Topic 2: Concrete Models and Tight Upper and Lower Bounds

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Theme: Tight Upper and Lower Bounds

- Number of comparisons to sort an array
- Number of exchanges to sort an array
- Number of comparisons needed to find the largest and second-largest elements in an array
- Number of probes into a graph needed to determine if the graph is connected

Formal Model

- Look at models which specify exactly which operations may be performed on the input, and what they cost
  - E.g., performing a comparison, or swapping a pair of elements
- An upper bound of f(n) means the algorithm takes at most f(n) steps on any input of size n
- A lower bound of g(n) means for any algorithm there exists an input for which the algorithm takes at least g(n) steps on that input

Sorting in the Comparison Model

- In the comparison model, we have n items in some initial order

  An algorithm may compare two items (asking is $a_i > a_j$?) at a cost of 1
  - Moving the items is free

- No other operations allowed, such as XORing, hashing, etc.

- Sorting: given an array a = $[a_1, \ldots, a_n]$, output a permutation π so that $[a_{\pi(1)}, \ldots, a_{\pi(n)}]$ in which the elements are in increasing order

Sorting Lower Bound

- Theorem: Any deterministic comparison-based sorting algorithm must perform at least $\lg_2(n!)$ comparisons to sort n elements in the worst case

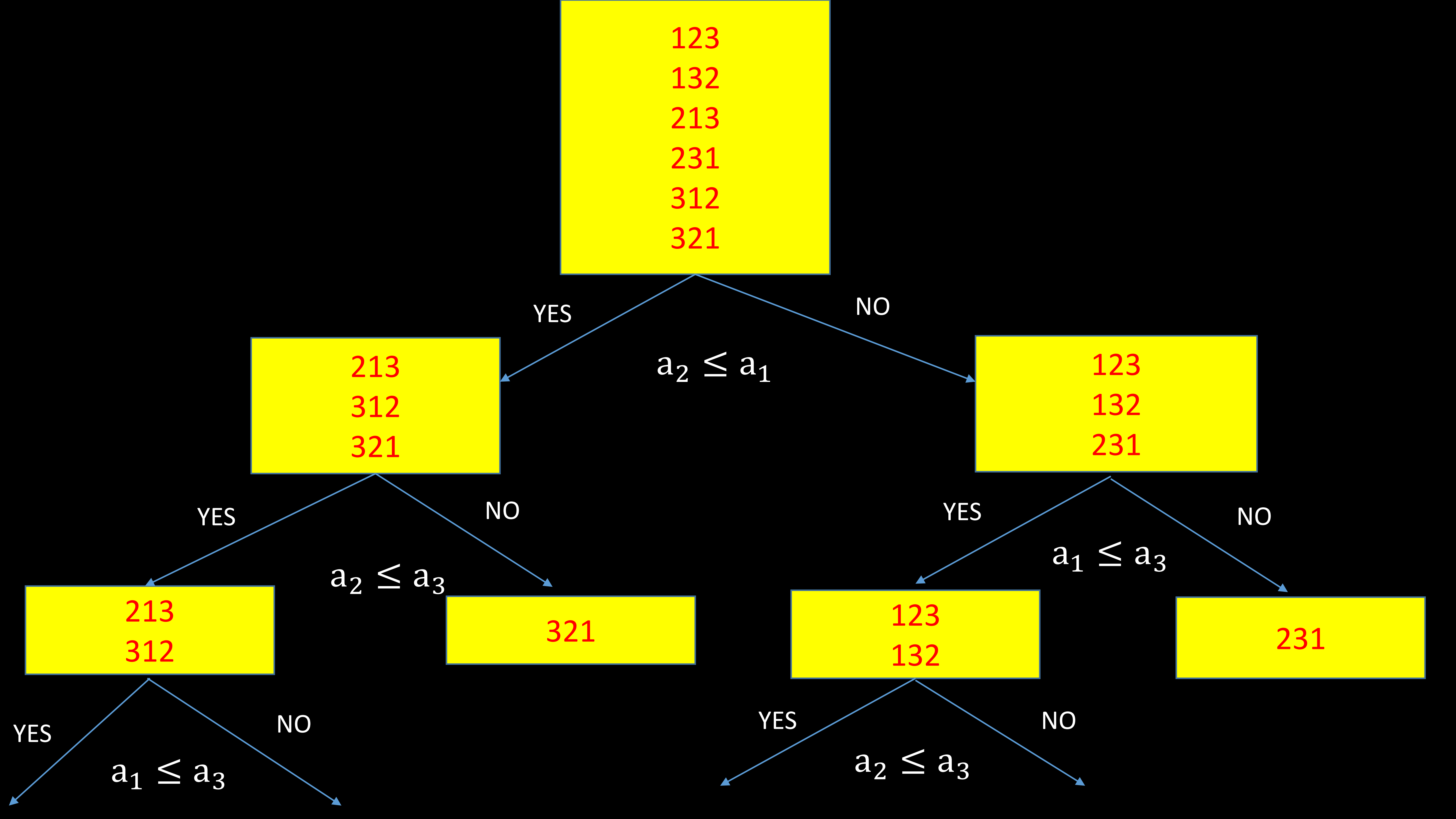
- I.e., for any sorting algorithm A and $n \geq 2$, there is an input I of size n so that A makes $\geq \lg(n!) = \Omega(n \log n)$ comparisons to sort I.

- Need to rule out any possible algorithm

- Proof is information-theoretic

Sorting Lower Bound

- Proof: Suppose there is a problem with M possible outputs
  - For sorting $\mathrm{M} = \mathrm{n}!$ since for each possible output permutation π, there is an input for which the output is π
- Suppose for each possible output, there is an input for which that output is the only correct answer
  - For sorting there are inputs for which π is the only correct answer
- Then there is a lower bound of $\lg_2 \mathrm{M}$
  - Consider a set of inputs in 1-to-1 correspondence with the M possible outputs
  - Algorithm needs to find out which of the M inputs we have
  - There's a path removing at most half of the possible inputs at each node


123
132
213
231
312
321
YES
NO
$a_2 \le a_1$
213
312
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123
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YES
NO
$a_2 \le a_3$
YES
NO
$a_1 \le a_3$
213
312
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123
132
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YES
NO
$a_1 \le a_3$
YES
NO
$a_2 \le a_3$

Sorting Lower Bound

- Information-theoretic: need $\lg(n!)$ bits of information about the input before we can correctly decide on the output

- $\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + \ldots + \lg(1) < n \lg n$
- $\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + \ldots + \lg(1) > \left(\frac{n}{2}\right) \lg\left(\frac{n}{2}\right) = \Omega(n \lg n)$

- $n! \in [\left(\frac{n}{e}\right)^n, n^n]$, so $n \lg n - n \lg e < \lg(n!) < n \lg n$

  $n \lg n - 1.443n < \lg(n!) < n \lg n$

- $\lg(n!) = (n \lg n)(1 - o(1))$

Sorting Upper Bounds

- Suppose for simplicity n is a power of 2

- Binary insertion sort: using binary search to insert each new element, the number of comparisons is $\sum_{k=2,\dots,n} \lceil \lg k \rceil \le n \lg n$
  - Note: may need to move items around a lot, but only counting comparisons

- Mergesort: merging two sorted lists of n/2 elements requires at most n-1 comparisons
  - Unrolling the recurrence, total number of comparisons is

$$(n-1) + 2\left(\frac{n}{2} - 1\right) + 4\left(\frac{n}{4} - 1\right) + \cdots + \frac{n}{2}(2-1) = n \lg n \; - (n-1) < n \lg n$$

Selection in the Comparison Model

- How many comparisons are necessary and sufficient to find the maximum of n elements in the comparison model?

- Claim: n-1 comparisons are sufficient
- Proof: scan from left to right, keep track of the largest element so far

- For lower bounds, what does our earlier information-theoretic argument give?
  - Only $\Omega(\log n)$, which is too weak

- Also, we have to look at all elements, otherwise we may have not looked at the largest, but that can be done with n/2 comparisons, also not tight

Lower Bound for Finding the Maximum

- Claim: n-1 comparisons are needed in the worst-case to find the maximum of n elements

- Proof: suppose A is an algorithm which finds the maximum of n distinct elements using fewer than n-1 comparisons
- Construct a graph G in which we join two elements by an edge if they are compared by A
- G has at least 2 connected components C_1 and C_2
- Suppose A outputs element u as the maximum, and $u \in C_1$
- Add a large positive number to each element in C_2
- Does not change any of the comparisons made by A, so will still output u
- But now u is not the maximum, so A is incorrect

Lower Bound for Finding the Maximum

- Recap: upper and lower bounds match at n-1
- Argument different from information-theoretic bound for sorting
- Instead,
  - if algorithm makes too few comparisons on some input In and outputs Out,
  - find another input In’ where the algorithm makes the same comparisons and also outputs Out,
  - but Out is not a correct output for In’

An Adversary Argument

- If algorithm makes “too few” comparisons, fool it into giving an incorrect answer

- Any deterministic algorithm sorting 3 elements requires at least 3 comparisons
  - If < 2 comparisons, some element not looked at and the algorithm is incorrect
  - After first comparison, 3 elements are w, l, and z, the winner and loser of the first comparison, as well as the uninvolved item
  - If the second query is between w and z, say
    - w is larger
  - If the second query is between l and z, say
    - l is smaller
  - Algorithm needs one more comparison for correctness

- Goal: answer comparisons so that (a) answers consistent with some input In, (b) answers make the algorithm perform “many” comparisons

First and Second Largest of n Elements

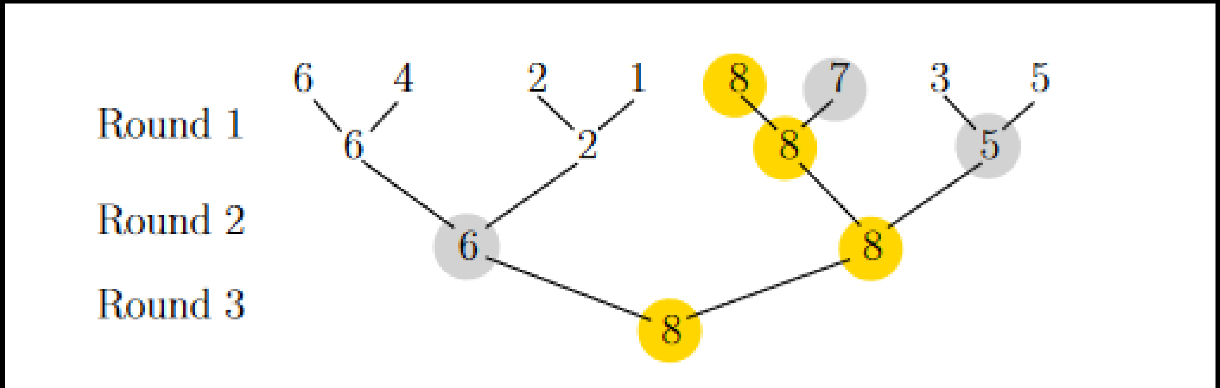
- How many comparisons are necessary (lower bound) and sufficient (upper bound) to find the first and second largest of n distinct elements?
- Claim: n-1 comparisons are needed in the worst-case
- Proof: need to at least find the maximum

What about Upper Bounds?

- Claim: 2n-3 comparisons are sufficient to find the first and second-largest of n elements
- Proof: find the largest using n-1 comparisons, then find the largest of the remainder using n-2 comparisons, so 2n-3 total
- Upper bound is 2n-3, and lower bound n-1, both are $\Theta(n)$ but can we get tight bounds?

Second Largest of n Elements Upper Bound

- Claim: $n + \lg n - 2$ comparisons are sufficient to find the first and second-largest of n elements
- Proof: find the maximum element using n-1 comparisons by grouping elements into pairs, finding the maximum in each pair, and recursing



- What can we say about the second maximum?
  - Must have been directly compared to the maximum and lost, so lg(n)-1 additional comparisons suffice. Kislitsyn (1964) shows this is optimal

Sorting in the Exchange Model

- Consider a shelf containing n unordered books to be arranged alphabetically. How many swaps do we need to order them?

- In the exchange model, you have n items and the only operation allowed on the items is to swap a pair of them at a cost of 1 step

  - All other work is free, e.g., the items can be examined and compared

- How many exchanges are necessary and sufficient?

Sorting in the Exchange Model

- Claim: n-1 exchanges is sufficient
- Proof: here's an algorithm:
- In first step, swap the smallest item with the item in the first location
- In second step, swap the second smallest item with the item in the second location
- In k-th step, swap the k-th smallest item with the item in the k-th location
  - If no swap is necessary, just skip a given step
- No swap ever undoes our previous work
- At the end, the last item must already be in the correct location

Lower Bound for Sorting in Exchange Model

- Claim: n-1 exchanges are necessary in the worst case
- Proof: create a directed graph in which the edge (i,j) means the book in location i must end up in location j

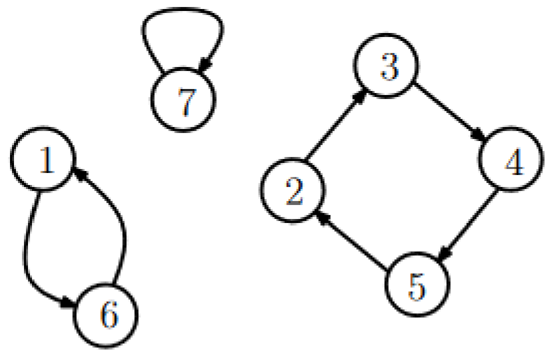
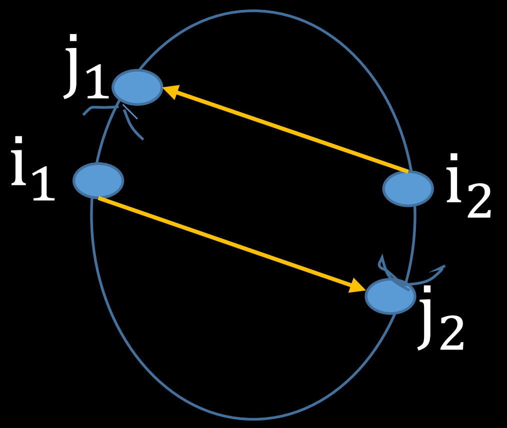


Figure 1: Graph for input [f c d e b a g]

- Graph is a set of cycles
  - Indegree and Outdegree of each node is 1

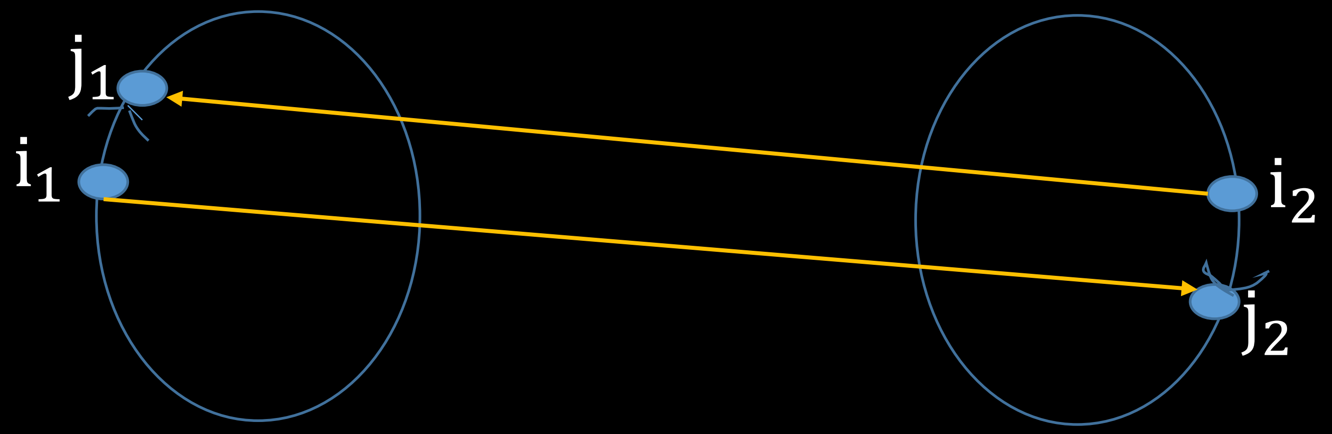
Lower Bound for Sorting in Exchange Model

- What is the effect of exchanging any two elements in the same cycle?
  - Suppose we have edges (i_1, j_1) and (i_2, j_2) and swap elements in locations i_1 and i_2
  - This replaces these edges with (i_2, j_1) and (i_1, j_2) since now the item in position i_2 need to go to j_1 and item in position i_1 need to go to j_2
  - Since i_1 and i_2 in the same cycle, now we get two disjoint cycles



Lower Bound for Sorting in Exchange Model

- What is the effect of exchanging any two elements in different cycles?
  - If we swap elements i_1 and i_2 in different cycles, similar argument shows this merges two cycles into one cycle



Lower Bound for Sorting in Exchange Model

- What is the effect of exchanging any two elements in the same cycle?
  - Get two disjoint cycles
- What is the effect of exchanging any two elements in different cycles?
  - Merges two cycles into one cycle
- Corner cases also result in self loop and create two disjoint cycles
- How many cycles are in the final sorted array?
  - n cycles
- Suppose we begin with an array [n, 1, 2, …, n-1] with one big cycle
- Each step increases the number of cycles by at most 1, so need n-1 steps

Optional content

Will not appear on the homework or exams

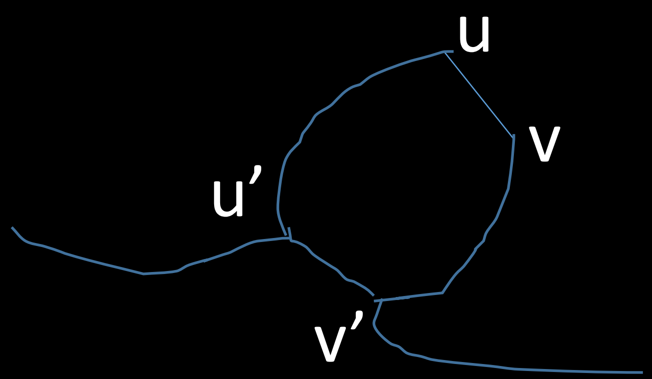
Query Models and Evasiveness

- Let G be the adjacency matrix of an n-node graph
  - G[i,j] = 1 if there is an edge between i and j, else G[i,j] = 0
- In 1 step, we can query any element of G. All other computation is free
- How many queries do we need to tell if G is connected?
- Claim: $n(n-1)/2$ queries suffice
- Proof: Just query every pair {i,j} to learn G, then check if G is connected

- *What about lower bounds?*

Connectivity is an Evasive Graph Property

- Theorem: n(n-1)/2 queries are necessary to determine connectivity
- Proof: adversary strategy: given a query G[u,v], answer 0 *unless* the graph consistent with all of your responses so far, which also satisfies G[u', v'] = 1 for each unasked pair {u',v'}, is disconnected
- Invariant: for any unasked pair {u,v}, the graph revealed so far has no path from u to v
- Reason: consider the last edge {u',v'} revealed on that path. Could have answered 0 and kept same connectivity by having edge {u,v} be present



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- Invariant: for any unasked pair {u,v}, the graph revealed so far has no path from u to v
- Suppose there is some unasked pair {u,v} by the algorithm
  - If algorithm says “connected”, we place all 0s on unasked pairs
  - If algorithm says “disconnected”, we place all 1s on unasked pairs
- So algorithm needs to query every pair