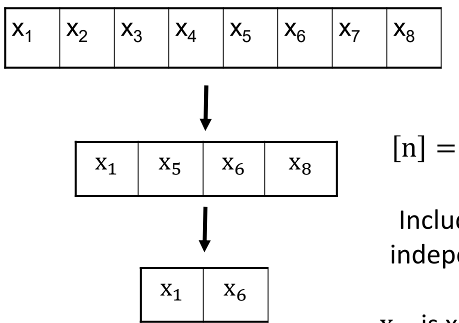
Subsampling



Uniformly sample the coordinates as nested subsets

$[n] = S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots \supseteq S_{\log_2 n}$

Include each item from S_{i-1} in S_i independently with probability 1/2

x_{S_i} is x restricted to coordinates in S_i

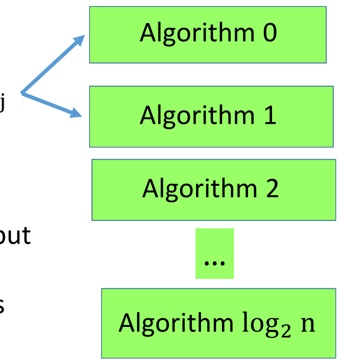
Subsampling in a Stream

Stream update $e_i \leftarrow e_i + \Delta_j$

If $i \in S_0, i \in S_1$ but $i \notin S_2$,

feed i to first 2 algorithms but

not to any other algorithms



Processing

$x_{S_0} = x$

x_{S_1}

x_{S_2}

...

$x_{S_{\log_2 n}}$

Algorithm for Finding a Non-Zero Item

- *If x has k non-zero entries, what's the expected number of non-zero entries in x_{S_i}?*
  - For each non-zero entry j in x, let $Z_j = 1$ if $j \in S_i$, and $Z_j = 0$ otherwise
  - $Z = \sum_j Z_j$,
  - $E[Z] = k \cdot E[Z_j] = \frac{k}{2^i}$
  - $Var[Z] = \sum_j Var[Z_j] = k \cdot Var[Z_1] = k \left(\frac{1}{2^i}\right)\left(1 - \frac{1}{2^i}\right) \le \frac{k}{2^i}$
- If $i = \lfloor \log_2 k \rfloor - 5$, then $32 \le E[Z] < 64$ and $Var[Z] < 64$
- By Chebyshev's inequality, $\Pr[|Z - E[Z]| \ge 32] \le \frac{Var[Z]}{32^2} \le \frac{1}{16}$
- If we run a k'-sparse algorithm with k' = 96 on x_{S_i}, we recover a non-zero item of x_{S_i} with probability at least 1-1/16 – 1/10 > 4/5, or output FAIL
- *But we don't know i?*

Algorithm for Finding a Non-Zero Item

- Run a k'=96-sparse vector algorithm on every x_{S_i} !
- For each x_{S_i}, our algorithm either returns a non-zero item of x_{S_i}, and hence of x, or outputs FAIL
- For $i = \lfloor \log_2 k \rfloor - 5$, with probability at least 4/5, we output a non-zero item of x_{S_i}, and hence of x
- Space is $(\log_2 n) \cdot O(k' \log n) = O(\log^2 n)$ bits!
  - (need to store $S_0, \ldots, S_{\log_2 n}$ but can use hash function for these)

Outline

- Sketching Model
  - Estimating the Euclidean norm of a vector
  - Finding a non-zero coordinate of a vector
- Graph sketching
  - Boruvka's spanning tree algorithm
  - Finding a spanning tree from a sketch

Sketching Graphs

Are there sketches for graphs? A_G is the n x n adjacency matrix of a graph G

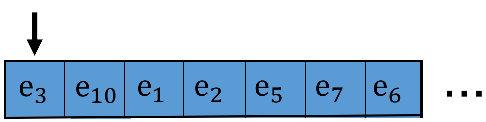
- $(A_G)_{i,j} = 1$ if {i,j} is an edge, and $(A_G)_{i,j} = 0$ otherwise

$$\begin{pmatrix} & S & \end{pmatrix} \begin{pmatrix} & A_G & \end{pmatrix} = \begin{pmatrix} & SA_G & \end{pmatrix} \longrightarrow \text{answer}$$

- Is there a distribution on matrices S with a small number of rows so that you can output a spanning tree of G, given SA_G, with high probability?

Application: Graph Streams

- Process a graph stream and see the edges of a graph $e_1, \ldots, e_m$ in an arbitrary order



- Make 1 pass over the stream
- Could store stream using $O(n^2)$ bits of memory
- Can we use only $n \cdot \text{poly}(\log n)$ bits of memory?
- How would you compute a spanning forest?

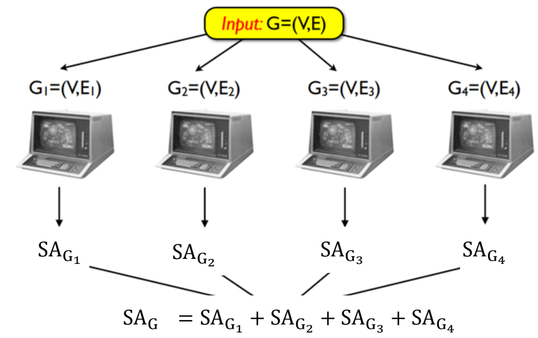
Computing a Spanning Forest

- For each edge e in the stream
  - If ______________________, store edge e
- ______________________ is "doesn't form a cycle"
- Store at most n-1 edges, so O(n log n) bits of memory
- *But what if you are allowed to delete edges? This is called a dynamic stream*

Handling Deletions with Sketching

- Given $S \cdot A_G$, if e is deleted, replace it with $S \cdot A_G - S \cdot A_e = S \cdot A_{G-e}$
- Memory to store $S \cdot A_G$ is (# of rows of S)$\cdot$ $n \cdot \log n$ bits
  - Also need to store S, which is (# of rows of S)$\cdot$ $n \cdot \log n$ bits
- Goal: find S with a small # of rows so that given $S \cdot A_G$, can output a spanning tree of G with high probability
- Theorem: there is a distribution on S with $O(\log^2 n)$ rows!

Parallel Computing



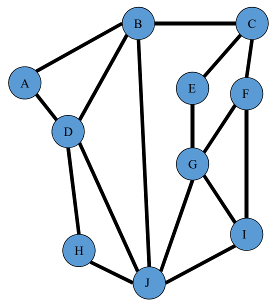
Outline

- Sketching Model
  - Estimating the Euclidean norm of a vector
  - Finding a non-zero coordinate of a vector
- Graph sketching
  - Boruvka's spanning tree algorithm
  - Finding a spanning tree from a sketch

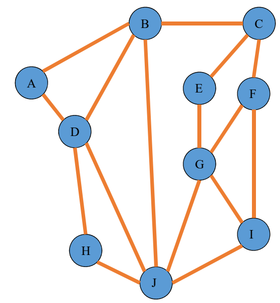
Boruvka's Spanning Tree Algorithm (Modified)

- Input graph is unweighted and connected
- Initialize edgeset E' to Ø
- Create a list of n groups of vertices, each initialized to a single vertex
- While the list has more than one group
  - For each group G, include in E' an edge e from a vertex in G to a vertex not in G
  - Merge groups connected by an edge in the previous step
- Find a spanning tree among the edges in E'

Input Graph



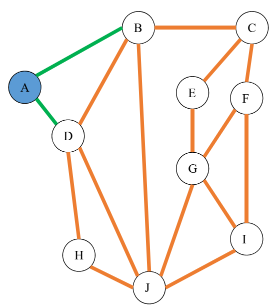
Groups at Beginning of Round 1



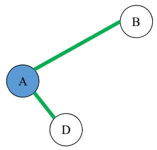
List of Groups

- A
- B
- C
- D
- E
- F
- G
- H
- I
- J

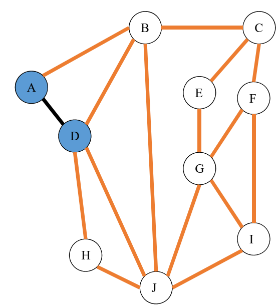
Round 1



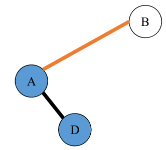
Group A

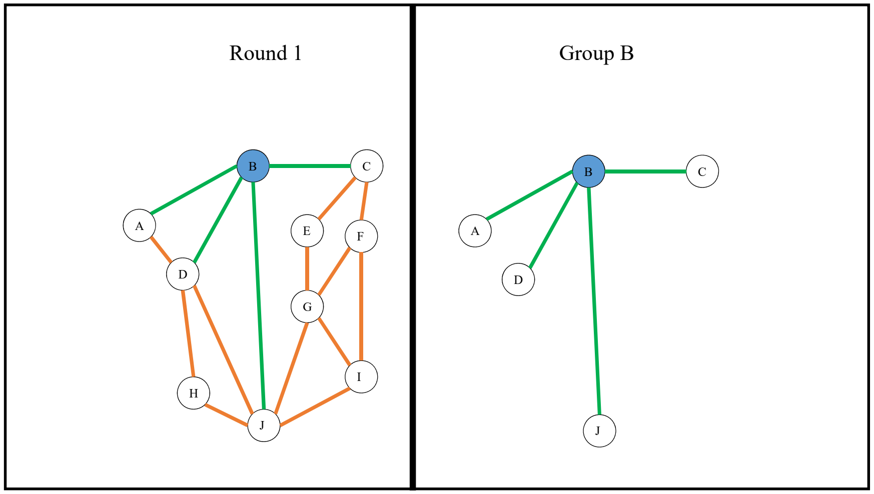


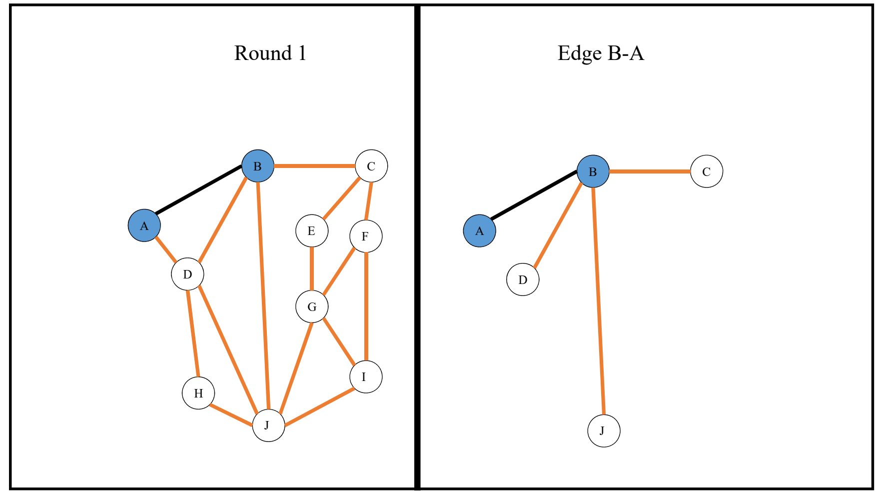
Round 1

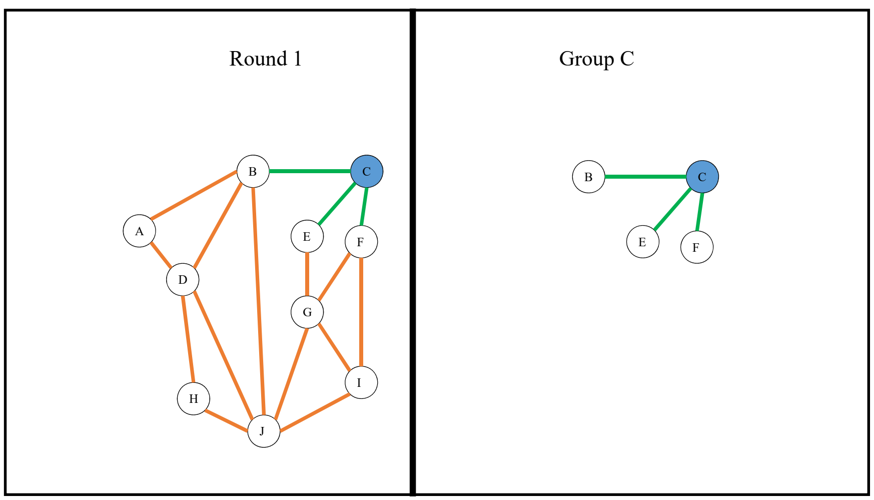


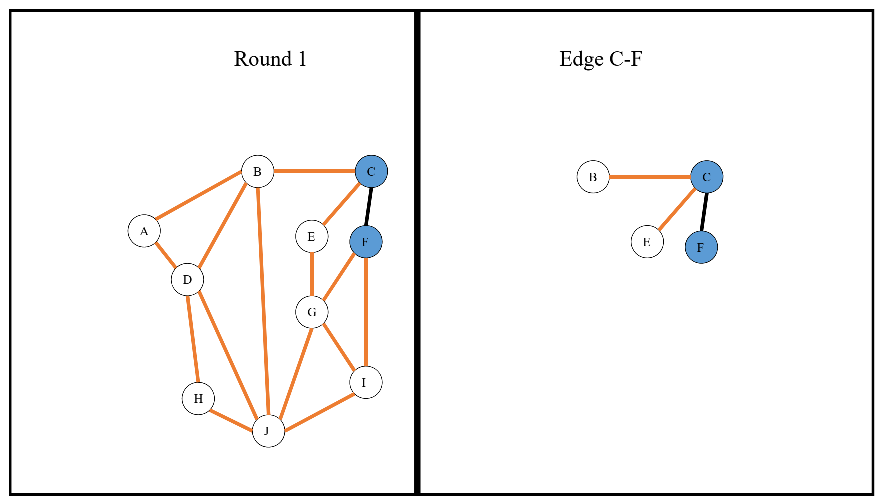
Edge A-D




Round 1
B
C
A
E
F
D
G
I
H
J
Group B
B
C
A
D
J


Round 1
B
C
A
E
F
D
G
I
H
J
Edge B-A
B
C
A
D
J


Round 1
B
C
A
E
F
D
G
I
H
J
Group C
B
C
E
F


Round 1
B
C
A
E
F
D
G
I
H
J
Edge C-F
B
C
E
F

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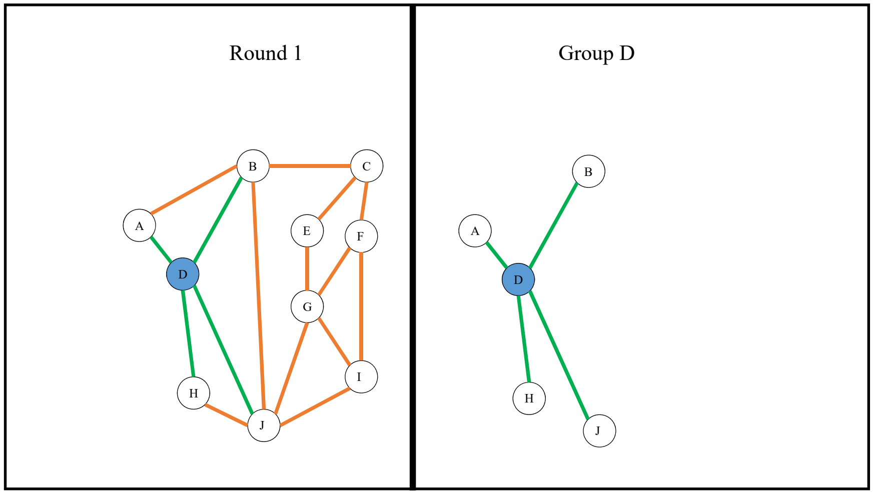
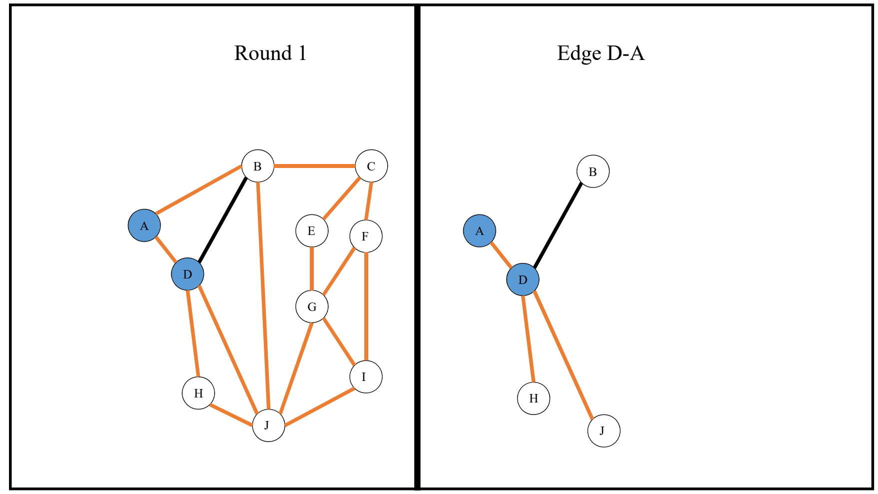
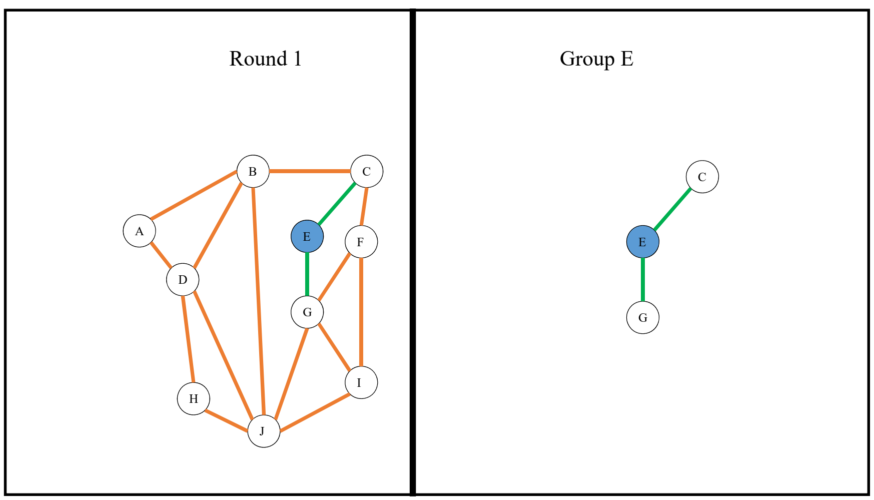
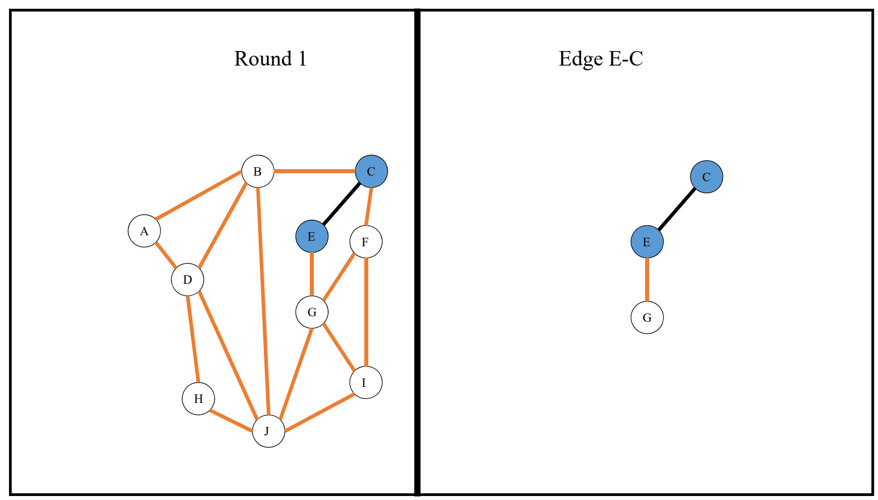
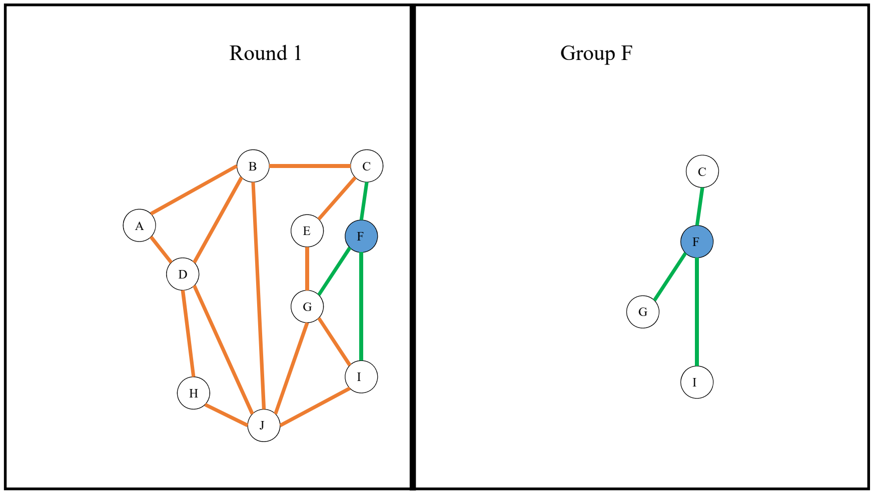
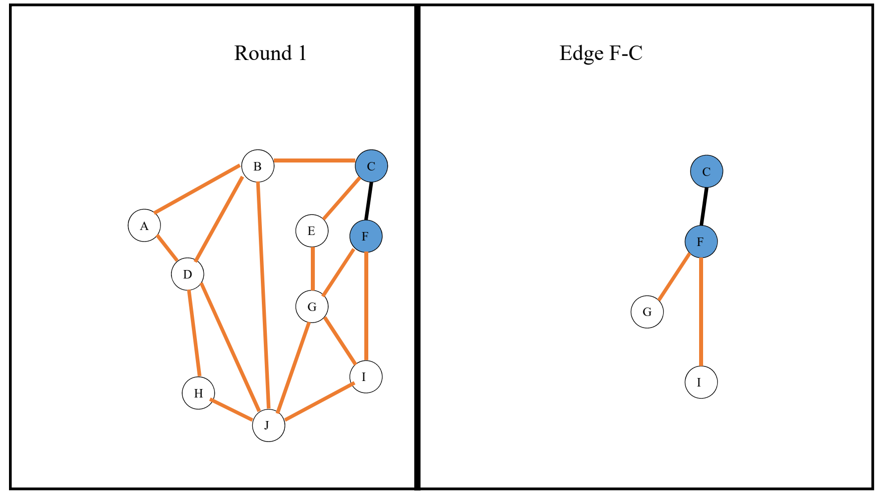

Round 1
Group D
A
B
C
D
E
F
G
H
I
J

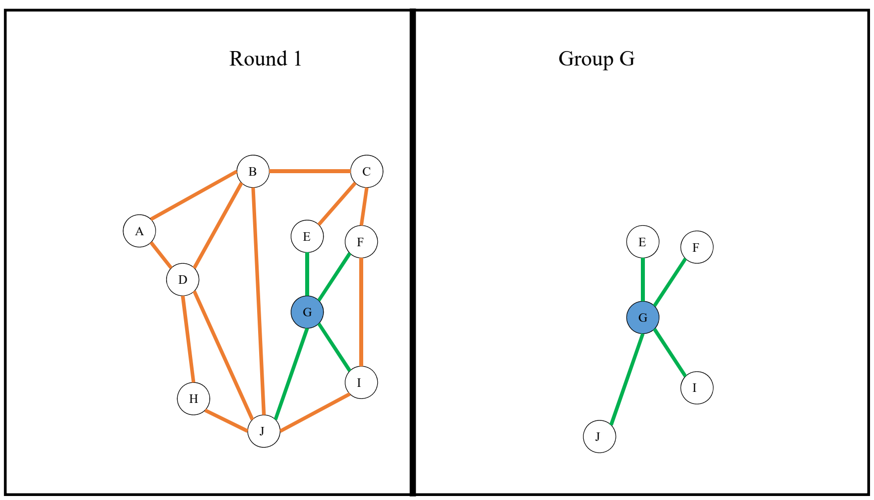

Round 1
Edge D-A
A
B
C
D
E
F
G
H
I
J

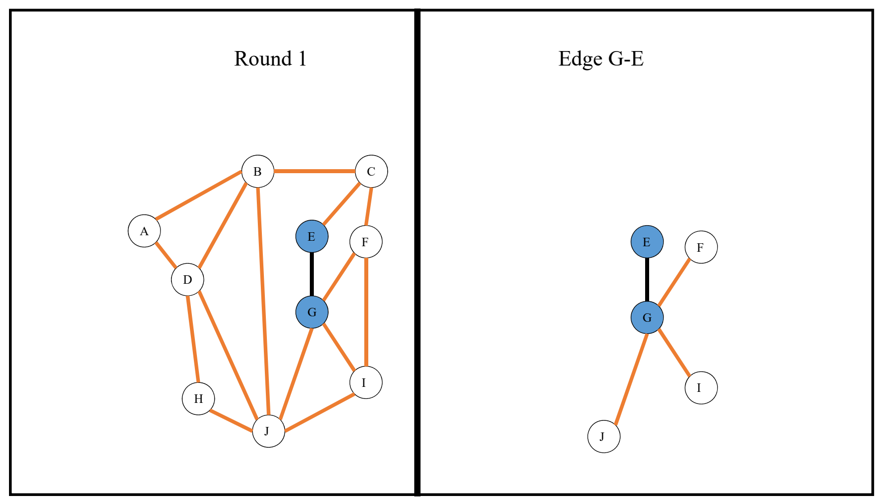

Round 1
Group E
A
B
C
D
E
F
G
H
I
J


Round 1
Edge E-C
A
B
C
D
E
F
G
H
I
J


Round 1
Group F
A B C D E F G H I J
C F G I


Round 1
Edge F-C
A B C D E F G H I J
C F G I


Round 1
Group G
A B C D E F G H I J
E F G I J


Round 1
Edge G-E
A B C D E F G H I J
E F G I J

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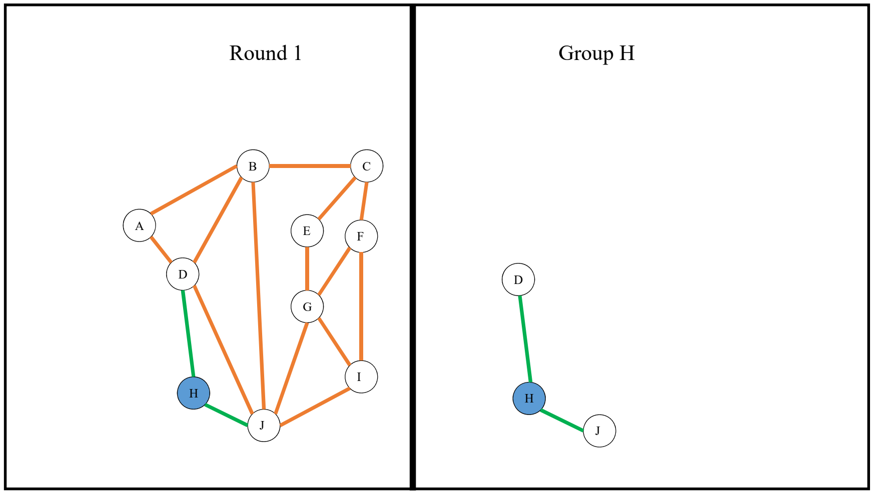
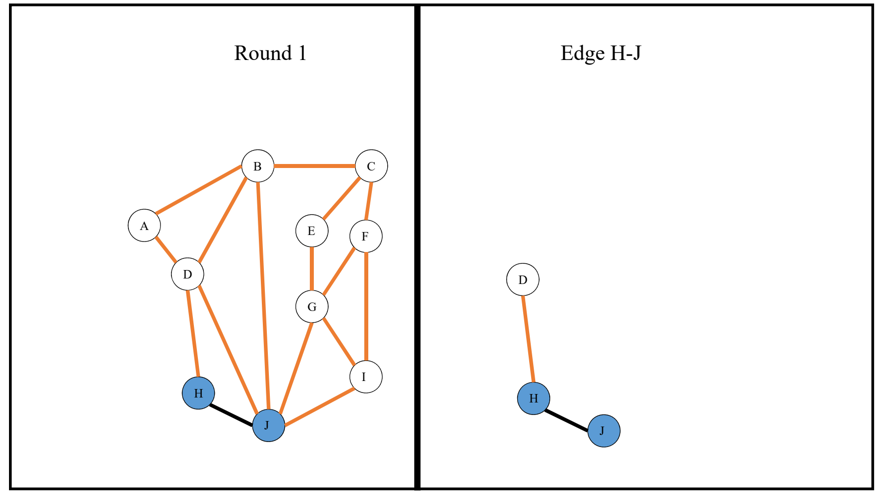
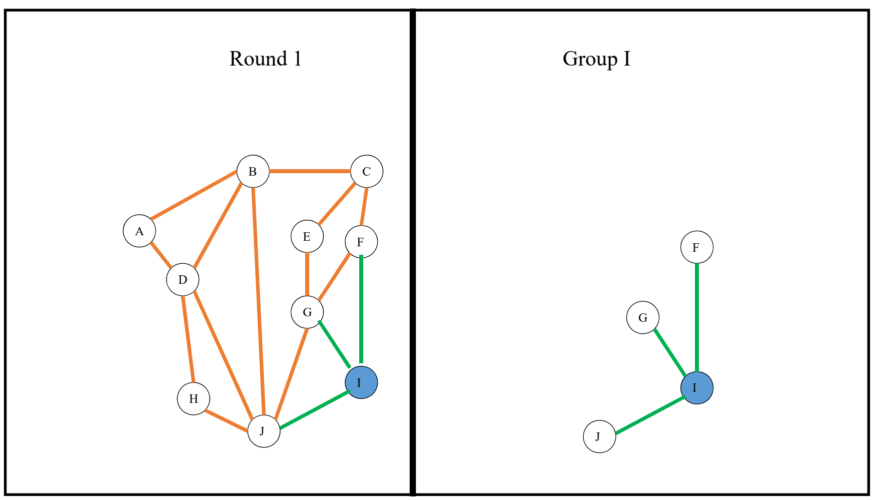
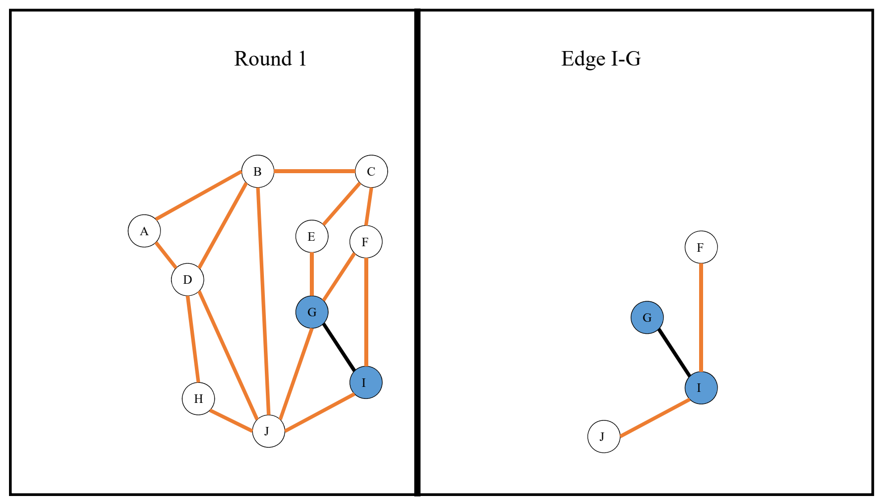
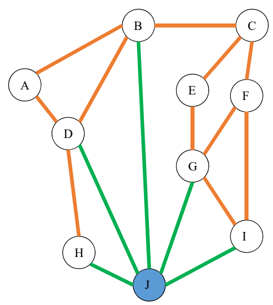

Round 1
Group H
A
B
C
D
E
F
G
H
I
J


Round 1
Edge H-J
A
B
C
D
E
F
G
H
I
J

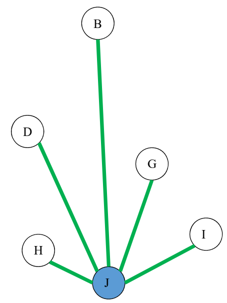

Round 1
Group I
A
B
C
D
E
F
G
H
I
J


Round 1
Edge I-G
A
B
C
D
E
F
G
H
I
J

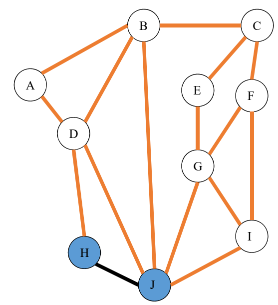
Round 1



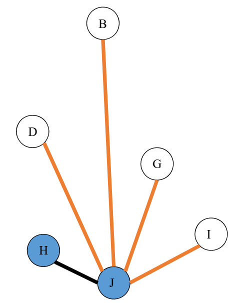
Group J



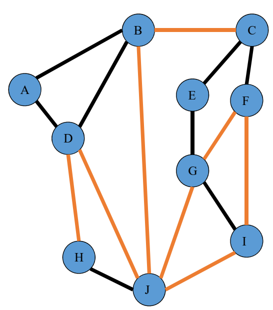
Round 1



Edge J-H



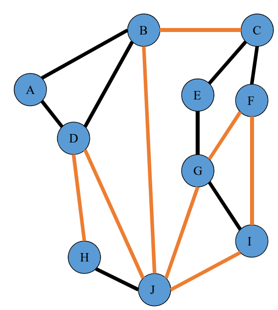
Round 1 Ends



List of Edges Added

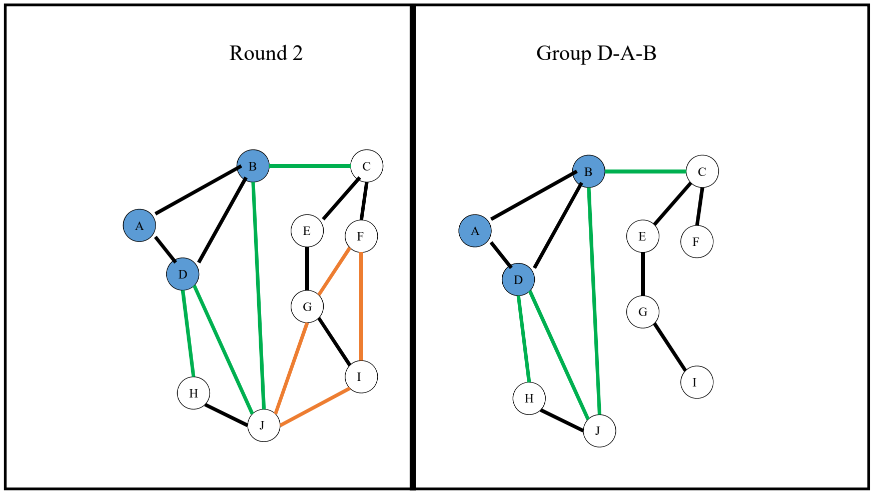
- A-D
- B-A
- C-F
- D-B
- E-C
- F-C
- G-E
- H-J
- I-G
- J-H

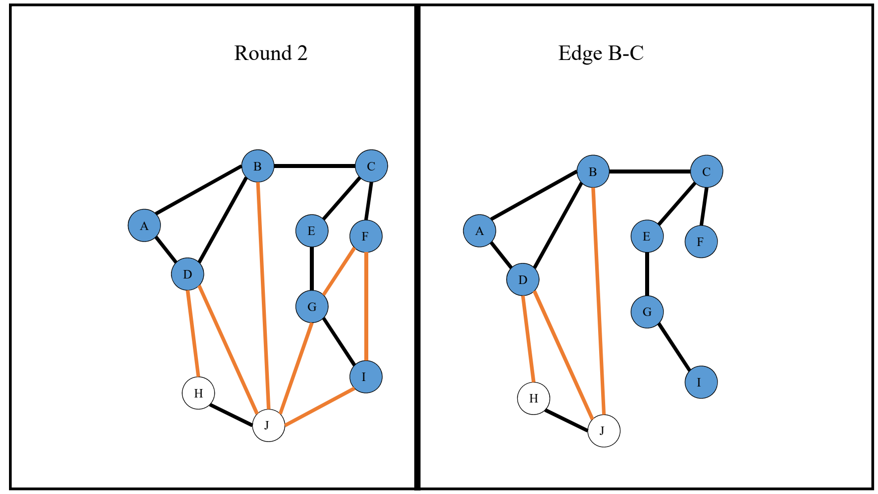
Groups at Beginning of Round 2

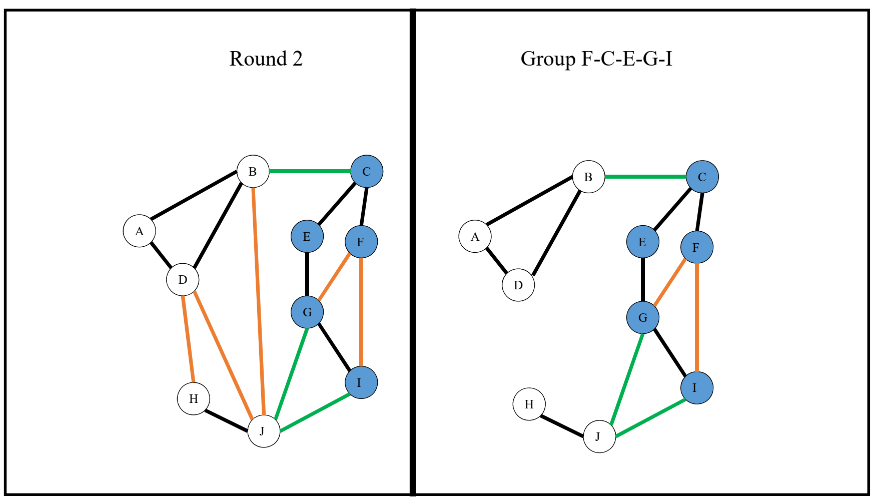


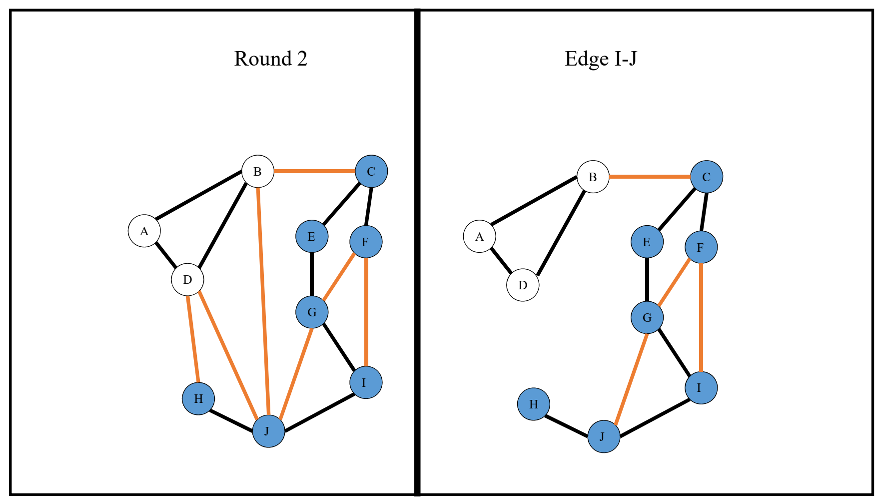
List of Groups

- D-A-B
- F-C-E-G-I
- H-J

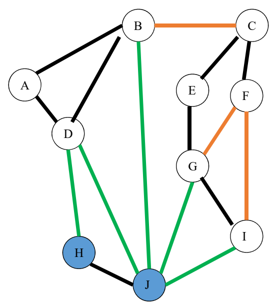

Round 2
Group D-A-B


Round 2
Edge B-C

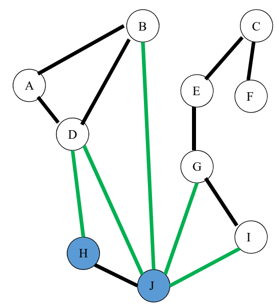

Round 2
Group F-C-E-G-I


Round 2
Edge I-J

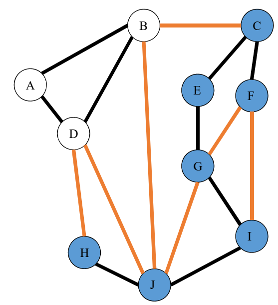
Round 2



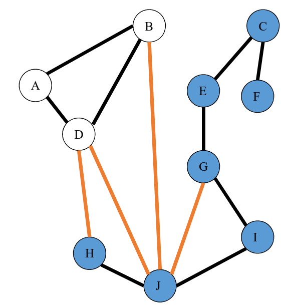
Group H-J



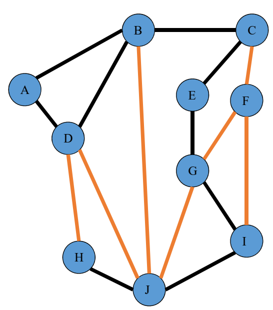
Round 2



Edge J-I



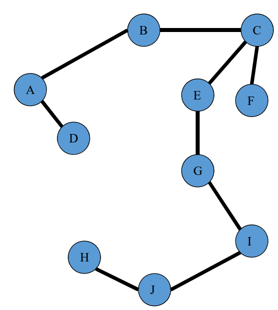
Round 2 Ends



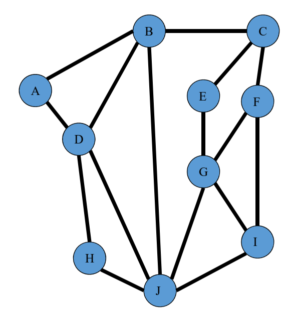
List of Edges Added

- B-C
- I-J
- J-I

Spanning Tree



Input Graph



Analysis

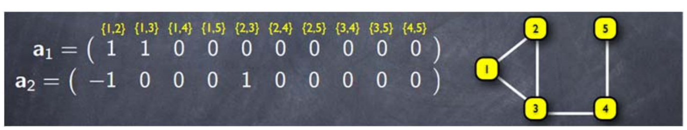
- If there are at least 2 groups in an iteration, then each group has an outgoing edge
  - Else, graph is disconnected
- If t groups at start of an iteration, at most t/2 groups at end of iteration
  - Consider graph with vertices $G_1, G_2, \ldots, G_r$ and r edges, where edges correspond to the groups we connect
  - Number of groups now at most number of connected components in H. *Why?*
- After $\log_2 n$ iterations, one group left
  - At most n + n/2 + n/4 + … + 1 $\leq 2n$ edges in E'
- E' contains a spanning tree
  - Invariant: the vertices in a group are connected

Outline

- Sketching Model
  - Estimating the Euclidean norm of a vector
  - Finding a non-zero coordinate of a vector
- Graph sketching
  - Boruvka's spanning tree algorithm
  - Finding a spanning tree from a sketch

Representing a Graph

- For node i, let a_i be a vector indexed by node pairs
- If {i,j} is an edge, $a_i[i,j] = 1$ if j > i, and $a_i[i,j] = -1$ if j < i
- If {i,j} is not an edge, $a_i[i,j] = 0$

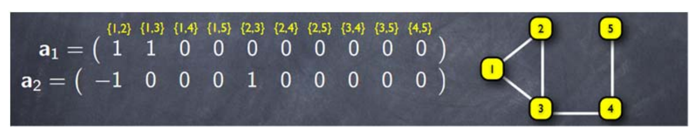


Representing a Graph

- Lemma: for a subset S of nodes,

  Support$(\sum_{i \in S} a_i) = E(S, V \backslash S)$

- Proof: for edge {i,j}, if i, j ∈ S, the sum of entries on {i,j}-th column is 0

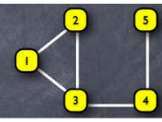


Spanning Tree Algorithm

{1,2} {1,3} {1,4} {1,5} {2,3} {2,4} {2,5} {3,4} {3,5} {4,5}

$\mathbf{a}_1 = (\ 1\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ )$

$\mathbf{a}_2 = (\ -1\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ )$



- If we delete edge {1, 2} in the stream, then a_1 and a_2 become:
  - $a_1 = (0\,1\,0\,0\,0\,0\,0\,0\,0\,0)$
  - $a_2 = (0\,0\,0\,0\,1\,0\,0\,0\,0\,0)$
- If we insert or delete edge {i,j} we just update a_i and a_j accordingly
- But we can't write down a_i since it is $\Theta(n^2)$-dimensional
- Million dollar question: what can we do to a vector if we can't store it?

Spanning Tree Algorithm

- Store a sketch of each a_j!
- We'll need more than 1 sketch of each a_j, let's take O(log n) sketches
- Maintain O(log n) sketches $C_1 \cdot a_j, \ldots, C_{O(\log n)} \cdot a_j$ for each a_j in a stream
- C_i squashes a_i down to $O(\log^2 n)$ bits
- $C_i \cdot a_j$ returns a non-zero item of a_j with probability 4/5, or returns FAIL
- A non-zero item of a_j is just an edge incident to vertex j!

Spanning Tree Algorithm

- Compute O(log n) sketches $C_1 \cdot a_j, \ldots, C_{O(\log n)} \cdot a_j$ for each a_j
- $C_i \cdot a_j$ outputs a non-zero item of a_j with probability > 4/5, or returns FAIL
- Idea: Run Boruvka's algorithm on sketches!
- For each node j, use $C_i \cdot a_j$ to get incident edge on j
- For i = 2, …, O(log n)
  - To get incident edge on group $G \subseteq V$, use

$$\sum_{j \in S} C_i \mathbf{a}_j = C_i \left(\sum_{j \in S} \mathbf{a}_j \right) \longrightarrow e \in \text{support}(\sum_{j \in S} \mathbf{a}_j) = E(S, V \setminus S)$$

Spanning Tree Wrapup

- O(n log n) sketches $C_i \cdot a_j$, as i and j vary, so $O(n \log^3 n)$ bits of space
- Note: a 1/5 fraction of sketches fail in each iteration in expectation, but a 4/5 fraction of groups get connected with other groups
- Expected number of iterations still O(log n)
- Since sketches are linear, can maintain with insertions and deletions of edges
- Overall, $O(n \log^3 n)$ bits of space to output a spanning tree!