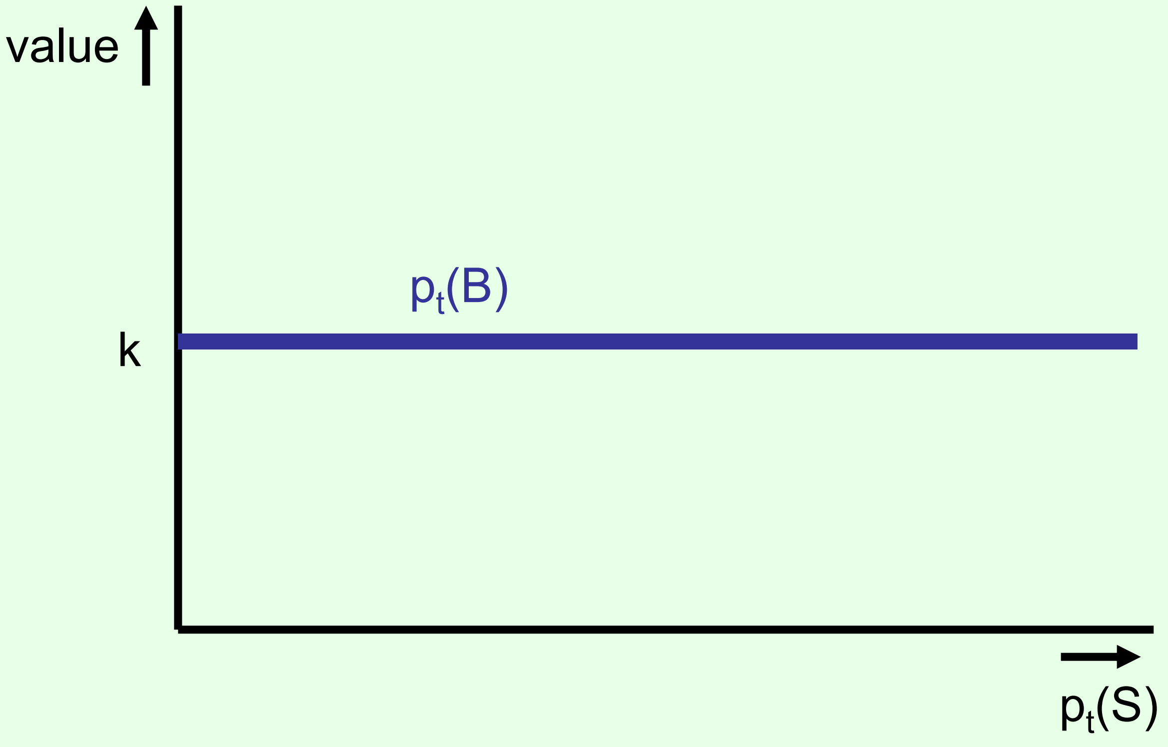
Put options

- A (European) put option $P(S, k, t)$ gives you the right to sell stock S at (strike) price k on (expiry) date t
- How much is a put option worth at time t (as a function of the price of the stock)?

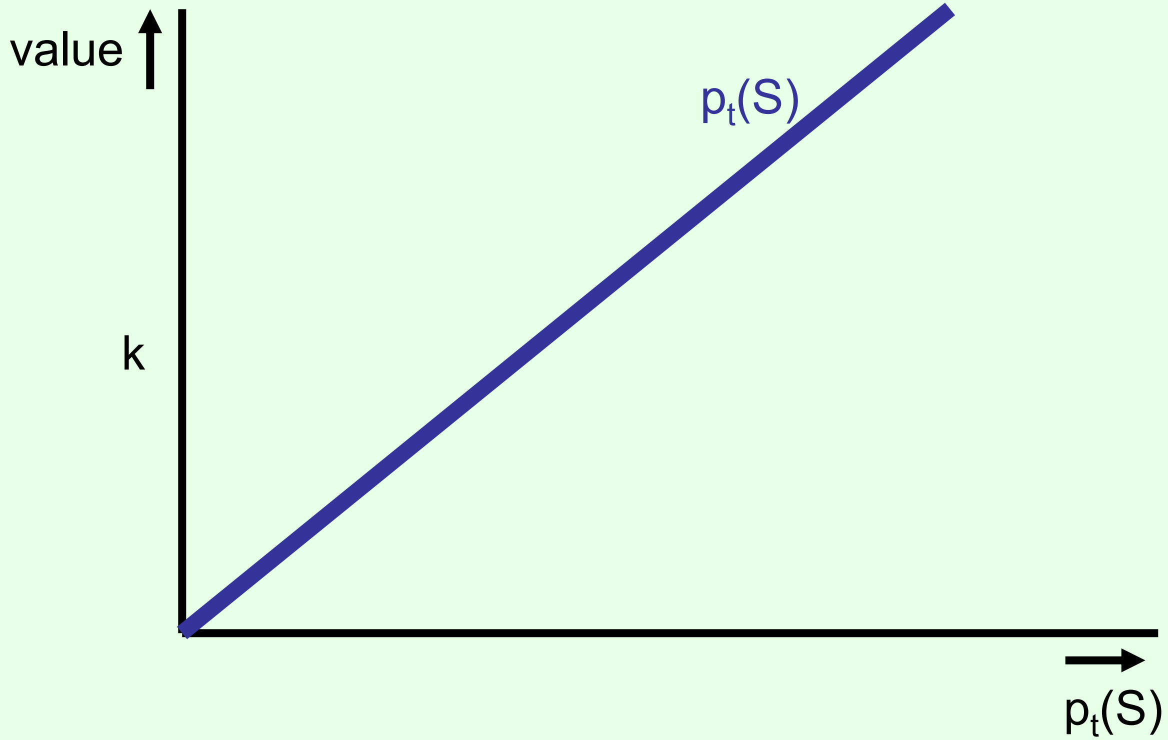


Bonds

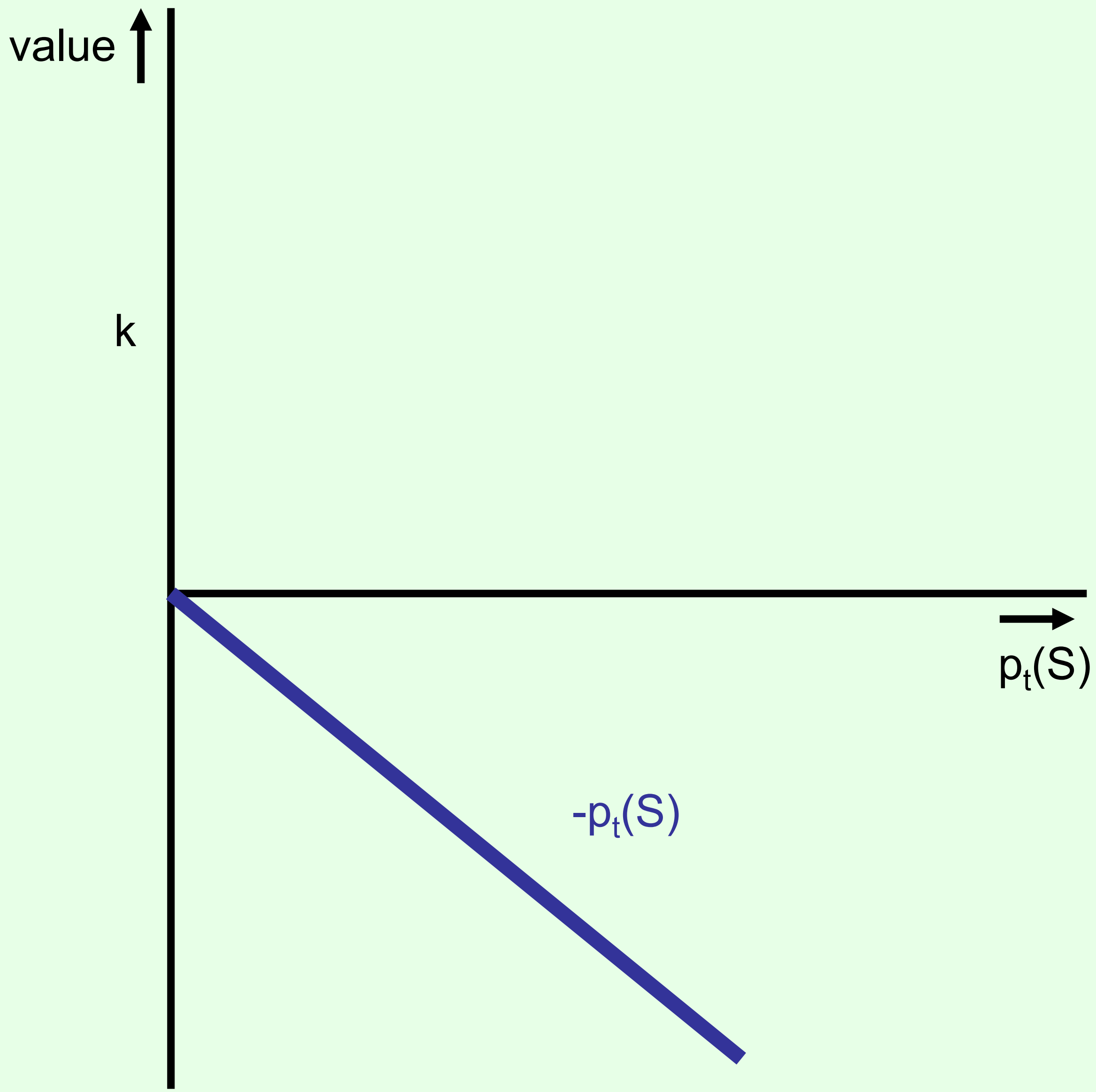
- A bond B(k, t) pays off k at time t



Stocks

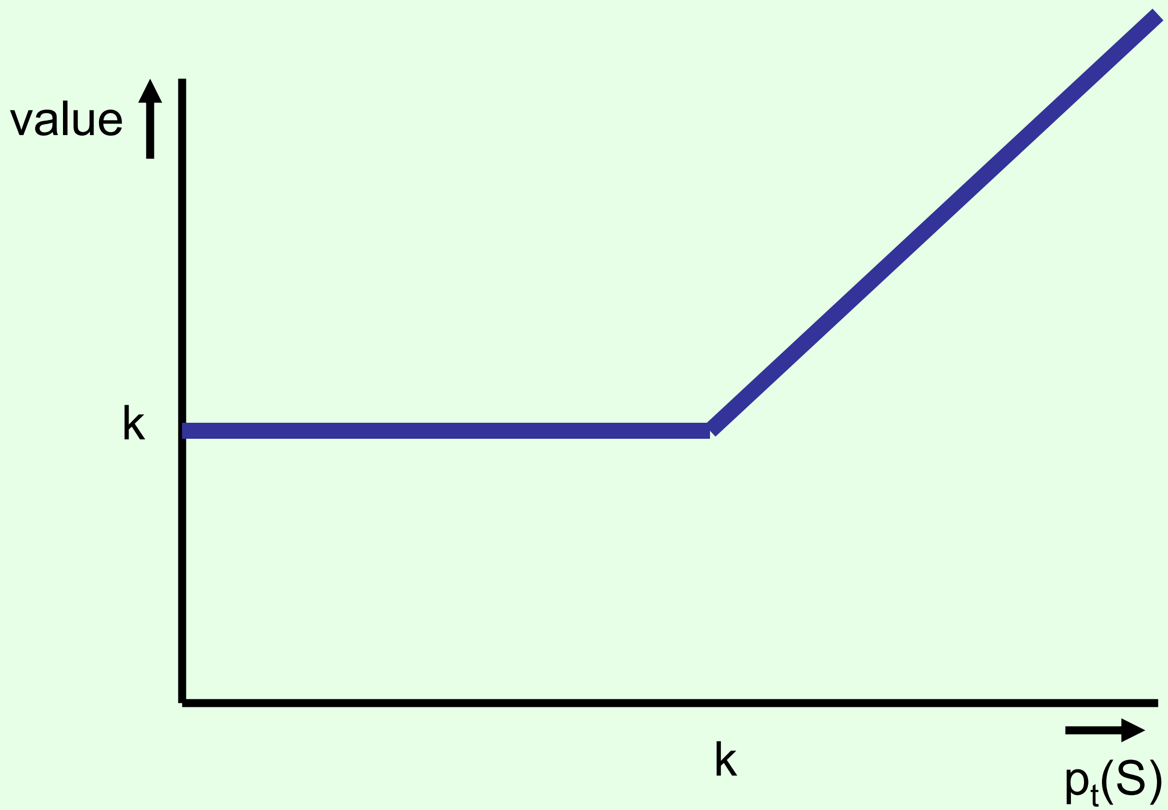


Selling a stock (short)



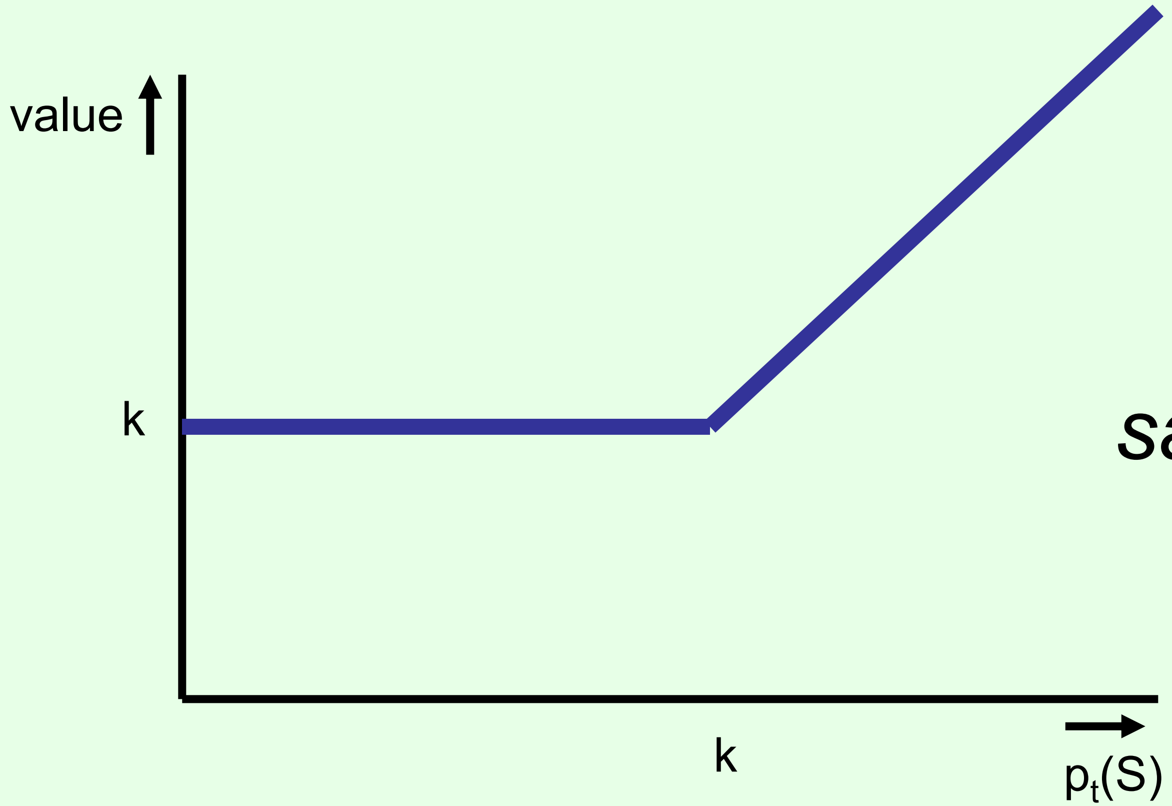
A portfolio

- One call option $C(S, k, t)$ + one bond $B(k, t)$



Another portfolio

- One put option P(S, k, t) + one stock S



same thing!

Put-call parity

- $C(S, k, t) + B(k, t)$ will have the same value at time t as $P(S, k, t) + S$ (regardless of the value of S)
- Assume stocks pay no dividends
- Then, portfolio should have the same value at any time before t as well
- I.e., for any $t' < t$, it should be that $p_{t'}(C(S, k, t)) + p_{t'}(B(k, t)) = p_{t'}(P(S, k, t)) + p_{t'}(S)$

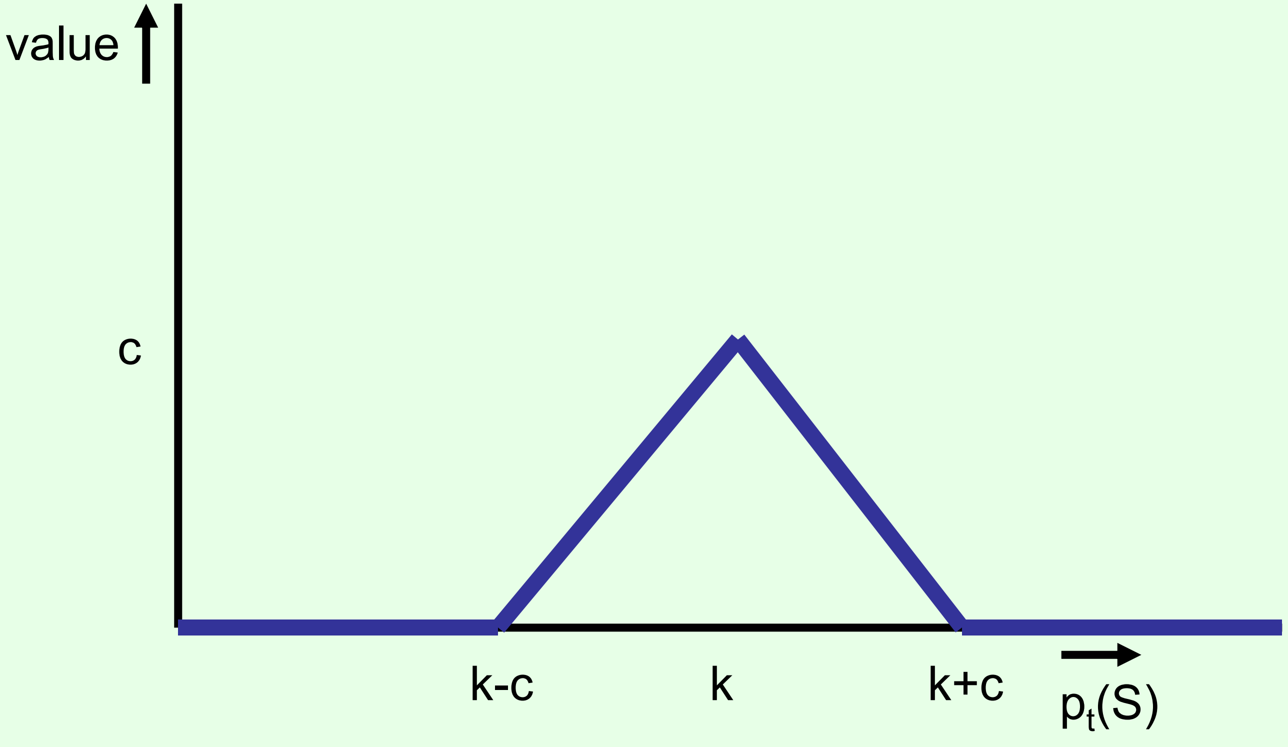
- Arbitrage argument: suppose (say) $p_{t'}(C(S, k, t)) + p_{t'}(B(k, t)) < p_{t'}(P(S, k, t)) + p_{t'}(S)$
- Then: buy $C(S, k, t) + B(k, t)$, sell (short) $P(S, k, t) + S$
- Value of portfolio at time t is 0
- Guaranteed profit!

Another perspective: auctioneer

- Auctioneer receives buy and sell offers, has to choose which to accept
- E.g.: offers received: buy(S, \$10); sell(S, \$9)
- Auctioneer can accept both offers, profit of \$1
- E.g. (put-call parity):
  - sell(C(S, k, t), \$3)
  - sell(B(k, t), \$4)
  - buy(P(S, k, t), \$5)
  - buy(S, \$4)
- Can accept all offers at no risk!

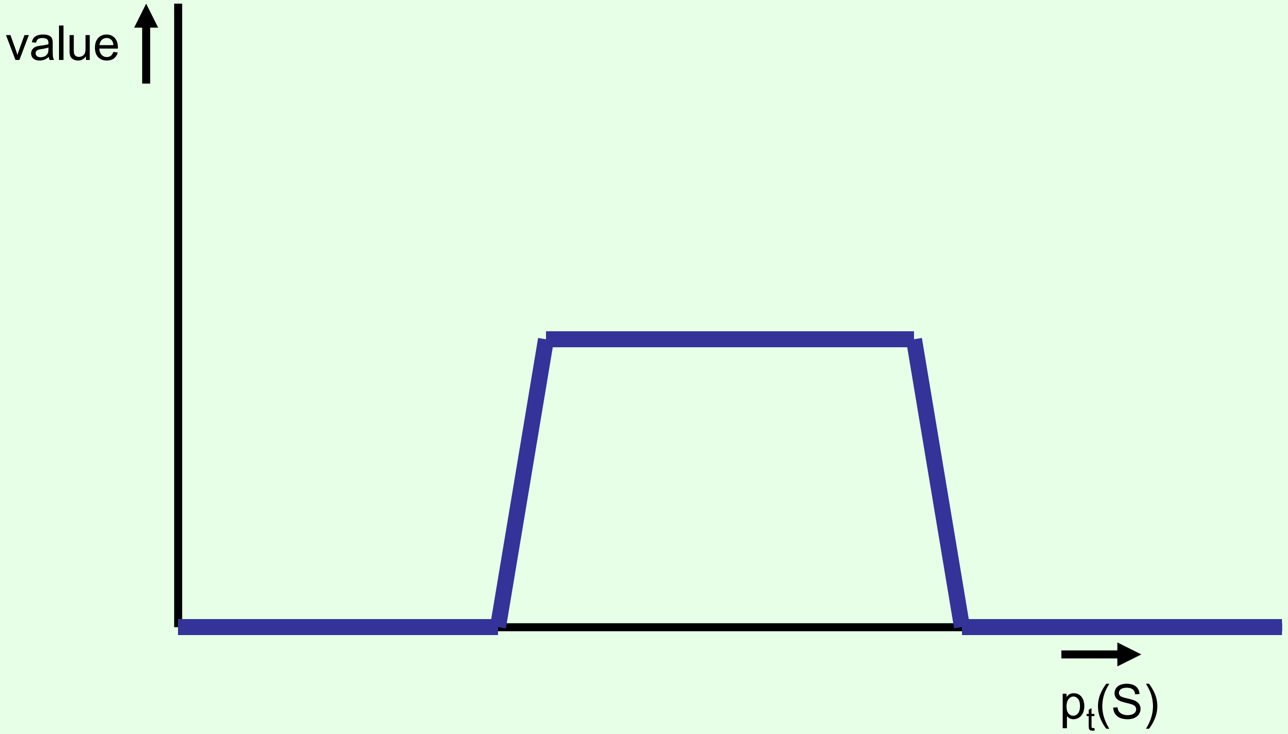
“Butterfly” portfolio

- 1 call at strike price k-c
- -2 calls at strike k
- 1 call at strike k+c



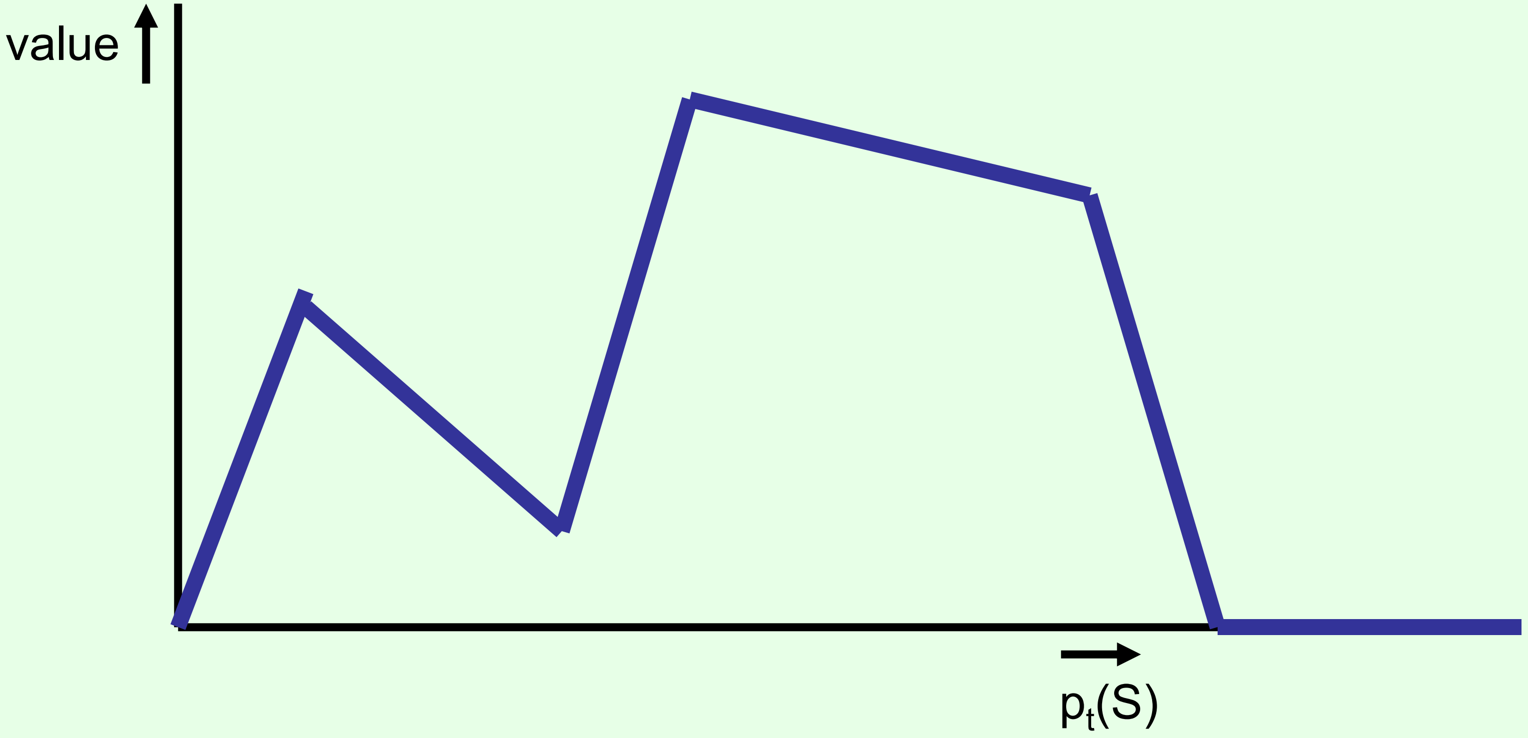
Another portfolio

- Can we create this portfolio?



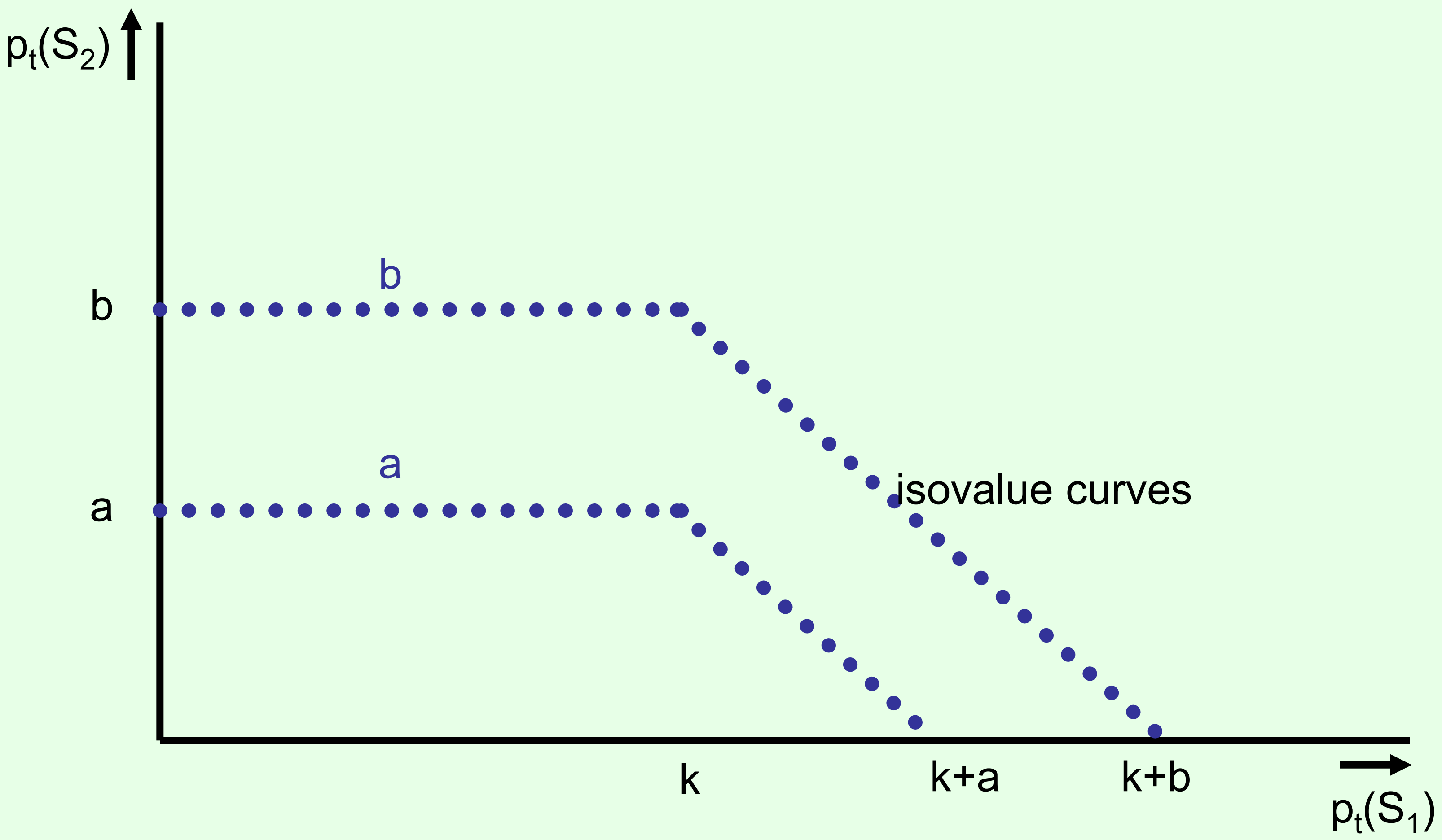
Yet another portfolio

- How about this one?



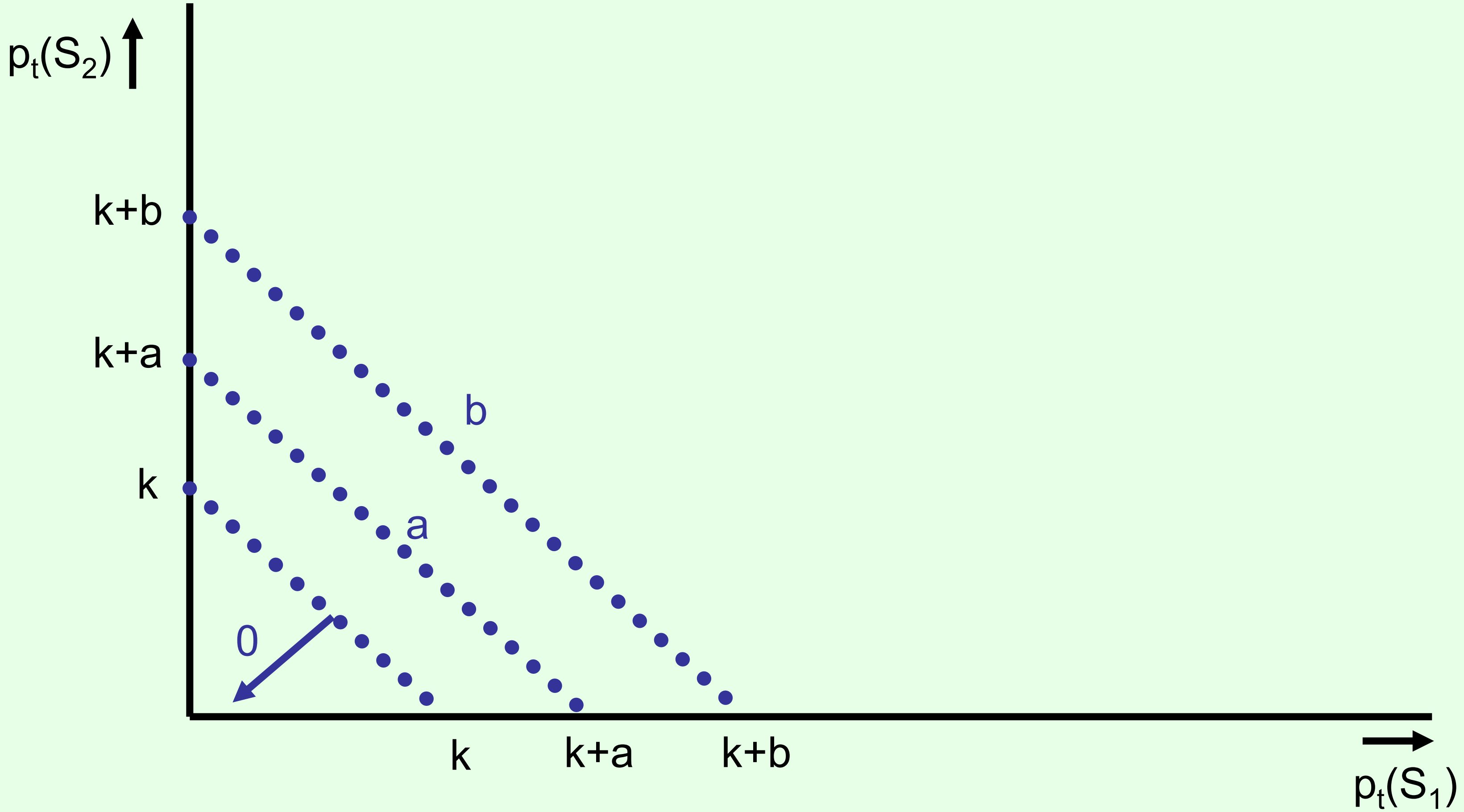
Two different stocks

- A portfolio with $C(S_1, k, t)$ and S_2



Another portfolio

- Can we create this portfolio?
  (In effect, a call option on S_1+S_2)

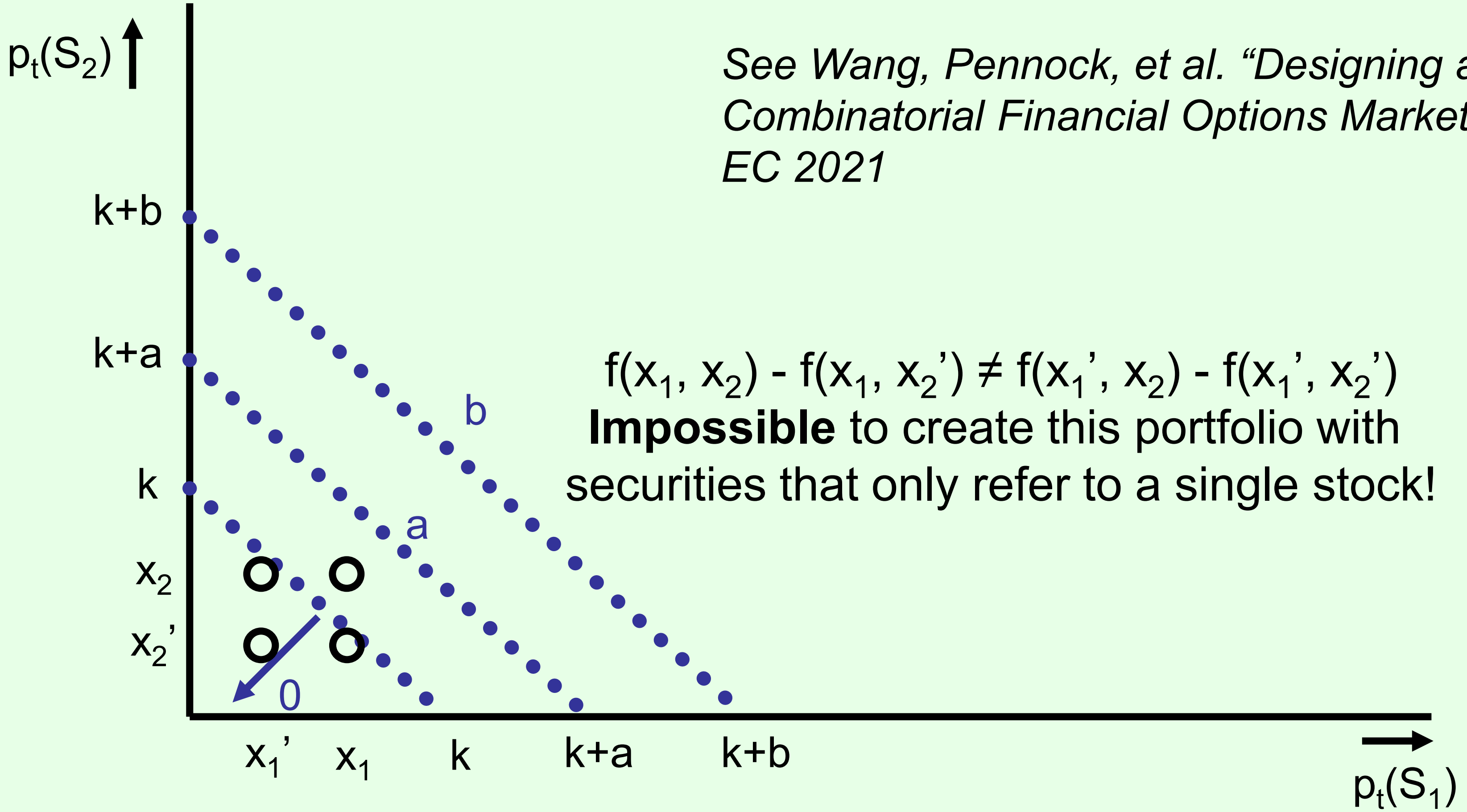


A useful property

- Suppose your portfolio pays off $f(p_t(S_1), p_t(S_2)) = f_1(p_t(S_1)) + f_2(p_t(S_2))$ (additive decomposition over stocks)
- This is all we know how to do
- Then: $f(x_1, x_2) - f(x_1, x_2') = f(x_1) + f(x_2) - f(x_1) - f(x_2') = f(x_2) - f(x_2') = f(x_1', x_2) - f(x_1', x_2')$

Portfolio revisited

- Can we create this portfolio?
  (In effect, a call option on S_1+S_2)



See Wang, Pennock, et al. "Designing a Combinatorial Financial Options Market" EC 2021

$$f(x_1, x_2) - f(x_1, x_2') \neq f(x_1', x_2) - f(x_1', x_2')$$

Impossible to create this portfolio with securities that only refer to a single stock!

Securities conditioned on finite set of outcomes

- E.g., PredictIt: security that pays off 1 if Trump is the Republican nominee in 2024
- Can we construct a portfolio that pays off 1 if Biden is the Democratic nominee AND Trump is the Republican nominee?

	Trump not nom.	Trump nom.
Biden not nom.	$0	$0
Biden nom.	$0	$1

Arrow-Debreu securities

- Suppose S is the set of all states that the world can be in tomorrow
- For each s in S, there is a corresponding Arrow-Debreu security that pays off 1 if s happens, 0 otherwise
- E.g., s could be: Biden is nominee and Trump is nominee and S_1 is at \$4 and S_2 at \$5 and butterfly 432123 flaps its wings in Peru and…
- Not practical, but conceptually useful
- Can think about Arrow-Debreu securities within a domain (e.g., states only involve stock trading prices)
- Practical for small number of states

With Arrow-Debreu securities you can do anything…

- Suppose you want to receive \$6 in state 1, \$8 in state 2, \$25 in state 3

- … simply buy 6 AD securities for state 1, 8 for state 2, 25 for state 3

- Linear algebra: Arrow-Debreu securities are a basis for the space of all possible securities

The auctioneer problem

- Tomorrow there must be one of   
- Agent 1 offers \$5 for a security that pays off \$10 if  or 
- Agent 2 offers \$8 for a security that pays off \$10 if  or 
- Agent 3 offers \$6 for a security that pays off \$10 if 
- Can we accept some of these at offers at no risk?

Reducing auctioneer problem to ~combinatorial exchange winner determination problem

- Let (x, y, z) denote payout under , , , respectively
- Previous problem’s bids:
  - 5 for (0, 10, 10)
  - 8 for (10, 0, 10)
  - 6 for (10, 0, 0)
- Equivalently:
  - (-5, 5, 5)
  - (2, -8, 2)
  - (4, -6, -6)
- Sum of accepted bids should be (≤0, ≤0, ≤0) to have no risk
- Sometimes possible to partially accept bids

A bigger instance (4 states)

- Objective: maximize our worst-case profit
- 3 for (0, 0, 11, 0)
- 4 for (0, 2, 0, 8)
- 5 for (9, 9, 0, 0)
- 3 for (6, 0, 0, 6)
- 1 for (0, 0, 0, 10)

- What if they are partially acceptable?

Settings with large state spaces

- Large = exponentially large
  - Too many to write down
- Examples:
- $S = S_1 \times S_2 \times \dots S_n$
  - E.g., S_1 = {Biden not nom., Biden nom.}, S_2 = {Trump not nom., Trump nom.}, S = {(-B, -T), (-B, +T), (+B, -T), (+B, +T)}
  - If all S_i have the same size k, there are k^n different states
- S is the set of all rankings of n candidates
  - E.g., outcomes of a horse race
  - n! different states (assuming no ties)

Bidding languages

- How should trader (bidder) express preferences?
- Logical bidding languages [Fortnow et al. 2004]:
  - (1) "If Trump nominated OR (DeSantis nominated AND Biden nominated), I want to receive \$10; I'm willing to pay \$6 for this."
- If the state is a ranking [Chen et al. 2007] :
  - (2a) "If horse A ranks 2nd, 3rd, or 4th I want to receive \$10; I'm willing to pay \$6 for this."
  - (2b) "If one of horses A, C, D rank 2nd, I want to receive \$10; I'm willing to pay \$6 for this."
  - (2c) "If horse A ranks ahead of horse C, I want to receive \$10; I'm willing to pay \$6 for this."
- Winner determination problem is NP-hard for all of these, except for (2a) and (2b) which are in P if bids can be partially accepted

A different computational problem

closely related to (separation problem for) winner determination

- Given that the auctioneer has accepted some bids, what is the worst-case outcome (state) for the auctioneer?
- For example:

- Must pay 2 to trader A if horse X or Z is first
- Must pay 3 to trader B if horse Y is first or second
- Must pay 6 to trader C if horse Z is second or third
- Must pay 5 to trader D if horse X or Y is third
- Must pay 1 to trader E if horse X or Z is second

Reduction to weighted bipartite matching

- Must pay 2 to trader A if horse X or Z is first
- Must pay 3 to trader B if horse Y is first or second
- Must pay 6 to trader C if horse Z is second or third
- Must pay 5 to trader D if horse X or Y is third
- Must pay 1 to trader E if horse X or Z is second