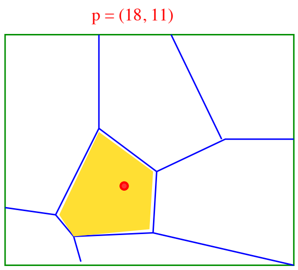
# Point Location

- Preprocess a planar, polygonal subdivision for point location queries.

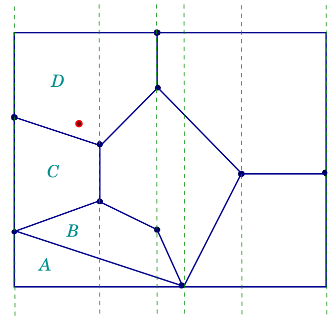



- Input is a subdivision $S$ of complexity $n$, say, number of edges.
- Build a data structure on $S$ so that for a query point $p = (x, y)$, we can find the face containing $p$ fast.
- Important metrics: space and query complexity.

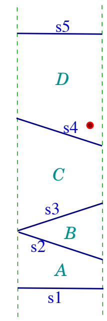


# The Slab Method

- Draw a vertical line through each vertex. This decomposes the plane into slabs.
- In each slab, the vertical order of line segments remains constant.




Partition into slabs



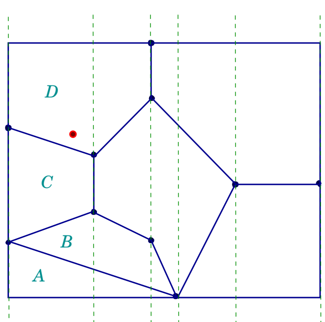

Slab 1

- If we know which slab $p = (x, y)$ lies, we can perform a binary search, using the sorted order of segments.

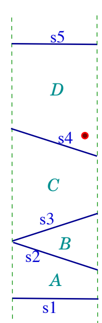


# The Slab Method

- To find which slab contains $p$, we perform a binary search on $x$, among slab boundaries.
- A second binary search in the slab determines the face containing $p$.




Partition into slabs




Slab 1

- Thus, the search complexity is $O(\log n)$.
- But the space complexity is $\Theta(n^2)$.

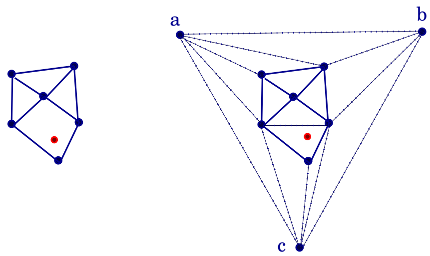


# Optimal Schemes

- There are other schemes ($kd$-tree, quad-trees) that can perform point location reasonably well, they lack theoretical guarantees. Most have very bad worst-case performance.
- Finding an optimal scheme was challenging. Several schemes were developed in 70's that did either $O(\log n)$ query, but with $O(n \log n)$ space, or $O(\log^2 n)$ query with $O(n)$ space.
- Today, we will discuss an elegant and simple method that achieved optimality, $O(\log n)$ time and $O(n)$ space [D. Kirkpatrick '83].
- Kirkpatrick's scheme however involves large constant factors, which make it less attractive in practice.
- Later we will discuss the use of persistent data structures to obtain a practical and almost optimal solution.

# Kirkpatrick's Algorithm

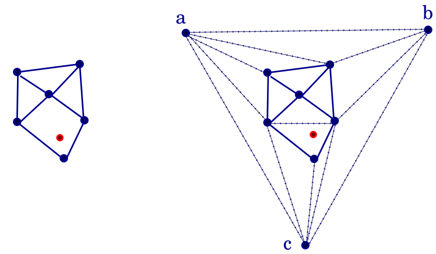
- Start with the assumption that planar subdivision is a triangulation.
- If not, triangulate each face, and label each triangular face with the same label as the original containing face.
- If the outer face is not a triangle, compute the convex hull, and triangulate the pockets between the subdivision and CH.
- Now put a large triangle $abc$ around the subdivision, and triangulate the space between the two.






# Modifying Subdivision

- By Euler'e formula, the final size of this triangulated subdivision is still $O(n)$.
- This transformation from $S$ to triangulation can be performed in $O(n \log n)$ time.




- If we can find the triangle containing $p$, we will know the original subdivision face containing $p$.



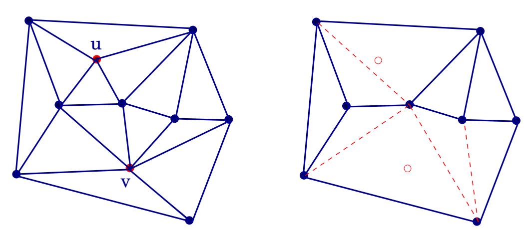
# Hierarchical Method

- Kirkpatrick's method is hierarchical: produce a sequence of increasingly coarser triangulations, so that the last one has $O(1)$ size.
- Sequence of triangulations $T_0, T_1, \ldots, T_k$, with following properties:
  1. $T_0$ is the initial triangulation, and $T_k$ is just the outer triangle $abc$.
  2. $k$ is $O(\log n)$.
  3. Each triangle in $T_{i+1}$ overlaps $O(1)$ triangles of $T_i$.
- Let us first discuss how to construct this sequence of triangulations.



# Building the Sequence

- Main idea is to delete some vertices of $T_i$.
- Their deletion creates holes, which we re-triangulate.




Vertex deletion and re−triangulation

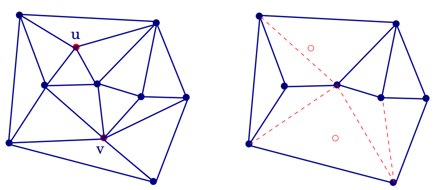
- We want to go from $O(n)$ size subdivision $T_0$ to $O(1)$ size subdivision $T_k$ in $O(\log n)$ steps.
- Thus, we need to delete a constant fraction of vertices from $T_i$.
- A critical condition is to ensure each new triangle in $T_{i+1}$ overlaps with $O(1)$ triangles of $T_i$.

# Independent Sets

- Suppose we want to go from $T_i$ to $T_{i+1}$, by deleting some points.
- Kirkpatrick's choice of points to be deleted had the following two properties:

[Constant Degree] Each deletion candidate has $O(1)$ degree in graph $T_i$.

- If $p$ has degree $d$, then deleting $p$ leaves a hole that can be filled with $d-2$ triangles.
- When we re-triangulate the hole, each new triangle can overlap at most $d$ original triangles in $T_i$.



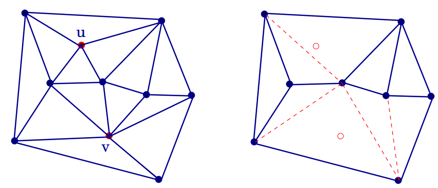

Vertex deletion and re−triangulation



# Independent Sets

[Independent Sets] No two deletion candidates are adjacent.

- This makes re-triangulation easier; each hole handled independently.




Vertex deletion and re−triangulation



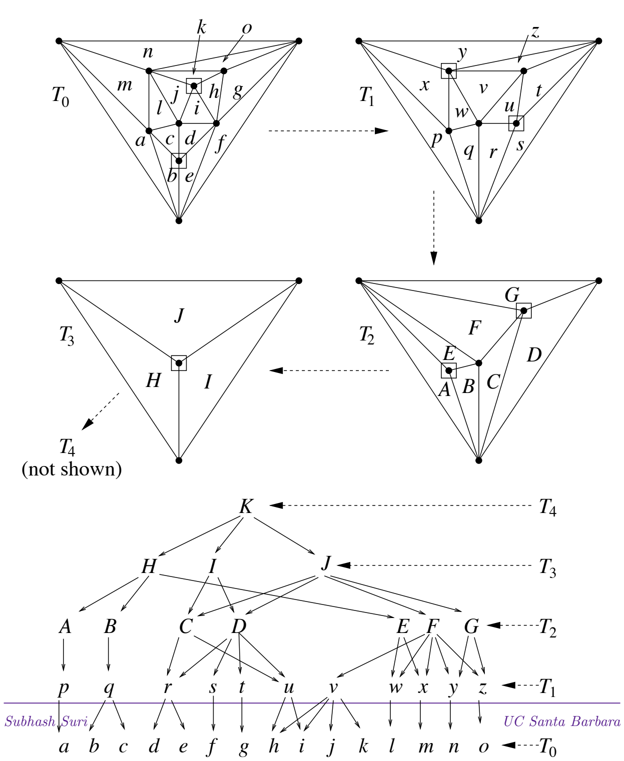
# I.S. Lemma

**Lemma:** Every planar graph on $n$ vertices contains an independent vertex set of size $n/18$ in which each vertex has degree at most $8$. The set can be found in $O(n)$ time.

- We prove this later. Let's use this now to build the triangle hierarchy, and show how to perform point location.
- Start with $T_0$. Select an ind set $S_0$ of size $n/18$, with max degree $8$. Never pick $a, b, c$, the outer triangle's vertices.
- Remove the vertices of $S_0$, and re-triangulate the holes.
- Label the new triangulation $T_1$. It has at most $\frac{17}{18}n$ vertices. Recursively build the hierarchy, until $T_k$ is reduced to $abc$.
- The number of vertices drops by $17/18$ each time, so the depth of hierarchy is $k = \log_{18/17} n \;\approx\; 12 \log n$

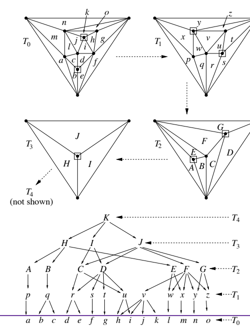


# Illustration



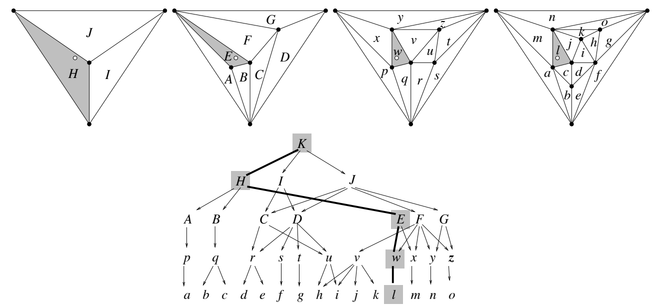
# The Data Structure

- Modeled as a DAG: the root corresponds to single triangle $T_k$.
- The nodes at next level are triangles of $T_{k-1}$.
- Each node for a triangle in $T_{i+1}$ has pointers to all triangles of $T_i$ that it overlaps.
- To locate a point $p$, start at the root. If $p$ outside $T_k$, we are done (exterior face). Otherwise, set $t = T_k$, as the triangle at current level containing $p$.





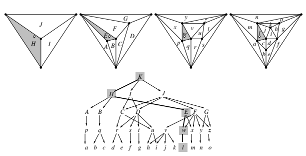
# The Search



- Check each triangle of $T_{k-1}$ that overlaps with $t$—at most 6 such triangles. Update $t$, and descend the structure until we reach $T_0$.
- Output $t$.



# Analysis



- Search time is $O(\log n)$—there are $O(\log n)$ levels, and it takes $O(1)$ time to move from level $i$ to level $i - 1$.
- Space complexity requires summing up the sizes of all the triangulations.
- Since each triangulation is a planar graph, it is sufficient to count the number of vertices.
- The total number of vertices in all triangulations is

$$n \left(1 + (17/18) + (17/18)^2 + (17/18)^3 + \cdots\right) \leq 18n.$$

- Kirkpatrick structure has $O(n)$ space and $O(\log n)$ query time.

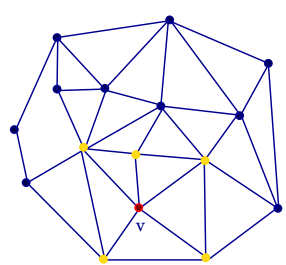


# Finding I.S.

- We describe an algorithm for finding the independent set with desired properties.
- Mark all nodes of degree $\geq 9$.
- While there is an unmarked node, do

1. Choose an unmarked node $v$.
2. Add $v$ to IS.
3. Mark $v$ and all its neighbors.

- Algorithm can be implemented in $O(n)$ time—keep unmarked vertices in list, and representing $T$ so that neighbors can be found in $O(1)$ time.



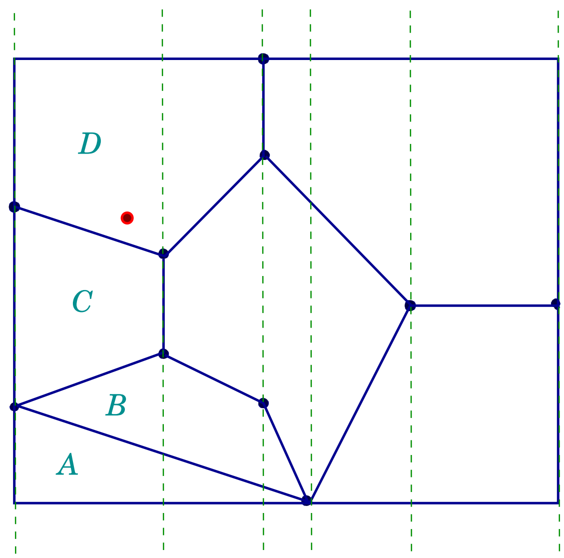
# I.S. Analysis

- **Existence of large size, low degree IS follows from Euler's formula for planar graphs.**
- **A triangulated planar graph on $n$ vertices has $e = 3n - 6$ edges.**
- **Summing over the vertex degrees, we get**
$$\sum_v deg(v) = 2e = 6n - 12 < 6n.$$
- **We now claim that at least $n/2$ vertices have degree $\leq 8$.**
- **Suppose otherwise. Then $n/2$ vertices all have degree $\geq 9$. The remaining have degree at least 3. (Why?)**
- **Thus, the sum of degrees will be at least $9\frac{n}{2} + 3\frac{n}{2} = 6n$, which contradicts the degree bound above.**
- **So, in the beginning, at least $n/2$ nodes are unmarked. Each chosen $v$ marks at most $8$ other nodes (total $9$ counting itself.)**
- **Thus, the node selection step can be repeated at least $n/18$ times.**
- **So, there is a I.S. of size $\geq n/18$, where each node has degree $\leq 8$.**

# 1 Point Location using Persistent Search Trees

In this section we describe another approach to the point-location problem, based on a fully-functional, or "persistent", representation of sets. We obtain an $O(\log n)$ query time and $O(n \log n)$ space solution. So it is not optimal in terms of space. This can be made optimal by a more sophisticated way to make data structures persistent.

Let's go back to the approach of dividing the polygonal subdivision into slabs. With the representation described earlier, doing a point location query took only $O(\log n)$ time. The problem was that the data structure took too much space.




Partition into slabs

Let's examine this problem a little more closely. The reason that the space is potentially quadratic in $n$ is that there is the possibility that a long horizontal segment can be divided up into many pieces, one for each slab. For example, the segment separating regions $A$ and $B$ is divided into three parts.

If there were some way that we could avoid having to store that redundantly in multiple slabs, we have a chance of controlling the space blow-up. This is the approach we take here.

With this in mind, let's examine the difference between consecutive slabs. Call the two slabs $S_i$ and $S_{i+1}$. The differences between the two slabs occur when there is a vertex $v$ of the subdivision along their boundary. To convert $S_i$ into $S_{i+1}$ we delete from $S_i$ the segments incident on $v$ from the left, and we insert into $S_i$ the segments incident on $v$ from the right. We do this for all the vertices on the boundary between the two slabs.

The total number of insertions and deletions that occur in the entire diagram is twice the number of segments, or $O(n)$. It turns out that using functional programming we can incurr a space coast of $O(\log n)$ per insertion or deletion.

### 1.1 Fully Functional Ordered Sets

A slab is represented by a set of non-vertical segments. For each non-vertical we're going to keep the equation of its line. So for segment $i$ we keep $m_i x + b_i$, where $m_i$ is the slope and $b_i$ is the intercept. This representation will allow us to determine which of two segments is higher (at a particular $x$ value).

A slab is a set of segments. We're going to store this in a balance binary search tree. It will be a fully functional implementation. This means, for example, that if we insert a segment $a$ into a set $S$, it returns a new set $S'$. The original set $S$ remains the same.

Our set data structure will support the following operations:

| | |
|---|---|
| Insert$(S, a)$: | Insert a segment $a$ into a set $S$. Return the result. |
| Delete$(S, a)$: | Delete a segment $a$ from the set $S$. Return the result. |
| Lookup$(S, p)$: | Given a point $p$ and a set of segments $S$, return a segment from $S$ that neighbors the region containing $p$. |

Fully functional implementations of sets based on balanced binary search trees are standard features of functional programming languages such as Haskell, Ocaml, and SML. Furthermore, it's easy to create fully functional implementations in any language. You just have to maintain the discipline of never modifying any fields of a note. Instead you create a new node initialized with the values you want in each field.

Because the sets are stored as a tree, the way lookup works is that it returns the last segment touched when searching down the tree for a segment containing point $p$.

The space used by Insert and Delete is $O(\log n)$. This happens automatically in a functional implementation of sets using balanced binary search trees. (You can also think of it as copying the path from the root to a node that changes.)

### 1.2 Building the Data Structure

Start out with an empty slab $S_0$. Process the vertices $v_0, \ldots, v_k$ in left-to-right order. To process vertex $v_i$ we take slab $S_i$ and delete the segments connecting to it on the left and insert the ones that connect to it on the right. After this, we have slab $S_{i+1}$. The space used by this is $O(n \log n)$.

As we do this we build a an array $A$, sorted by $x$, with pointers to the slabs that we constructed in this scan. So given $x$ we can find in $O(\log n)$ time the slab containing $x$.

We're also going to build a dictionary $D$ which keeps, for each non-vertical segment in the diagram, the name of the region above it and below it.

### 1.3 Doing Point Location

Given a point $p = (x, y)$ here is how we find the region containing $p$.

First we find the slab containing $p$ by doing binary search in the array $A$. Say it's in slab $S_i$. Now we do a lookup of $p$ in $S_i$. This gives us a segment $s$ bounding the region containing point $p$. Now we determine if the point $p$ is above or below $s$. So we can then lookup in the dictionary $D$ the name of the region containing the point p.

This process is $O(\log n)$ time.

### 1.4 Achieving $O(n)$ Space

Optimal space can be achieved by using the "fat node" method of making data structures persistent. It's described in this paper by Sarnak and Tarjan.

`www.link.cs.cmu.edu/15859-f07/papers/point-location.pdf`

The details are beyond the scope of this course, but feel free to talk to me about how it works.