Instructor: Minchen Li

$$R(\mathbf{X},t)J(\mathbf{X},t) = R(\mathbf{X},0) \quad \text{Conservation of mass}$$

$$R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t) = \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X},t) + R(\mathbf{X},0)\boldsymbol{g} \quad \text{Conservation of momentum}$$

?

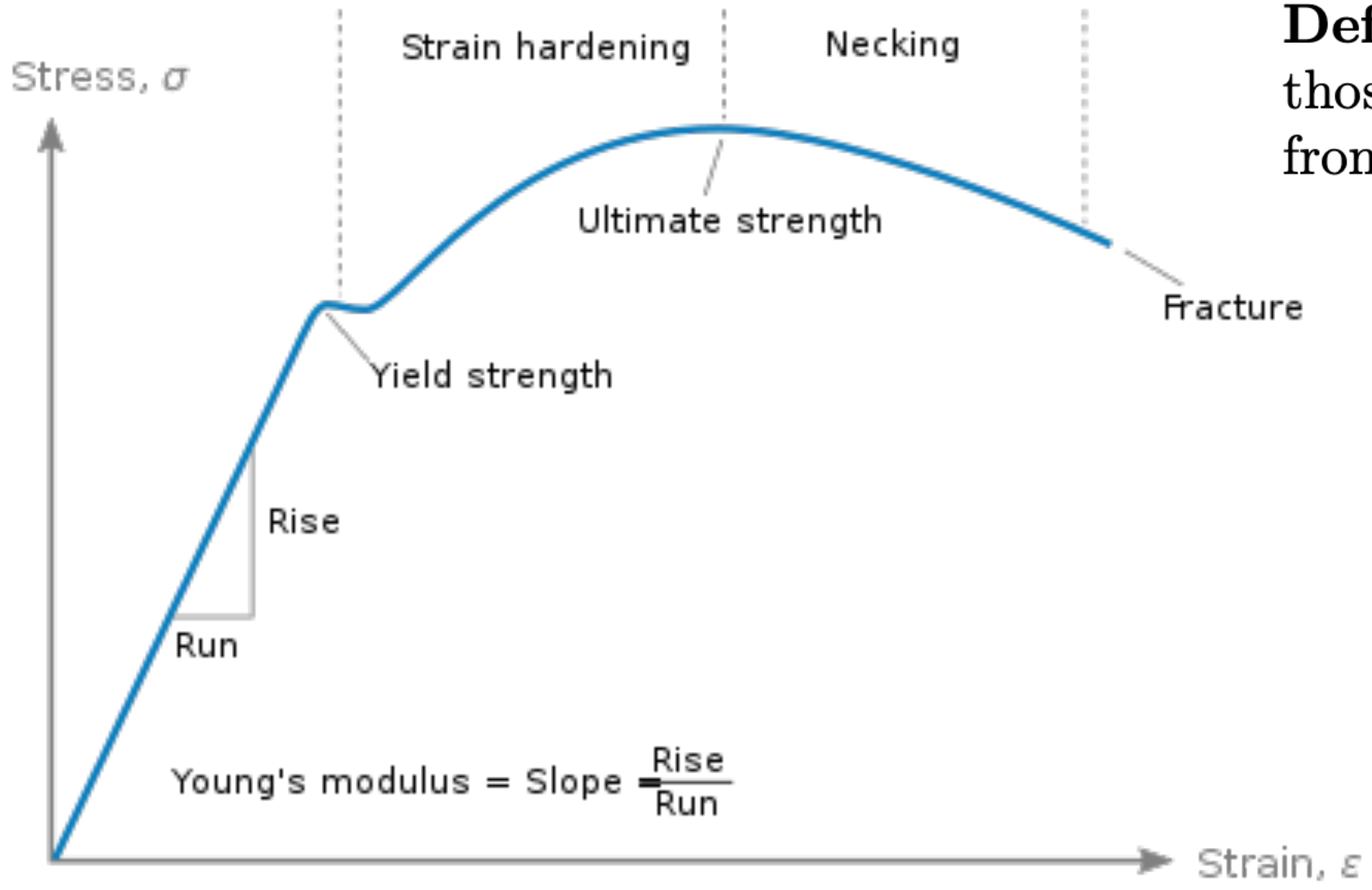
$$M(x^{n+1} - (x^n + \Delta t v^n)) - \Delta t^2 f(x^{n+1}) = 0$$

Lec 10: Governing Equations

15-769: Physically-based Animation of Solids and Fluids (F23)

Recap: Stress

- a tensor field (like F) measuring pressure (unit: force per area)
- related to F through a constitutive relationship, e.g. neo-Hookean model



Definition (Hyperelastic Materials)**.** Hyperelastic materials are those elastic solids whose **first Piola-Kirchoff stress P** can be derived from an strain energy density function $\Psi(\mathbf{F})$ via

$$\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}} \qquad P_{ij} = \frac{\partial \Psi}{\partial F_{ij}}$$

- **Cauchy stress**

$$\sigma = \frac{1}{J}\mathbf{P}\mathbf{F}^T = \frac{1}{\det(\mathbf{F})}\frac{\partial \Psi}{\partial \mathbf{F}}\mathbf{F}^T$$

Calculation in the diagonal space (isotropic): $\mathbf{P}(\mathbf{F}) = \mathbf{P}(\mathbf{U}\Sigma\mathbf{V}^T) = \mathbf{U}\mathbf{P}(\Sigma)\mathbf{V}^T = \mathbf{U}\hat{\mathbf{P}}\mathbf{V}^T$.

Recap: Stress Derivative

$$(\delta \mathbf{P})_{ij} = U_{ik} \left(\frac{\partial \mathbf{P}}{\partial \mathbf{F}}(\Sigma) \right)_{klmn} U_{rm} \delta F_{rs} V_{sn} V_{jl}, \quad \text{and} \quad (\delta \mathbf{P})_{ij} = \left(\frac{\partial \mathbf{P}}{\partial \mathbf{F}}(\mathbf{F}) \right)_{ijrs} \delta F_{rs}$$

$$\frac{\partial \mathbf{P}}{\partial \mathbf{F}}(\Sigma) = \begin{bmatrix} A & & & \\ & B_{12} & & \\ & & B_{23} & \\ & & & B_{31} \end{bmatrix}$$

(With flattening and permutation)

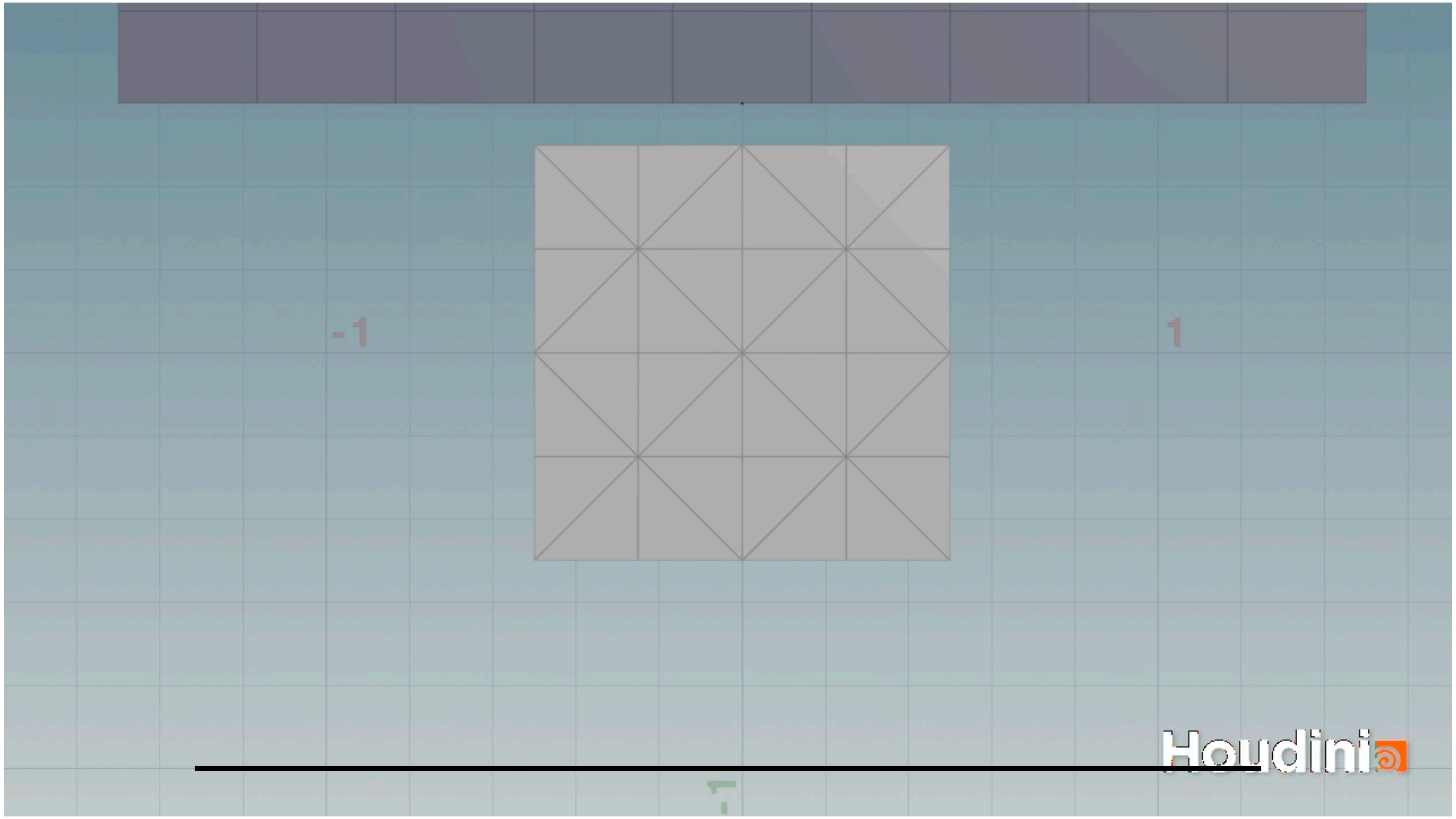
$$\mathbf{A} = \begin{pmatrix} \hat{\Psi}_{,\sigma_1\sigma_1} & \hat{\Psi}_{,\sigma_1\sigma_2} & \hat{\Psi}_{,\sigma_1\sigma_3} \\ \hat{\Psi}_{,\sigma_2\sigma_1} & \hat{\Psi}_{,\sigma_2\sigma_2} & \hat{\Psi}_{,\sigma_2\sigma_3} \\ \hat{\Psi}_{,\sigma_3\sigma_1} & \hat{\Psi}_{,\sigma_3\sigma_2} & \hat{\Psi}_{,\sigma_3\sigma_3} \end{pmatrix}$$

$$\mathbf{B}_{ij} = \frac{1}{\sigma_i^2 - \sigma_j^2} \begin{pmatrix} \sigma_i \hat{\Psi}_{,\sigma_i} - \sigma_j \hat{\Psi}_{,\sigma_j} & \sigma_j \hat{\Psi}_{,\sigma_i} - \sigma_i \hat{\Psi}_{,\sigma_j} \\ \sigma_j \hat{\Psi}_{,\sigma_i} - \sigma_i \hat{\Psi}_{,\sigma_j} & \sigma_i \hat{\Psi}_{,\sigma_i} - \sigma_j \hat{\Psi}_{,\sigma_j} \end{pmatrix}$$

$$\mathbf{B}_{ij} = \frac{1}{2} \frac{\hat{\Psi}_{,\sigma_i} - \hat{\Psi}_{,\sigma_j}}{\sigma_i - \sigma_j} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} \frac{\hat{\Psi}_{,\sigma_i} + \hat{\Psi}_{,\sigma_j}}{\sigma_i + \sigma_j} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- **Other ways to compute: Analytic Eigensystems for Isotropic Distortion Energies [Smith et al. 2019]**
  - **Modes with negative Eigenvalues are directly projected out**

Recap: Inversion-Free Elastodynamics



Strong Form
Definition

Definition (Strong Form)**.** Letting $\mathbf{V}(\mathbf{X},t) = \frac{\partial \phi(\mathbf{X},t)}{\partial t} = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}$ be the velocity defined over $\mathbf{X}$, the equations are [Gonzalez and Stuart, 2008]

$$R(\mathbf{X},t)J(\mathbf{X},t) = R(\mathbf{X},0) \quad \text{Conservation of mass,}$$

$$R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t) = \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X},t) + R(\mathbf{X},0)\boldsymbol{g} \quad \text{Conservation of momentum,}$$

where $\mathbf{X} \in \Omega_0$ and $t \geq 0$. Here R is the mass density, $J(\mathbf{X},t) = \det \mathbf{F}(\mathbf{X},t)$, $\mathbf{P}$ is the first Piola-Kirchoff stress, and $\boldsymbol{g}$ is the constant gravitational acceleration. Note that $J(\mathbf{X},0) = 1$, and the mass conservation can also be written as $\frac{\partial}{\partial t}\left(R(\mathbf{X},t)J(\mathbf{X},t)\right) = 0$.

Strong Form — Conservation of Mass

- Density: $R(\mathbf{X},t) = \lim_{\epsilon \to +0} \frac{\text{mass}(B_\epsilon^t)}{\text{volume}(B_\epsilon^t)} = \lim_{\epsilon \to +0} \frac{\text{mass}(B_\epsilon^t)}{\int_{B_\epsilon^t} d\mathbf{x}}$

  B_ϵ^t is the ball of radius ϵ surrounding an arbitrary $\mathbf{X} \in \Omega^0$

- Conservation of Mass: B_ϵ^t is constant over time
  - the material takes more or less space, but the amount (mass) won't change

$$\textbf{mass}(B_\epsilon^t) = \int_{B_\epsilon^t} R(\mathbf{X}(\mathbf{x}),t)d\mathbf{x} = \int_{B_\epsilon^0} R(\mathbf{X},t)J(\mathbf{X},t)d\mathbf{X} = \textbf{mass}(B_\epsilon^0) = \int_{B_\epsilon^0} R(\mathbf{X},0)d\mathbf{X}$$

$$\forall\ B_\epsilon^0 \subset \Omega^0 \text{ and } t \geq 0$$

$$\boxed{R(\mathbf{X},t)J(\mathbf{X},t) = R(\mathbf{X},0), \quad \forall\ \mathbf{X} \in \Omega^0 \text{ and } t \geq 0}$$

- **Automatically satisfied in Lagrangian methods**
- **Needs to be explicitly considered in Eulerian methods**

Strong Form — Conservation of Momentum

Traction

- Types of forces:
  - body forces (or external forces, e.g. gravity)
  - surface forces (or internal forces, which is stress-based, e.g. elasticity)
    - defined via traction (force per area)

$\mathbf{T}(\cdot, \mathbf{N}, t) : \Omega^0 \to \mathbb{R}^d$ is defined via the relation

$$\text{force}_S(B_\epsilon^0) = \int_{\partial B_\epsilon^0} \mathbf{T}(\mathbf{X}, \mathbf{N}) ds(\mathbf{X})$$

where $\mathbf{N}$ is the outward-pointing normal direction in the material space, and $\text{force}_S(B_\epsilon^0)$ is the net force on an arbitrary B_ϵ^0 exerted from material outside ∂B_ϵ^0 on material inside B_ϵ^0.

Strong Form — Conservation of Momentum

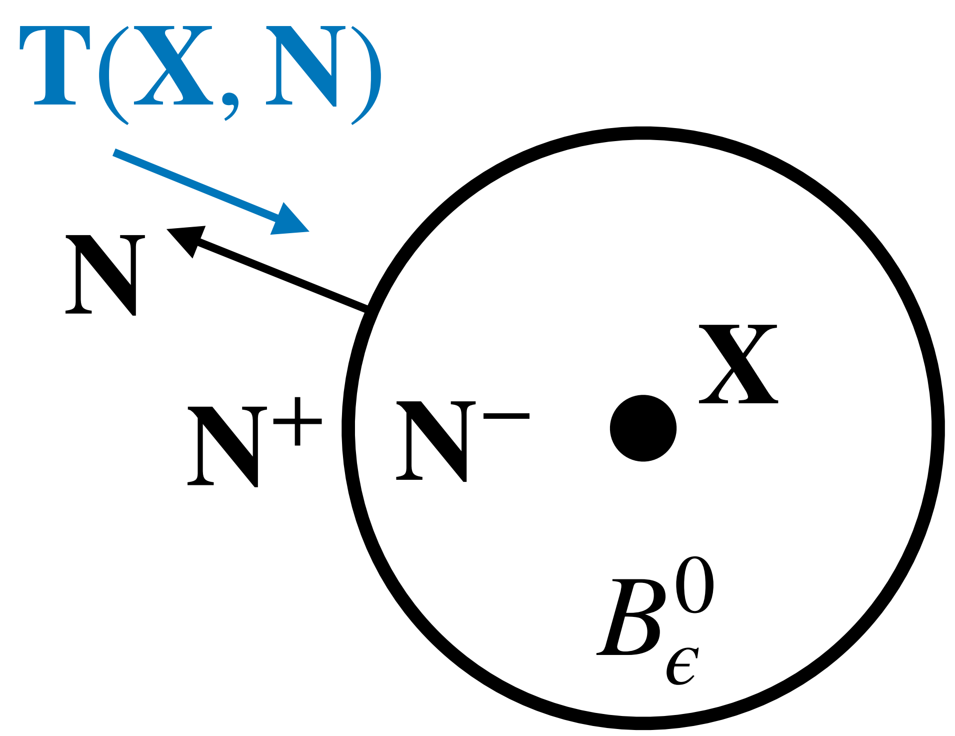
Traction and Stress

- Traction

$$\text{force}_S(B_\epsilon^0) = \int_{\partial B_\epsilon^0} \mathbf{T}(\mathbf{X}, \mathbf{N}) ds(\mathbf{X})$$

  - **the force per unit area(3D)/length(2D) that material in $\mathbf{N}^+$ exerts on material in $\mathbf{N}^-$**

    (Internal forces inside B_ϵ^0 are self-balanced)



- Stress (based on Cauchy's Stress Theorem)

(first Piola-Kirchoff stress) $\mathbf{P}(\cdot, t) : \Omega^0 \to \mathbb{R}^{d\times d}$

$$\mathbf{T}(\mathbf{X}, \mathbf{N}, t) = \mathbf{P}(\mathbf{X}, t)\mathbf{N}.$$

Strong Form — Conservation of Momentum

Derivation — Applying Newton's 2nd Law

Then, by applying Newton's second law on B_ϵ^0, we can express the conservation of momentum as

$$\int_{B_\epsilon^0} R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t)d\mathbf{X}$$
$$= \int_{\partial B_\epsilon^0} \mathbf{P}(\mathbf{X},t)\mathbf{N}(\mathbf{X})ds(\mathbf{X}) + \int_{B_\epsilon^0} R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t)d\mathbf{X},$$
$$\forall\ B_\epsilon^0 \subset \Omega^0 \text{ and } t \geq 0.$$

Here we have added the contribution of external body force (such as gravity) with its acceleration $\mathbf{A}^{\text{ext}}$ to the change of momentum.

Strong Form — Conservation of Momentum

Derivation — Applying Divergence Theorem

$$\int_{B_\epsilon^0} R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t)d\mathbf{X} = \int_{\partial B_\epsilon^0} \mathbf{P}(\mathbf{X},t)\mathbf{N}(\mathbf{X})ds(\mathbf{X}) + \int_{B_\epsilon^0} R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t)d\mathbf{X}$$

$$\forall\ B_\epsilon^0 \subset \Omega^0 \text{ and } t \geq 0.$$

Applying Divergence Theorem:

$$\int_{B_\epsilon^0} R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t)d\mathbf{X} = \int_{B_\epsilon^0} \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X},t)d\mathbf{X} + \int_{B_\epsilon^0} R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t)d\mathbf{X}, \quad \forall\ B_\epsilon^0 \subset \Omega^0 \text{ and } t \geq 0$$

Definition 16.2 (Divergence Theorem)**.** For a vector-valued function $\mathbf{f}(\mathbf{x}) : \Omega \to \mathbb{R}^d$ defined on a closed domain Ω, let $\mathbf{n}(\mathbf{x})$ be the outward-pointing normal on the boundary of this domain, the following equality holds:

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n} ds(\mathbf{x}) = \int_{\Omega} \nabla \cdot \mathbf{f} d\mathbf{x}. \tag{16.13}$$

Strong Form — Conservation of Momentum

Derivation — Extract the Integrand

$$\int_{B_\epsilon^0} R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t)d\mathbf{X} = \int_{B_\epsilon^0} \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X},t)d\mathbf{X} + \int_{B_\epsilon^0} R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t)d\mathbf{X}, \quad \forall\, B_\epsilon^0 \subset \Omega^0 \text{ and } t \geq 0.$$

Here the divergence operator $\nabla\cdot$ acts on every row vector of $\mathbf{P}$ independently and result in a column vector. Since Equation also holds for arbitrary B_ϵ^0, we arrive at the strong form of the force balance equation by removing the integration:

$$R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t) = \nabla^{\mathbf{X}}\cdot\mathbf{P}(\mathbf{X},t) + R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t), \quad \forall\, \mathbf{X} \in \Omega^0 \text{ and } t \geq 0.$$

Weak Form Derivation

Applying Test Function

$$R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t) = \nabla^{\mathbf{X}}\cdot\mathbf{P}(\mathbf{X},t)+R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t), \quad \forall\, \mathbf{X}\in\Omega^0 \text{ and } t\geq 0.$$

Ignoring external force for simplicity

For arbitrary test function $\mathbf{Q}(\,\cdot\,,t):\Omega^0\to\mathbb{R}^d$, compute the dot product to both sides and integrate

$$\int_{\Omega^0} R(\mathbf{X},0)\mathbf{Q}(\mathbf{X},t)\cdot\mathbf{A}(\mathbf{X},t)d\mathbf{X} = \int_{\Omega^0}\mathbf{Q}(\mathbf{X},t)\cdot(\nabla^{\mathbf{X}}\cdot\mathbf{P}(\mathbf{X},t))d\mathbf{X},$$
$$\forall\ \mathbf{Q}(\cdot,t):\Omega^0\to\mathbb{R}^d \text{ and } t\geq 0.$$

Here we denote $\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t)$.

— equivalent to strong form as it needs to hold for arbitrary $\mathbf{Q}$

In index notation: $$\int_{\Omega^0} R(\mathbf{X},0)\sum_i Q_i(\mathbf{X},t)A_i(\mathbf{X},t)d\mathbf{X} = \int_{\Omega^0}\sum_i Q_i(\mathbf{X},t)\sum_j P_{ij,j}(\mathbf{X},t)d\mathbf{X}.$$

Duplicate indices for summation: $$\int_{\Omega^0} R(\mathbf{X},0)Q_i(\mathbf{X},t)A_i(\mathbf{X},t)d\mathbf{X} = \int_{\Omega^0} Q_i(\mathbf{X},t)P_{ij,j}(\mathbf{X},t)d\mathbf{X}.$$

Weak Form Derivation

Applying Integration by Parts

$$\int_{\Omega^0} R(\mathbf{X},0)Q_i(\mathbf{X},t)A_i(\mathbf{X},t)d\mathbf{X} = \int_{\Omega^0} Q_i(\mathbf{X},t)P_{ij,j}(\mathbf{X},t)d\mathbf{X}$$

Now applying Integration By Parts on the right-hand side,

$$\int_{\Omega^0} Q_i(\mathbf{X},t)P_{ij,j}(\mathbf{X},t)d\mathbf{X}.$$

$$= \int_{\Omega^0} (\nabla \cdot (Q_i(\mathbf{X},t)\mathbf{P}_i(\mathbf{X},t)) - \nabla Q_i(\mathbf{X},t) \cdot \mathbf{P}_i(\mathbf{X},t))d\mathbf{X}$$

$$= \int_{\Omega^0} ((Q_i(\mathbf{X},t)P_{ij}(\mathbf{X},t))_{,j} - Q_{i,j}(\mathbf{X},t)P_{ij}(\mathbf{X},t))d\mathbf{X}.$$

Definition 16.3 (Integration By Parts)**.** For a scalar-valued function $u(\mathbf{x})$ and a vector-valued function (vector field) $\mathbf{V}(\mathbf{x})$, the product rule for divergence states that

$$\nabla \cdot (u(\mathbf{x})\mathbf{V}(\mathbf{x})) = u(\mathbf{x})\nabla \cdot \mathbf{V}(\mathbf{x}) + \nabla u(\mathbf{x}) \cdot \mathbf{V}(\mathbf{x}). \tag{16.19}$$

Integrating both sides on domain Ω then gives

$$\int_\Omega \nabla \cdot (u(\mathbf{x})\mathbf{V}(\mathbf{x}))d\mathbf{x} = \int_\Omega u(\mathbf{x})\nabla \cdot \mathbf{V}(\mathbf{x})d\mathbf{x} + \int_\Omega \nabla u(\mathbf{x}) \cdot \mathbf{V}(\mathbf{x})d\mathbf{x}. \tag{16.20}$$

Weak Form Derivation

Applying Divergence Theorem

Then if we further apply divergence theorem on the first part of $\int_{\Omega^0}((Q_i(\mathbf{X},t)P_{ij}(\mathbf{X},t))_{,j} - Q_{i,j}(\mathbf{X},t)P_{ij}(\mathbf{X},t))d\mathbf{X}$.

$$\int_{\Omega^0} R(\mathbf{X},0)Q_i(\mathbf{X},t)A_i(\mathbf{X},t)d\mathbf{X}$$
$$= \int_{\partial\Omega^0} Q_i(\mathbf{X},t)P_{ij}(\mathbf{X},t)N_j(\mathbf{X})ds(\mathbf{X}) - \int_{\Omega^0} Q_{i,j}(\mathbf{X},t)P_{ij}(\mathbf{X},t)d\mathbf{X}.$$

The quantity $P_{ij}N_j$ would be specified as a boundary condition. If we let $\mathbf{T}(\mathbf{X},t)$ be the boundary force per unit reference area (traction) with $T_i = P_{ij}N_j$, then we can say that the conservation of momentum implies that $\forall \mathbf{Q}(\cdot,t) : \Omega^0 \to \mathbb{R}^d$

$$\int_{\Omega^0} R(\mathbf{X},0)Q_i(\mathbf{X},t)A_i(\mathbf{X},t)d\mathbf{X}$$
$$= \int_{\partial\Omega^0} Q_i(\mathbf{X},t)T_i(\mathbf{X},t)ds(\mathbf{X}) - \int_{\Omega^0} Q_{i,j}(\mathbf{X},t)P_{ij}(\mathbf{X},t)d\mathbf{X}.$$

Weak Form Discretization

Spatial Sampling and Interpolation

Looking at a specific moment $t = t^n$:

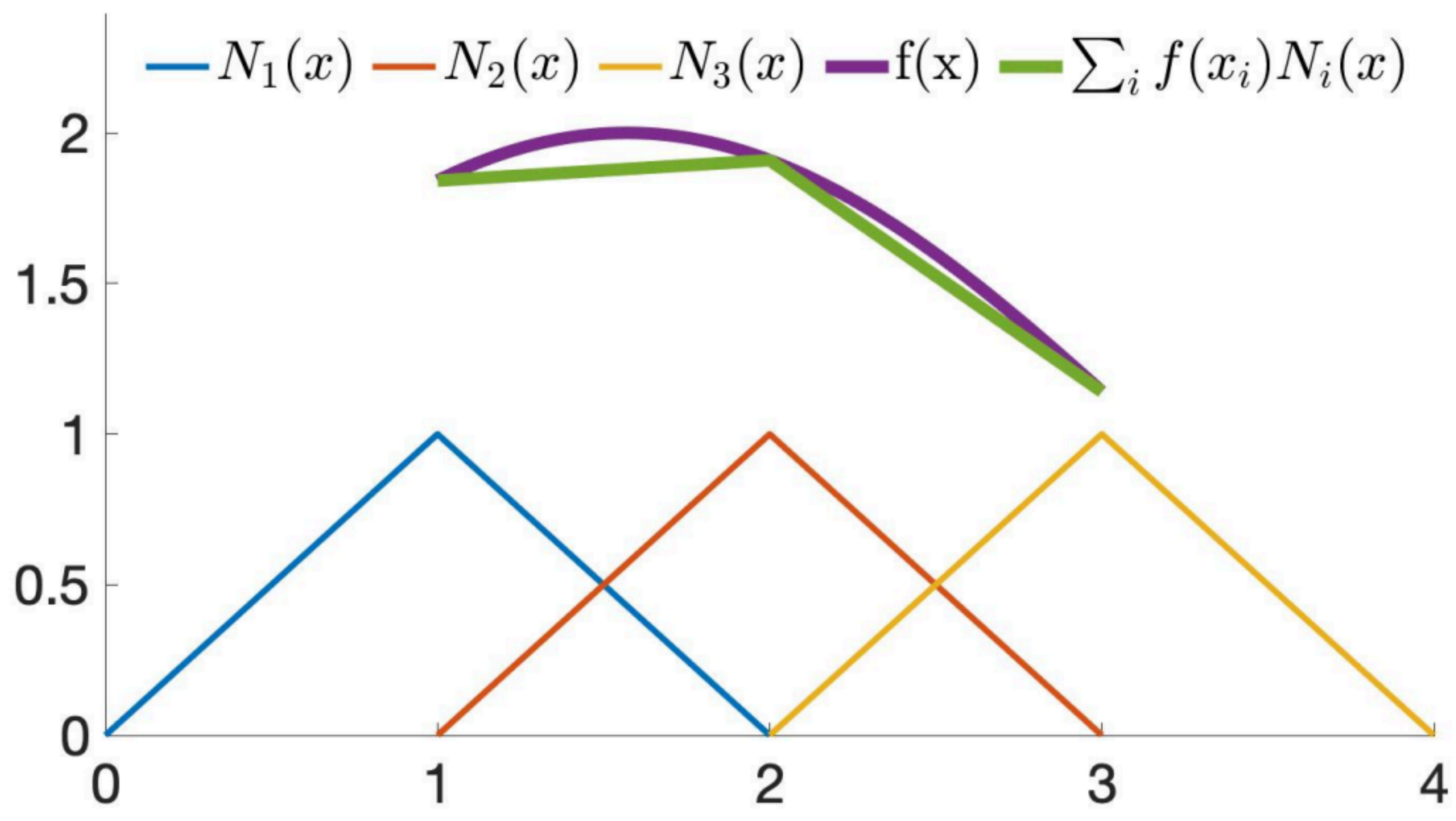
$$\int_{\Omega^0} R^0(\mathbf{X}) Q_i^n(\mathbf{X}) A_i^n(\mathbf{X}) d\mathbf{X}$$
$$= \int_{\partial\Omega^0} Q_i^n(\mathbf{X}) T_i^n(\mathbf{X}) ds(\mathbf{X}) - \int_{\Omega^0} Q_{i,j}^n(\mathbf{X}) P_{ij}^n(\mathbf{X}) d\mathbf{X}$$

Given a set of sample points indexed by a or b in the simulation domain, we can approximate the test function $\mathbf{Q}$ and the DOF $\mathbf{x}$ as

$$Q_i(\mathbf{X}, t^n) \approx \sum_a Q_{a|i}(t^n) N_a(\mathbf{X}) = \sum_a Q_{a|i}^n N_a(\mathbf{X}),$$
$$\mathbf{x}_i(\mathbf{X}, t^n) \approx \sum_b \mathbf{x}_{b|i}(t^n) N_b(\mathbf{X}) = \sum_b \mathbf{x}_{b|i}^n N_b(\mathbf{X})$$

where $Q_{a|i}^n = Q_{a|i}(t^n)$ refers to the i-th dimension of $\mathbf{Q}$ evaluated at sample point a at time t^n, and $N_a(\mathbf{X}) : \Omega^0 \to \mathbb{R}$ is the interpolation function at sample point a.

Example in 1D:



N **can be higher-order, or globally supported.**

Weak Form Discretization

Spatial Sampling and Interpolation (Cont.)

$$Q_i(\mathbf{X}, t^n) \approx \sum_a Q_{a|i}(t^n) N_a(\mathbf{X}) = \sum_a Q^n_{a|i} N_a(\mathbf{X}),$$

$$\mathbf{x}_i(\mathbf{X}, t^n) \approx \sum_b \mathbf{x}_{b|i}(t^n) N_b(\mathbf{X}) = \sum_b \mathbf{x}^n_{b|i} N_b(\mathbf{X})$$

Then

$$A_i(\mathbf{X}, t^n) \approx \sum_b A_{b|i}(t^n) N_b(\mathbf{X}) = \sum_b A^n_{b|i} N_b(\mathbf{X})$$

$$\int_{\Omega^0} R^0(\mathbf{X}) Q^n_i(\mathbf{X}) A^n_i(\mathbf{X}) d\mathbf{X} = \int_{\partial\Omega^0} Q^n_i(\mathbf{X}) T^n_i(\mathbf{X}) ds(\mathbf{X}) - \int_{\Omega^0} Q^n_{i,j}(\mathbf{X}) P^n_{ij}(\mathbf{X}) d\mathbf{X}$$

$$\Downarrow$$

$$\int_{\Omega^0} R(\mathbf{X}, 0) Q^n_{a|i} N_a(\mathbf{X}) A^n_{b|i} N_b(\mathbf{X}) d\mathbf{X} = \int_{\partial\Omega^0} Q^n_{a|i} N_a(\mathbf{X}) T_i(\mathbf{X}, t^n) ds(\mathbf{X}) - \int_{\Omega^0} Q^n_{a|i} N_{a,j}(\mathbf{X}) P_{ij}(\mathbf{X}, t^n) d\mathbf{X}.$$

Weak Form Discretization

Mass Matrix

$$\int_{\Omega^0} R(\mathbf{X},0) Q^n_{a|i} N_a(\mathbf{X}) A^n_{b|i} N_b(\mathbf{X}) d\mathbf{X}$$
$$= \int_{\partial\Omega^0} Q^n_{a|i} N_a(\mathbf{X}) T_i(\mathbf{X}, t^n) ds(\mathbf{X}) - \int_{\Omega^0} Q^n_{a|i} N_{a,j}(\mathbf{X}) P_{ij}(\mathbf{X}, t^n) d\mathbf{X}.$$

On the left-hand-side, we see that the sample values $Q^n_{a|i}$ and $A^n_{b|i}$ are in fact independent of the integration, so we can move them out of the integral and obtain

$$M_{ab} Q^n_{a|i} A^n_{b|i}$$
$$= \int_{\partial\Omega^0} Q^n_{a|i} N_a(\mathbf{X}) T_i(\mathbf{X}, t^n) ds(\mathbf{X}) - \int_{\Omega^0} Q^n_{a|i} N_{a,j}(\mathbf{X}) P_{ij}(\mathbf{X}, t^n) d\mathbf{X}$$

where

$$M_{ab} = \int_{\Omega^0} R(\mathbf{X},0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X}$$

is the mass matrix.

Remark 17.1. The mass matrix M (Equation 17.7) is symmetric positive semi-definite because it can be written as $\int_{\Omega^0} BB^T d\mathbf{X}$ where $B_i = \sqrt{R(\mathbf{X},0)} N_i(\mathbf{X})$ so that $z^T M z = \int_{\Omega^0} (z^T B)^2 d\mathbf{X} \geq 0$ for any vector z. In practice, this mass matrix may be singular. Therefore, we usually apply a "mass lumping" strategy to approximate the mass matrix with a diagonal and positive definite one by taking the sum of each row and obtain $M^{\text{lump}}_{ab} = \delta_{ab} \sum_c M_{ac}$. □

Weak Form Discretization

Choosing Test Functions

$$M_{ab}Q^n_{a|i}A^n_{b|i} = \int_{\partial\Omega^0} Q^n_{a|i}N_a(\mathbf{X})T_i(\mathbf{X},t^n)ds(\mathbf{X}) - \int_{\Omega^0} Q^n_{a|i}N_{a,j}(\mathbf{X})P_{ij}(\mathbf{X},t^n)d\mathbf{X}$$

- Our discretization limit our solution space to the d*n interpolation functions
  - Test function can be chosen to generate d*n functions

Therefore, for $\hat{a}$ traversing all sample points, and for $\hat{i} = 1, 2, ..., d$, we can respectively assign the test function

$$Q^n_{a|i} = \begin{cases} 1, & a = \hat{a} \text{ and } i = \hat{i} \\ 0, & \text{otherwise} \end{cases}$$

to obtain nd equations

$$M_{\hat{a}b}A^n_{b|\hat{i}} = \int_{\partial\Omega^0} N_{\hat{a}}(\mathbf{X})T_{\hat{i}}(\mathbf{X},t^n)ds(\mathbf{X}) - \int_{\Omega^0} N_{\hat{a},j}(\mathbf{X})P_{\hat{i}j}(\mathbf{X},t^n)d\mathbf{X}.$$

Weak Form Discretization

Time Discretization

Discretization in time connects $\mathbf{A}$ to our DOF $\mathbf{x}$. In the continuous setting, $\mathbf{A}(\mathbf{X}, t) = \frac{\partial^2 \mathbf{x}}{\partial t^2}(\mathbf{X}, t)$. Let us now split time into small intervals $t^0, t^2, ..., t^n, ...$ as mentioned in the first chapter of this book. With finite difference formula, we can conveniently approximate $\mathbf{A}$ using $\mathbf{x}$, for example, with backward Euler,

$$\mathbf{A}^n(\mathbf{X}) = \frac{\mathbf{V}^n(\mathbf{X}) - \mathbf{V}^{n-1}(\mathbf{X})}{t^n - t^{n-1}},$$

$$\mathbf{V}^n(\mathbf{X}) = \frac{\mathbf{x}^n(\mathbf{X}) - \mathbf{x}^{n-1}(\mathbf{X})}{t^n - t^{n-1}},$$

which gives us

$$\mathbf{A}^n(\mathbf{X}) = \frac{\mathbf{x}^n(\mathbf{X}) - (\mathbf{x}^{n-1}(\mathbf{X}) + h\mathbf{V}^{n-1}(\mathbf{X}))}{\Delta t^2},$$

where $\Delta t = t^n - t^{n-1}$. Applying this relation at the sample points

$$M_{\hat{a}b} \frac{x^n_{b|\hat{i}} - (x^{n-1}_{b|\hat{i}} + hV^{n-1}_{b|\hat{i}})}{\Delta t^2} = \int_{\partial\Omega^0} N_{\hat{a}}(\mathbf{X}) T_{\hat{i}}(\mathbf{X}, t^n) ds(\mathbf{X}) - \int_{\Omega^0} N_{\hat{a},j}(\mathbf{X}) P_{\hat{i}j}(\mathbf{X}, t^n) d\mathbf{X}$$

Different time discretization gives different time integration rules

Weak Form Discretization

Zero Traction Boundary Condition

$$M_{\hat{a}b}\frac{x^n_{b|\hat{i}} - (x^{n-1}_{b|\hat{i}} + hV^{n-1}_{b|\hat{i}})}{\Delta t^2} = \int_{\partial\Omega^0} N_{\hat{a}}(\mathbf{X}) T_{\hat{i}}(\mathbf{X}, t^n) ds(\mathbf{X}) - \int_{\Omega^0} N_{\hat{a},j}(\mathbf{X}) P_{\hat{i}j}(\mathbf{X}, t^n) d\mathbf{X}$$

By applying mass lumping and zero traction boundary condition $\mathbf{T}(\mathbf{X}, t) = 0$, we get

$$M(x^{n+1} - (x^n + \Delta t v^n)) - \Delta t^2 f(x^{n+1}) = 0$$

with elasticity force $f(x^{n+1})$ obtained by evaluating

$$-\int_{\Omega^0} N_{\hat{a},j}(\mathbf{X}) P_{\hat{i}j}(\mathbf{X}, t) d\mathbf{X}$$

Remarks

- Strong form: d-dimensional equation for arbitrary X
- Weak form: 1-dimensional equation for arbitrary Q
- Discrete weak form: 1-dimensional equation for arbitrary $Q_{a|i}$
- Discrete form (after choosing $Q_{a|i}$): dn-dimensional equation (n is # of sample points)

- Discretization on the weak form:
  - FEM, MPM
- Directly discretize the strong form:
  - Finite difference method
  - Smoothed-particle hydrodynamics (SPH)

Next Lecture: Finite Element Discretization

- Evaluation of elasticity forces $-\int_{\Omega^0} N_{\hat{a},j}(\mathbf{X}) P_{\hat{i}j}(\mathbf{X},t) d\mathbf{X}$
- Boundary treatment

Image Sources

- https://en.wikipedia.org/wiki/Stress%E2%80%93strain_curve