15326 - Computational Microeconomics

Homework 0: Probability background (due Sep. 8 by 5pm Eastern)

Please read the rules for assignments on the course web page (`http://www.cs.cmu.edu/~15326-f23/`). Use Piazza for questions and Gradescope to turn this in. Throughout, please provide both exact expressions and (possibly approximate) numerical values for answers.

1 Conditional probability

We have three different toys, a red one (r), a blue one (b) and a green one (g). Children have preferences over these toys, which determine which one they choose to play with when offered a choice between two of them. For example, the ordering $b \succ g \succ r$ indicates a preference for the blue over the green, the green over the red, and (by transitivity) the blue over the red. No child is ever indifferent between between two toys. All preference orderings are equally likely. For example, there are just as many children with preferences $b \succ g \succ r$ as there are with preferences $r \succ g \succ b$.

Now imagine that we offer a child with unknown preferences two of the toys, say the red and the blue one. The child picks one, say, the blue one, and plays with it. After a while the child gets bored with the toy and so we offer the two remaining toys, i.e., the green and the red one. **What** is the probability that in this second round the child will choose to play with the toy that we already offered in the first round (i.e., the red one)? Try to use formal notation in your answer, e.g., $P(r \succ g \mid b \succ r)$.

2 Probability density functions

Now assume we have three bidders in a one-shot auction: Alice, Bob, and Carol. They bid independently and simultaneously on the item according to their valuations (i.e., they bid their true values, unless constrained by their budgets), which follow certain probability density functions (PDF).

2.1 Calculate expectation of bid

Assume Alice's valuation for the item is drawn from a uniform PDF: $v_{\text{A}} \sim U[1,3]$, but she only has a budget of 2. She cannot bid higher than his budget. Thus, her bid will be $b_{\text{A}} = \min(v_{\text{A}}, 2)$. **What** is the expected value of her bid b_{A}?

2.2 Calculate probability

The rule of the auction is easy: whoever bids highest wins. Assume that Bob's bid b_{B} and Carol's bid b_{C} have no budget constraints and are drawn independently from a uniform distribution: $b_{\text{B}}, b_{\text{C}} \sim U[0,3]$. **What** is then the cumulative distribution function (CDF) of the highest bid between Bob and Carol? (Recall that the CDF gives the probability that a random variable is below a certain value. Thus, this CDF gives the probability that both their bids are below that value.)

Now suppose that Alice, for some reason, does not get to evaluate the item, so she simply bids her expected value $\mathbb{E}[v_{\text{A}}]$. **What** is the probability that Alice wins the auction with $\mathbb{E}[v_{\text{A}}]$?

Bonus question: **What** is the probability that Alice wins, if she does evaluate the item, and bids b_{A} (keep in mind the budget constraint) instead of $\mathbb{E}[v_{\text{A}}]$?

3 Combinatorics

We have 10 students with all different heights. Suppose they have to sit in a single line of 10 chairs, behind each other, facing a board. We are hoping that shorter students always sit in front of taller ones, so that everyone can see the board clearly. Suppose the seats are initially assigned randomly (uniformly at random). Then, we want the students to switch seats to sit in order. **What** is the probability that the following number of students have to change their seats? You may use expressions such as 10! and $\binom{10}{2}$ in your answer.

- No one.
- Exactly one student. (There is a simple commonsense answer to this.)
- Exactly two students.