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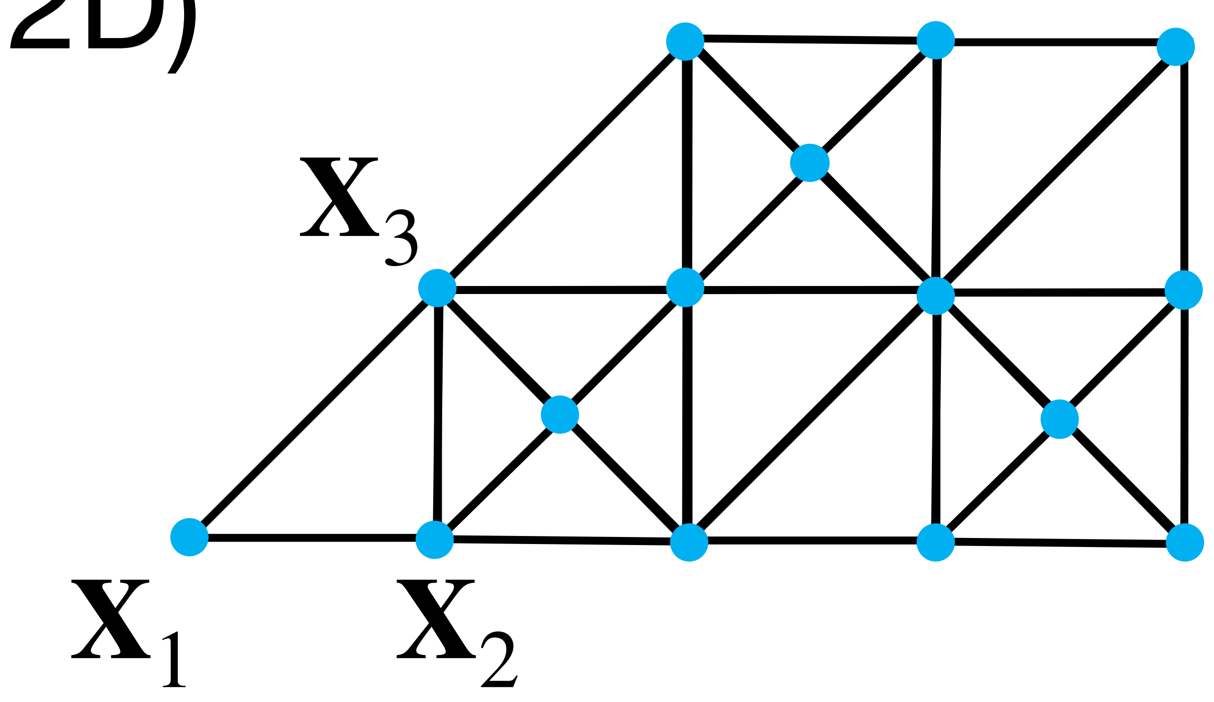
Lec 12: Frictional Self-Contact

15-769: Physically-based Animation of Solids and Fluids (F23)

Recap: Piecewise Linear Displacement

- We partition the space into simplex elements (triangles in 2D)
- Approximate the world-space coordinates (DOF) via interpolation:



$$\hat{\mathbf{x}}(\mathbf{X}) = \mathbf{x}(\mathbf{X}_1)N_1(\mathbf{X}) + \mathbf{x}(\mathbf{X}_2)N_2(\mathbf{X}) + \mathbf{x}(\mathbf{X}_3)N_3(\mathbf{X})$$

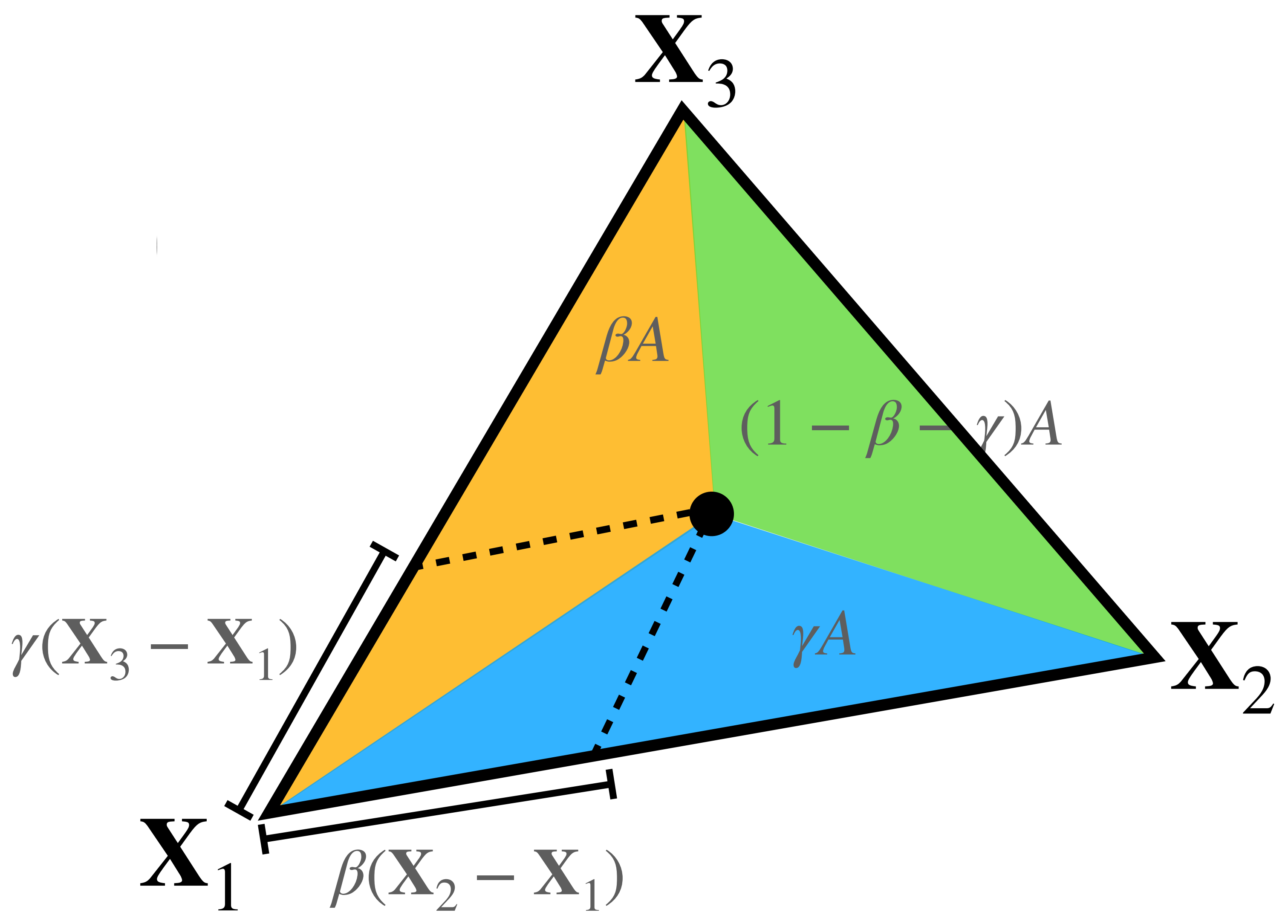
Let $\beta, \gamma \in [0,1]$ and $\beta + \gamma = 1$, we can use them to express the material space coordinate of an arbitrary point $\mathbf{X}$ in element $\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3$ as

$$\begin{aligned}\mathbf{X}(\beta,\gamma) &= \mathbf{X}_1 + \beta(\mathbf{X}_2 - \mathbf{X}_1) + \gamma(\mathbf{X}_3 - \mathbf{X}_1)\\ &= (1-\beta-\gamma)\mathbf{X}_1 + \beta\mathbf{X}_2 + \gamma\mathbf{X}_3.\end{aligned}$$

Similarly, in world space:

$$\begin{aligned}\mathbf{x}(\beta,\gamma) \approx \hat{\mathbf{x}}(\beta,\gamma) &= \mathbf{x}_1 + \beta(\mathbf{x}_2 - \mathbf{x}_1) + \gamma(\mathbf{x}_3 - \mathbf{x}_1)\\ &= (1-\beta-\gamma)\mathbf{x}_1 + \beta\mathbf{x}_2 + \gamma\mathbf{x}_3,\end{aligned}$$

$$N_1(\beta,\gamma) = 1-\beta-\gamma, \quad N_2(\beta,\gamma) = \beta, \quad N_3(\beta,\gamma) = \gamma.$$



Recap: Mass Matrix

$$M_{ab} = \int_{\Omega^0} R(\mathbf{X}, 0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X}$$

➡

With the solid domain discretized into triangles $\mathcal{T}$, we have

$$M_{ab} = \sum_{e \in \mathcal{T}} \int_{\Omega_e^0} R(\mathbf{X}, 0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X},$$

where Ω_e^0 represents the material space of triangle e.

N_i is nonzero only on the incident triangles of node i

Let us change the integration variable from $\mathbf{X}$ to (β, γ), which gives

$$\begin{aligned}
\int_{\Omega_e^0} R(\mathbf{X}, 0) N_a(\mathbf{X}) N_b(\mathbf{X}) d\mathbf{X} =& 2RA_e \int_0^1 \int_0^{1-\beta} \beta\gamma d\gamma d\beta \\
=& 2RA_e \int_0^1 \frac{1}{2}\beta\gamma^2|_{\gamma=0}^{\gamma=1-\beta} d\beta \\
=& RA_e \int_0^1 \beta(1-\beta)^2 d\beta \\
=& RA_e(\frac{\beta^2}{2} - \frac{2\beta^3}{3} + \frac{\beta^4}{4})|_{\beta=0}^{\beta=1} = \frac{1}{12}RA_e
\end{aligned}$$

(Assuming uniform density R,
a and b are the 2nd and 3rd vertices,
A_e is the triangle area)

Recap: Lumped Mass Matrix

$$
\begin{aligned}
M_{aa}^{\text{lump}} &= \sum_{e \in \mathcal{T}(a)} 2RA_e\Big(\int_0^1 \int_0^{1-\beta} \beta(1-\beta-\gamma)d\gamma d\beta + \int_0^1 \int_0^{1-\beta} \beta^2 d\gamma d\beta \\
&\quad + \int_0^1 \int_0^{1-\beta} \beta\gamma d\gamma d\beta\Big) \\
&= \sum_{e \in \mathcal{T}(a)} 2RA_e \int_0^1 \int_0^{1-\beta} \beta d\gamma d\beta = \sum_{e \in \mathcal{T}(a)} 2RA_e \int_0^1 \beta\gamma|_{\gamma=0}^{\gamma=1-\beta} d\beta \\
&= \sum_{e \in \mathcal{T}(a)} 2RA_e \int_0^1 \beta(1-\beta)d\beta = \sum_{e \in \mathcal{T}(a)} 2RA_e\Big(\frac{\beta^2}{2} - \frac{\beta^3}{3}\Big)|_{\beta=0}^{\beta=1} \\
&= \sum_{e \in \mathcal{T}(a)} \frac{1}{3}RA_e,
\end{aligned}
$$

where $\mathcal{T}(a)$ denotes the set of triangles incident to node a.

Recap: Elasticity

$$\int_{\Omega^0} N_{\hat{a},j}(\mathbf{X}) P_{\hat{i}j}(\mathbf{X},t^n) d\mathbf{X} = \int_{\Omega^0} (\mathbf{P}(\mathbf{X},t^n)\nabla^{\mathbf{X}} N_{\hat{a}}(\mathbf{X}))_{\hat{i}}\, d\mathbf{X} = \sum_{e\in\mathcal{T}} \int_{\Omega^0_e} (\mathbf{P}(\mathbf{X},t^n)\nabla^{\mathbf{X}} N_{\hat{a}}(\mathbf{X}))_{\hat{i}}\, d\mathbf{X}$$

Analogously, this summation also only needs to involve the incident triangles of node $\hat{a}$.

- $\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}}$ **can be calculated with** $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$**:**

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial(\beta,\gamma)}\left(\frac{\partial \mathbf{X}}{\partial(\beta,\gamma)}\right)^{-1} \approx \frac{\partial \hat{\mathbf{x}}}{\partial(\beta,\gamma)}\left(\frac{\partial \mathbf{X}}{\partial(\beta,\gamma)}\right)^{-1}$$
$$= [\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1][\mathbf{X}_2 - \mathbf{X}_1, \mathbf{X}_3 - \mathbf{X}_1]^{-1},$$

$$\nabla^{\mathbf{X}} N_1(\mathbf{X}) = \frac{\partial(1-\beta-\gamma)}{\partial \mathbf{X}} = \left(\frac{\partial(1-\beta-\gamma)}{\partial(\beta,\gamma)}\left(\frac{\partial \mathbf{X}}{\partial(\beta,\gamma)}\right)^{-1}\right)^T$$
$$= ([-1,-1][\mathbf{X}_2 - \mathbf{X}_1, \mathbf{X}_3 - \mathbf{X}_1]^{-1})^T$$

piecewise constant in Ω^0, so does $\mathbf{P}$

$$\nabla^{\mathbf{X}} N_2(\mathbf{X}) = \frac{\partial \beta}{\partial \mathbf{X}} = \left(\frac{\partial \beta}{\partial(\beta,\gamma)}\left(\frac{\partial \mathbf{X}}{\partial(\beta,\gamma)}\right)^{-1}\right)^T = ([1,0][\mathbf{X}_2 - \mathbf{X}_1, \mathbf{X}_3 - \mathbf{X}_1]^{-1})^T$$

$$\nabla^{\mathbf{X}} N_3(\mathbf{X}) = \frac{\partial \gamma}{\partial \mathbf{X}} = \left(\frac{\partial \gamma}{\partial(\beta,\gamma)}\left(\frac{\partial \mathbf{X}}{\partial(\beta,\gamma)}\right)^{-1}\right)^T = ([0,1][\mathbf{X}_2 - \mathbf{X}_1, \mathbf{X}_3 - \mathbf{X}_1]^{-1})^T$$

Higher-order N and non-simplex elements are both possible

$$\frac{\partial \Psi_e}{\partial \mathbf{x}_{\hat{a}}} = \frac{\partial \Psi_e}{\partial \mathbf{F}}\frac{\partial \mathbf{F}}{\partial \hat{\mathbf{x}}}\frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{x}_{\hat{a}}} = \mathbf{P}\frac{\partial \hat{\mathbf{x}}/\partial \mathbf{X}}{\partial \hat{\mathbf{x}}} N_{\hat{a}} = \mathbf{P}\nabla^{\mathbf{X}} N_{\hat{a}}$$

Recap: Boundary Conditions

$$R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t) = \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X},t) + R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t),$$
$$\forall\ \mathbf{X} \in \Omega^0 \text{ and } t \geq 0;$$
$$\mathbf{x} = \mathbf{x}_D(\mathbf{X},t), \qquad (18.1)$$
$$\forall\ \mathbf{X} \in \Gamma_D \text{ and } t \geq 0;$$
$$\mathbf{P}(\mathbf{X},t)\mathbf{N}(\mathbf{X}) = \mathbf{T}_N(\mathbf{X},t),$$
$$\forall\ \mathbf{X} \in \Gamma_N \text{ and } t \geq 0.$$

Here Γ_N and Γ_D are the Neumann and Dirichlet boundaries respectively, $\Gamma_N \cup \Gamma_D = \partial\Omega_0$, $\Gamma_N \cap \Gamma_D = \emptyset$, and $\mathbf{x}_D$ and $\mathbf{T}_N$ are given. After we derive the weak from of the momentum conservation (Equation 18.1 1st line), the boundary term $\int_{\partial\Omega^0} Q_i(\mathbf{X},t)T_i(\mathbf{X},t)ds(\mathbf{X})$ can be separately considered for Dirichlet and Neumann boundaries:

$$\int_{\partial\Omega^0} Q_i(\mathbf{X},t)T_i(\mathbf{X},t)ds(\mathbf{X})$$
$$= \int_{\Gamma_D} Q_i(\mathbf{X},t)T_{D|i}(\mathbf{X},t)ds(\mathbf{X}) + \int_{\Gamma_N} Q_i(\mathbf{X},t)T_{N|i}(\mathbf{X},t)ds(\mathbf{X}).$$

Dirichlet:

$$\hat{\mathbf{x}}(\mathbf{X}_i) = \mathbf{x}_D(\mathbf{X}_i) \quad \forall\ \mathbf{X}_i \in \Gamma_D$$

Neumann:

$$\int_{\Gamma_N} N_{\hat{a}}(\mathbf{X})T_{\hat{i}}(\mathbf{X},t^n)ds(\mathbf{X})$$
$$= \sum_{e\in\mathcal{T}} \int_{\partial\Omega_e^0\cap\Gamma_N} N_{\hat{a}}(\mathbf{X})T_{\hat{i}}(\mathbf{X},t^n)ds(\mathbf{X})$$

assume the 2nd and 3rd vertices are on the boundary:

$$\int_0^1 \beta T_{\hat{i}}(\beta\mathbf{X}_2 + (1-\beta)\mathbf{X}_3, t^n)\left|\frac{\partial s}{\partial \beta}\right|d\beta$$

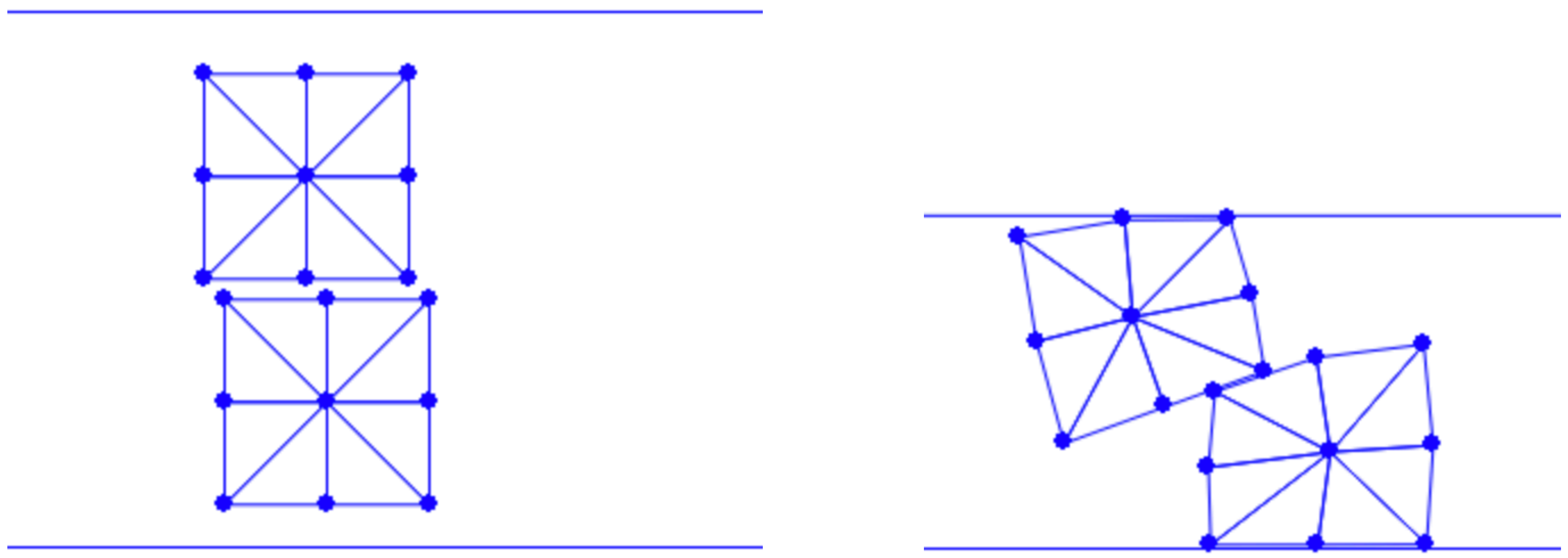
assume constant T:

$$T_{\hat{i}}^n \int_0^1 \beta\left|\frac{\partial s}{\partial \beta}\right|d\beta = \frac{1}{2}\|\mathbf{X}_2 - \mathbf{X}_3\|T_{\hat{i}}^n$$

Today: Frictional Self-Contact

$$\int_{\partial\Omega^0} Q_i(\mathbf{X},t)T_i(\mathbf{X},t)ds(\mathbf{X})$$
$$= \int_{\Gamma_D} Q_i(\mathbf{X},t)T_{D|i}(\mathbf{X},t)ds(\mathbf{X}) + \int_{\Gamma_N} Q_i(\mathbf{X},t)T_{N|i}(\mathbf{X},t)ds(\mathbf{X})$$
$$+ \int_{\Gamma_C} Q_i(\mathbf{X},t)T_{C|i}(\mathbf{X},t)ds(\mathbf{X}) + \int_{\Gamma_C} Q_i(\mathbf{X},t)T_{F|i}(\mathbf{X},t)ds(\mathbf{X}).$$

(Here Γ_C can overlap with Γ_D or Γ_N)



Normal Contact

Approximation with Conservative Force

Two disjoint colliding boundaries $\Gamma_1 \subset \Gamma_C, \Gamma_2 \subset \Gamma_C$

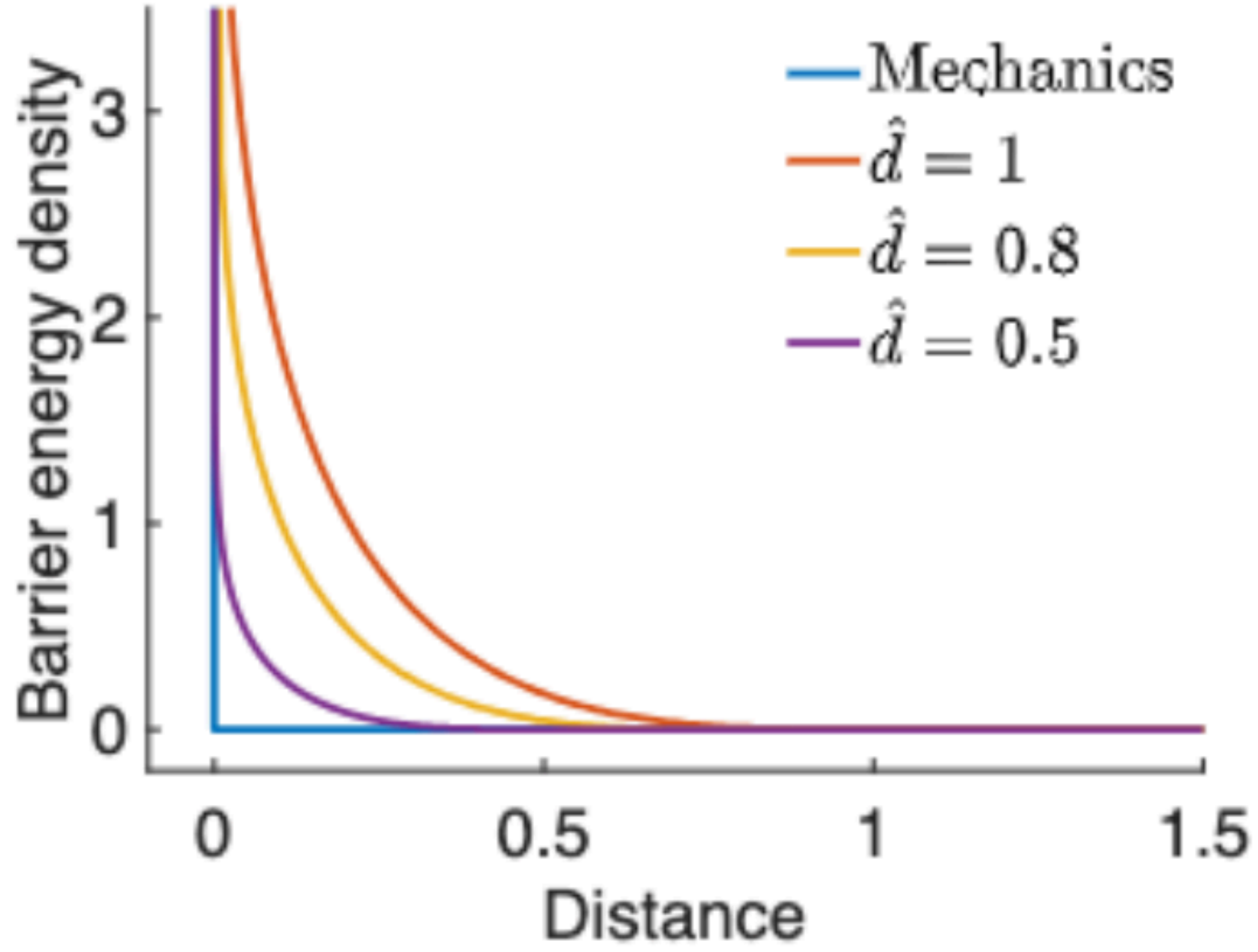
Nonpenetration: $\forall \mathbf{X}_1 \in \Gamma_1, \mathbf{X}_2 \in \Gamma_2, \; \|\mathbf{x}_1 - \mathbf{x}_2\| > 0$

$$\forall \mathbf{X}_1 \in \Gamma_1, \; \min_{\mathbf{X}_2 \in \Gamma_2} \|\mathbf{x}_1 - \mathbf{x}_2\| > 0$$

$$b(d, \hat{d}) = \begin{cases} \frac{\kappa}{2} \hat{d} (\frac{d}{\hat{d}} - 1) \ln \frac{d}{\hat{d}} & d < \hat{d} \\ 0 & d \geq \hat{d} \end{cases}$$

Approximate with conservative force:

$$\mathbf{T}_C(\mathbf{X}_1, t) = -\frac{\partial b(\min_{\mathbf{X}_2 \in \Gamma_2} \|\mathbf{x}(\mathbf{X}_1, t) - \mathbf{x}(\mathbf{X}_2, t)\|, \hat{d})}{\partial \mathbf{x}(\mathbf{X}_1, t)}$$



Remarks:

- **Needs accumulation for multiple minima**
- $\min()$ **is non-smooth**

Normal Contact

Barrier Potential

The above two regions colliding case results in a boundary integral

$$\int_{\Gamma_1} Q_i(\mathbf{X}_1,t)T_{C1|i}(\mathbf{X}_1,t)ds(\mathbf{X}) + \int_{\Gamma_2} Q_i(\mathbf{X}_2,t)T_{C2|i}(\mathbf{X}_2,t)ds(\mathbf{X})$$

$$\mathbf{T}_{C1}(\mathbf{X}_1,t) = -\frac{\partial b(\min_{\mathbf{X}_2\in\Gamma_2}\|\mathbf{x}(\mathbf{X}_1,t)-\mathbf{x}(\mathbf{X}_2,t)\|,\hat{d})}{\partial\mathbf{x}(\mathbf{X}_1,t)}$$

$$\mathbf{T}_{C2}(\mathbf{X}_2,t) = -\frac{\partial b(\min_{\mathbf{X}_1\in\Gamma_1}\|\mathbf{x}(\mathbf{X}_2,t)-\mathbf{x}(\mathbf{X}_1,t)\|,\hat{d})}{\partial\mathbf{x}(\mathbf{X}_2,t)}$$

Merge $\mathbf{T}_{C1}$ and $\mathbf{T}_{C2}$:

$$\mathbf{T}_C(\mathbf{X},t) = -\frac{\partial b(\min_{\mathbf{X}_2\in\Gamma_C-\mathcal{N}(\mathbf{X})}\|\mathbf{x}(\mathbf{X},t)-\mathbf{x}(\mathbf{X}_2,t)\|,\hat{d})}{\partial\mathbf{x}(\mathbf{X},t)}$$

where $\mathcal{N}(\mathbf{X}) = \{\mathbf{X}_N \in \mathbb{R}^d \mid \|\mathbf{X}_N - \mathbf{X}\| < r\}$ is an infinitesimal circle around $\mathbf{X}$ with the radius r sufficiently small to avoid unnecessary contact forces between a point and its geodesic neighbors.

Barrier Potential:

$$\int_{\Gamma_C} \frac{1}{2} b(\min_{\mathbf{X}_2\in\Gamma_C-\mathcal{N}(\mathbf{X})}\|\mathbf{x}(\mathbf{X},t)-\mathbf{x}(\mathbf{X}_2,t)\|,\hat{d})ds(\mathbf{X})$$

Need $\hat{d}\to 0$, $r\to 0$, and $\hat{d}/r\to 0$.

Friction

Approximation with Constitutive Model

Maximum Dissipation Principle:

$$\mathbf{T}_F(\mathbf{X},t) = \underset{\boldsymbol{\beta}\in\mathbb{R}^d}{\operatorname{argmin}}\, \boldsymbol{\beta}^T \mathbf{V}_F(\mathbf{X},t)$$

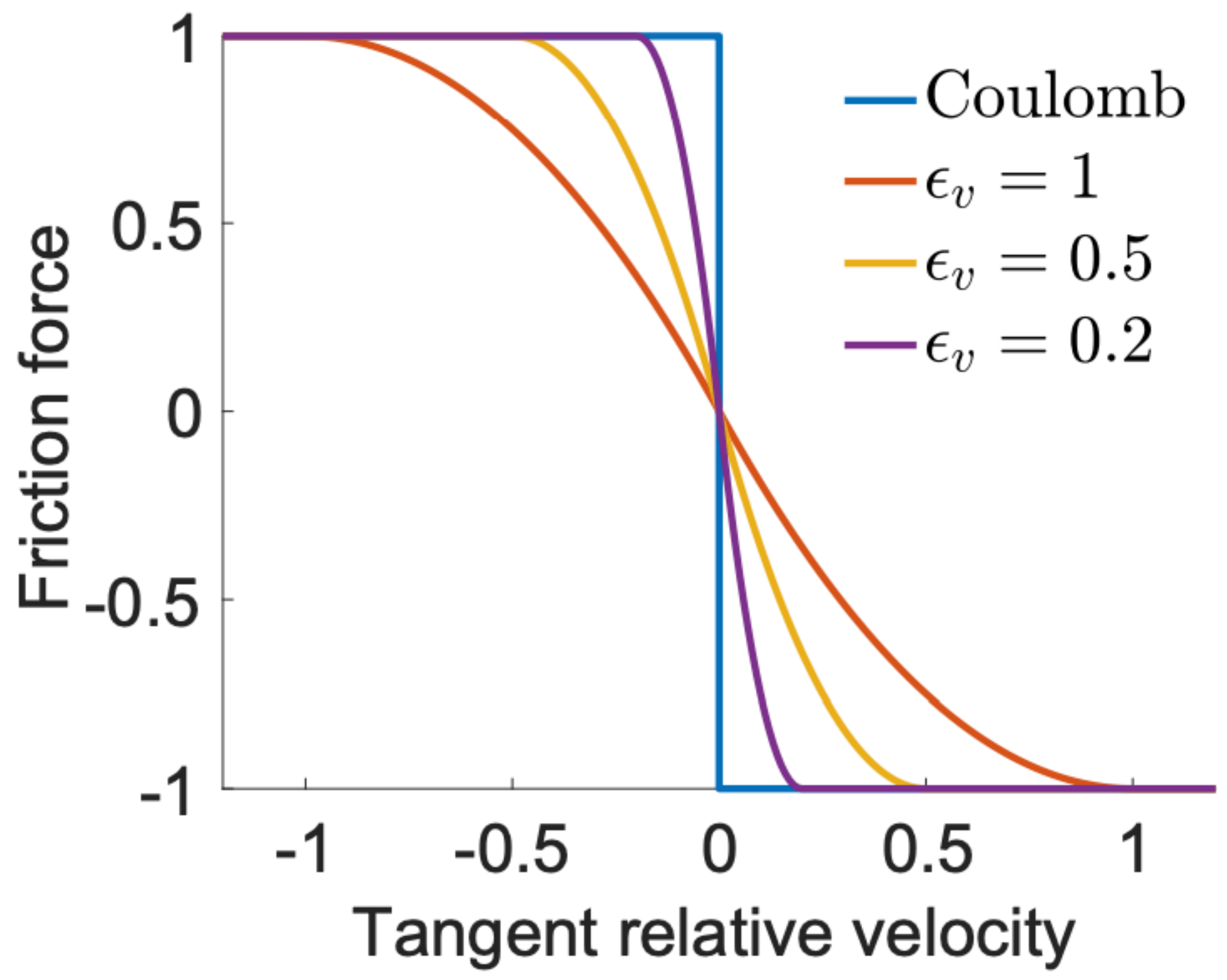
$$\text{s.t.} \quad \|\boldsymbol{\beta}\| \le \mu \|\mathbf{T}_C(\mathbf{X},t)\| \quad \text{and} \quad \boldsymbol{\beta}\cdot\mathbf{N}(\mathbf{X},t) = 0.$$

Here $\mathbf{V}_F(\mathbf{X},t) = \mathbf{V}(\mathbf{X},t) - \mathbf{V}(\mathbf{X}_2,t)$ is the relative sliding velocity between $\mathbf{X}$ and the closest point $\mathbf{X}_2 = \arg\min_{\mathbf{X}_2\in\Gamma_C - \mathcal{N}(\mathbf{X})} \|\mathbf{X}-\mathbf{X}_2\|$, μ is the coefficient of friction, $\mathbf{T}_C$ is the normal contact force per unit area, and $\mathbf{N}$ is the normal direction.

$$\mathbf{T}_F(\mathbf{X},t) = -\mu\|\mathbf{T}_C(\mathbf{X},t)\|\; f(\|\mathbf{V}_F(\mathbf{X},t)\|)\; \mathbf{s}(\mathbf{V}_F(\mathbf{X},t))$$

with $\mathbf{s}(\mathbf{V}_F) = \frac{\mathbf{V}_F}{\|\mathbf{V}_F\|}$ when $\|\mathbf{V}_F\| > 0$, while $\mathbf{s}(\mathbf{V}_F)$ takes any unit vector orthogonal to $\mathbf{N}(\mathbf{X},t)$ when $\|\mathbf{V}_F\| = 0$.

Approximate f **with** $f_1(y) = \begin{cases} -\frac{y^2}{\epsilon_v^2} + \frac{2y}{\epsilon_v}, & y \in [0,\epsilon_v) \\ 1, & y \ge \epsilon_v, \end{cases}$



$$\boxed{\mathbf{T}_F(\mathbf{X},t) \approx -\mu\|\mathbf{T}_C(\mathbf{X},t)\| f_1(\|\mathbf{V}_F(\mathbf{X},t)\|)\mathbf{s}(\mathbf{V}_F(\mathbf{X},t))}$$

Strong Form with All Boundary Effects

$$
\begin{aligned}
&R(\mathbf{X},0)\frac{\partial \mathbf{V}}{\partial t}(\mathbf{X},t) = \nabla^{\mathbf{X}} \cdot \mathbf{P}(\mathbf{X},t) + R(\mathbf{X},0)\mathbf{A}^{\text{ext}}(\mathbf{X},t), \quad \forall\ \mathbf{X} \in \Omega^0;\\
&\mathbf{x} = \mathbf{x}_D(\mathbf{X},t), \quad \forall\ \mathbf{X} \in \Gamma_D;\\
&\mathbf{P}(\mathbf{X},t)\mathbf{N}(\mathbf{X}) = \mathbf{T}_N(\mathbf{X},t) + \mathbf{T}_C(\mathbf{X},t) + \mathbf{T}_F(\mathbf{X},t), \quad \forall\ \mathbf{X} \in \Gamma_N;\\
&\phi(\mathbf{X},t) : \Omega^0 \to \Omega^t \text{ is bijective}, \quad \forall\ \mathbf{X} \in \Omega^0;\\
&\mathbf{T}_F(\mathbf{X},t) = \operatorname*{argmin}_{\boldsymbol{\beta} \in \mathbb{R}^d} \boldsymbol{\beta}^T \mathbf{V}_F(\mathbf{X},t)\\
&\text{s.t.} \quad \|\boldsymbol{\beta}\| \le \mu \|\mathbf{T}_C(\mathbf{X},t)\| \ \text{ and } \ \boldsymbol{\beta} \cdot \mathbf{N}(\mathbf{X},t) = 0, \quad \forall\ \mathbf{X} \in \Gamma_C.
\end{aligned}
$$

After deriving the weak form of the momentum equation, the boundary integral term can be considered separately as

$$
\begin{aligned}
&\int_{\partial\Omega^0} Q_i(\mathbf{X},t) T_i(\mathbf{X},t) ds(\mathbf{X})\\
&= \int_{\Gamma_D} Q_i(\mathbf{X},t) T_{D|i}(\mathbf{X},t) ds(\mathbf{X}) + \int_{\Gamma_N} Q_i(\mathbf{X},t) T_{N|i}(\mathbf{X},t) ds(\mathbf{X})\\
&+ \int_{\Gamma_C} Q_i(\mathbf{X},t) T_{C|i}(\mathbf{X},t) ds(\mathbf{X}) + \int_{\Gamma_C} Q_i(\mathbf{X},t) T_{F|i}(\mathbf{X},t) ds(\mathbf{X}).
\end{aligned}
$$

Solid-Obstacle Contact

Barrier Potential

$$P_C = \int_{\Gamma_C} \frac{1}{2} b(\min_{\mathbf{X}_2 \in \Gamma_C - \mathcal{N}(\mathbf{X})} \|\mathbf{x}(\mathbf{X}, t) - \mathbf{x}(\mathbf{X}_2, t)\|, \hat{d}) ds(\mathbf{X})$$

where $b()$ is barrier energy density function and $\mathcal{N}(\mathbf{X})$ is an infinitesimal region around $\mathbf{X}$ where contact is ignored for theoretical soundness.

For normal contact between simulated solids and collision obstacles (ignoring self-contact for now), P_C can be written in a much simpler form

$$\begin{aligned} P_C &= \int_{\Gamma_S} \frac{1}{2} b(\min_{\mathbf{X}_2 \in \Gamma_O} \|\mathbf{x}(\mathbf{X}, t) - \mathbf{x}(\mathbf{X}_2, t)\|, \hat{d}) ds(\mathbf{X}) \\ &+ \int_{\Gamma_O} \frac{1}{2} b(\min_{\mathbf{X}_2 \in \Gamma_S} \|\mathbf{x}(\mathbf{X}, t) - \mathbf{x}(\mathbf{X}_2, t)\|, \hat{d}) ds(\mathbf{X}) \\ &= \int_{\Gamma_S} b(\min_{\mathbf{X}_2 \in \Gamma_O} \|\mathbf{x}(\mathbf{X}, t) - \mathbf{x}(\mathbf{X}_2, t)\|, \hat{d}) ds(\mathbf{X}) \\ &= \int_{\Gamma_S} b(d^{PO}(\mathbf{x}(\mathbf{X}, t), O), \hat{d}) ds(\mathbf{X}). \end{aligned}$$

Γ_S **: boundary of the simulated solids**

Γ_O **: boundary of the obstacles**

Symmetry

Solid-Obstacle Contact

Discretizing the Barrier Potential

Triangulation:

$$\int_{\Gamma_S} b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})ds(\mathbf{X}) \approx \sum_{e\in\mathcal{T}} \int_{\partial\Omega_e^0\cap\Gamma_S} b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})ds(\mathbf{X})$$

Assume the 2nd and 3rd vertices are on the boundary:

$$\int_{\partial\Omega_e^0\cap\Gamma_S} b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})ds(\mathbf{X}) = \int_0^1 b(d^{PO}(\mathbf{x}(\beta\mathbf{X}_2+(1-\beta)\mathbf{X}_3,t),O),\hat{d})|\frac{\partial s}{\partial\beta}|d\beta$$

$$\approx \frac{1}{2}b(d^{PO}(\mathbf{x}(\mathbf{X}_2,t),O),\hat{d})|\frac{\partial s}{\partial\beta}| + \frac{1}{2}b(d^{PO}(\mathbf{x}(\mathbf{X}_3,t),O),\hat{d})|\frac{\partial s}{\partial\beta}|$$

(Using triangle vertices as quadrature)

Assume $\mathbf{X}_{\hat{a}-1}$ and $\mathbf{X}_{\hat{a}+1}$ are the two neighbors of $\mathbf{X}_{\hat{a}}$ on the boundary:

$$\int_{\Gamma_S} b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})ds(\mathbf{X}) \approx \sum_{\hat{a}} \frac{\|\mathbf{X}_{\hat{a}}-\mathbf{X}_{\hat{a}-1}\| + \|\mathbf{X}_{\hat{a}}-\mathbf{X}_{\hat{a}+1}\|}{2} b(d^{PO}(\mathbf{x}_{\hat{a}},O),\hat{d})$$

Solid-Obstacle Contact

Normal Contact Boundary Integral Revisited

$$
\begin{aligned}
& -\frac{\partial(\sum_{e\in\mathcal{T}}\int_{\partial\Omega_e^0\cap\Gamma_S} b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})ds(\mathbf{X}))}{\partial\mathbf{x}_{\hat{a}}} \\
=& \sum_{e\in\mathcal{T}}\int_{\partial\Omega_e^0\cap\Gamma_S} -\frac{\partial b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{x}_{\hat{a}}}ds(\mathbf{X}) \\
=& \sum_{e\in\mathcal{T}}\int_{\partial\Omega_e^0\cap\Gamma_S} -\frac{\partial b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})}{\partial\mathbf{x}}N_{\hat{a}}(\mathbf{X})ds(\mathbf{X}).
\end{aligned}
$$

$$
\begin{aligned}
& \int_{\partial\Omega^0} Q_i(\mathbf{X},t)T_i(\mathbf{X},t)ds(\mathbf{X}) \\
=& \int_{\Gamma_D} Q_i(\mathbf{X},t)T_{D|i}(\mathbf{X},t)ds(\mathbf{X}) + \int_{\Gamma_N} Q_i(\mathbf{X},t)T_{N|i}(\mathbf{X},t)ds(\mathbf{X}) \\
&+ \int_{\Gamma_C} Q_i(\mathbf{X},t)T_{C|i}(\mathbf{X},t)ds(\mathbf{X}) + \int_{\Gamma_C} Q_i(\mathbf{X},t)T_{F|i}(\mathbf{X},t)ds(\mathbf{X}).
\end{aligned}
$$

Then we also verified that $\mathbf{T}_C(\mathbf{X},t) = -\frac{\partial b(d^{PO}(\mathbf{x}(\mathbf{X},t),O),\hat{d})}{\partial\mathbf{x}}$ here.

Normal Self-Contact

Discretizing the Barrier Potential

Triangulation:

$$\int_{\Gamma_C} \frac{1}{2} b\left(\min_{\mathbf{X}_2 \in \Gamma_C - \mathcal{N}(\mathbf{X})} \|\mathbf{x}(\mathbf{X}, t) - \mathbf{x}(\mathbf{X}_2, t)\|, \hat{d}\right) ds(\mathbf{X}) \iff \int_{\Gamma_C} \frac{1}{2} \max_{e \in \mathcal{E} - I(\mathbf{X})} b\left(d^{\mathrm{PE}}(\mathbf{x}(\mathbf{X}, t), e), \hat{d}\right) ds(\mathbf{X})$$

$$\approx \int_{\Gamma_C} \frac{1}{2} b\left(\min_{e \in \mathcal{E} - I(\mathbf{X})} \min_{\mathbf{X}_2 \in e} \|\mathbf{x}(\mathbf{X}, t) - \mathbf{x}(\mathbf{X}_2, t)\|, \hat{d}\right) ds(\mathbf{X})$$

$$= \int_{\Gamma_C} \frac{1}{2} b\left(\min_{e \in \mathcal{E} - I(\mathbf{X})} d^{\mathrm{PE}}(\mathbf{x}(\mathbf{X}, t), e), \hat{d}\right) ds(\mathbf{X}).$$

Here $I(\mathbf{X})$ is the set of edges that contains $\mathbf{X}$.

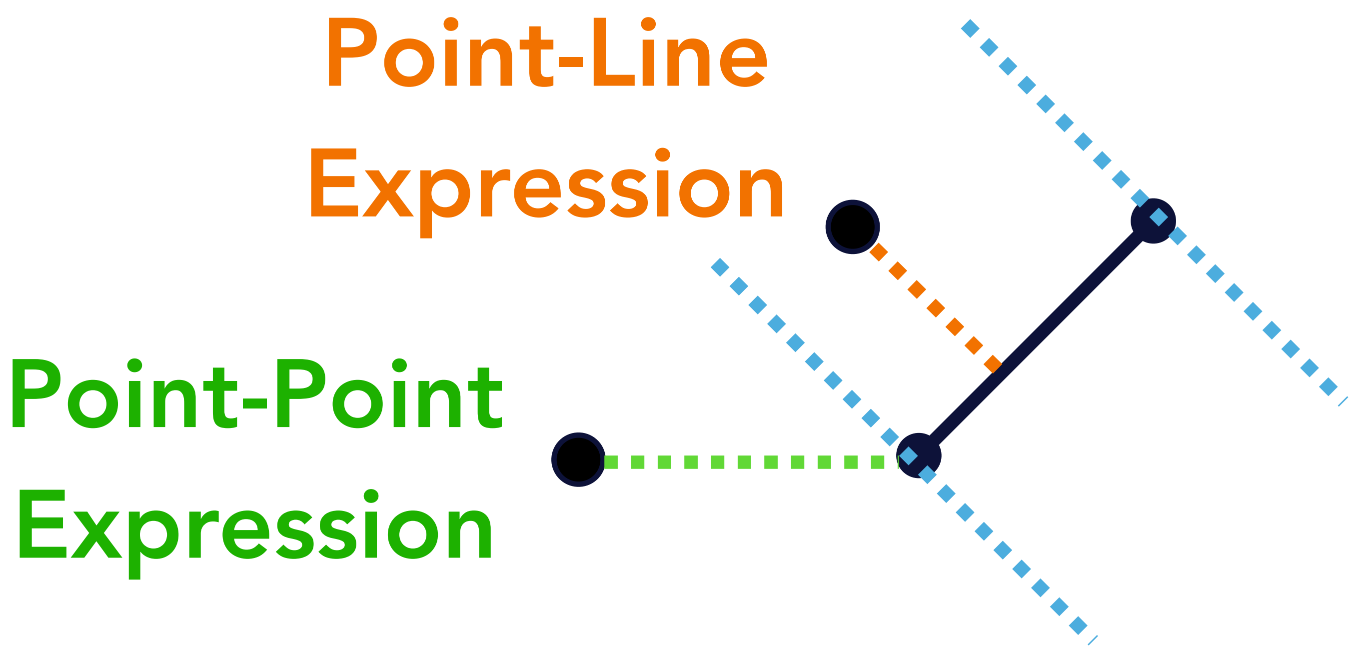
$b()$ is monotonically decreasing,

$$\max(a_1, a_2, \ldots, a_n) \approx (a_1^p + a_2^p + \ldots + a_n^p)^{\frac{1}{p}}$$

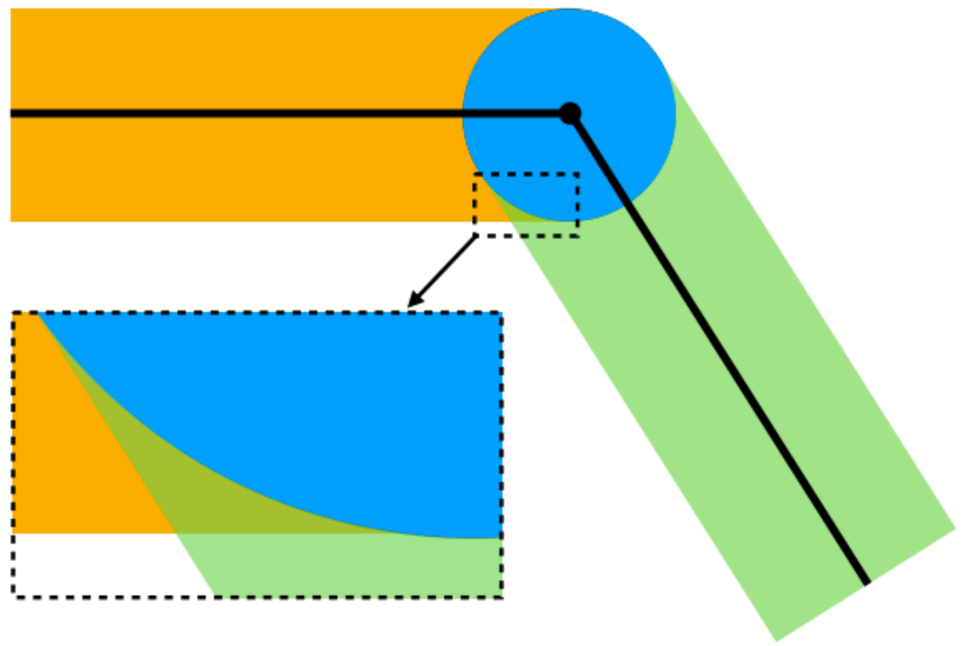
Accurate when $p \to \infty$: Expensive!

d^{PE} **is at least C^1 smooth everywhere:**

But $\min()$ is non-smooth!

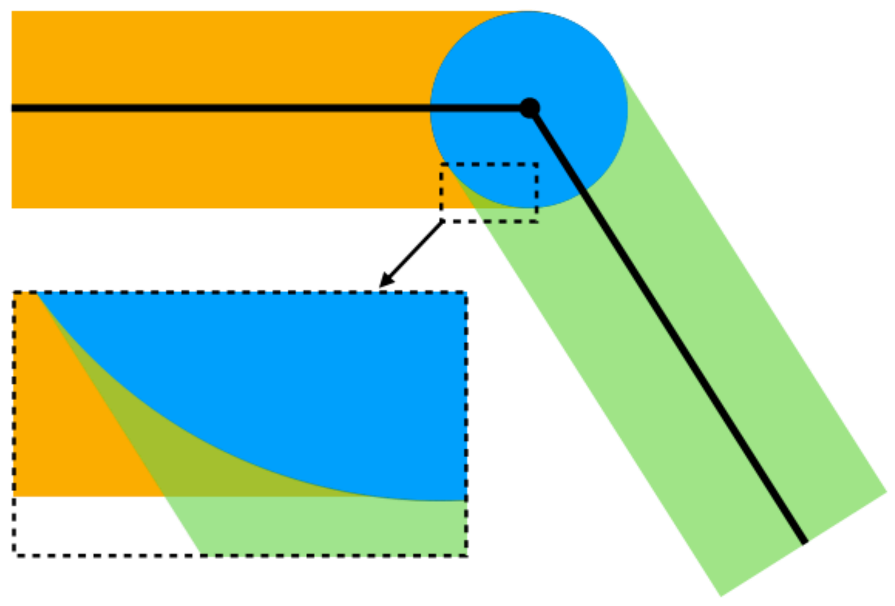


When $p = 1$:



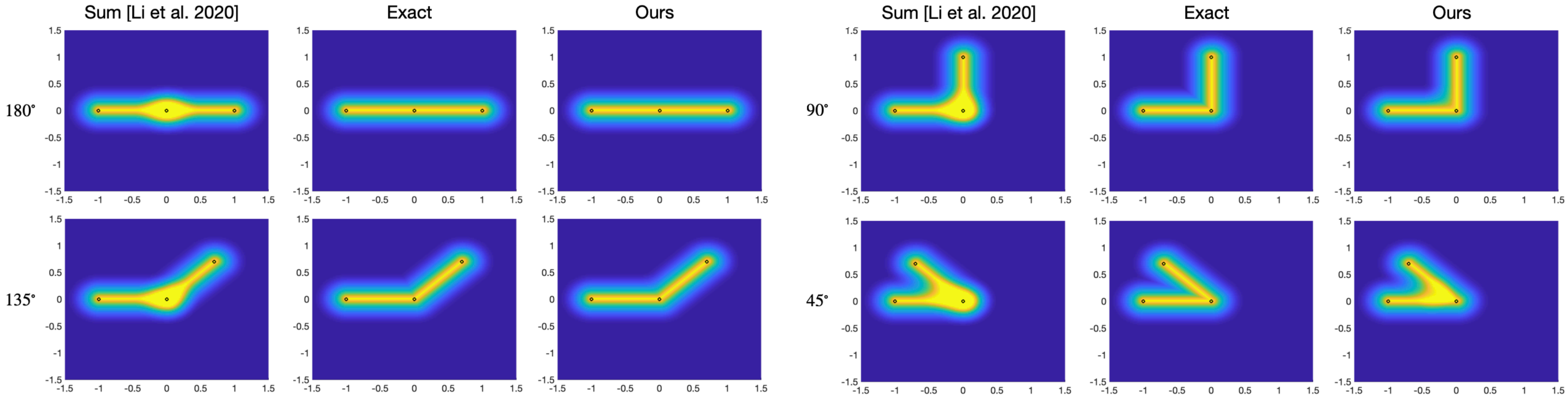
Normal Self-Contact

Smoothly Approximating the Barrier Potential



Can subtract the duplicate point-point barrier [Li et al. 2023]:

$$\Psi_c(x) = \sum_{e \in E \setminus x} b(d(x,e),\hat{d}) \quad - \sum_{x_2 \in V_{int} \setminus x} b(d(x,x_2),\hat{d}) \approx \max_{e \in E \setminus x} b(d(x,e),\hat{d})$$



Minchen Li, Zachary Ferguson, Teseo Schneider, Timothy Langlois, Denis Zorin, Daniele Panozzo, Chenfanfu Jiang, Danny M. Kaufman. Convergent Incremental Potential Contact. Arxiv 2307.15908.

Normal Self-Contact

Discretizing the Smoothly Approximated Barrier Potential

For simplicity, we use $p = 1$:

$$\int_{\Gamma_C} \frac{1}{2} \max_{e \in \mathcal{E} - I(\mathbf{X})} b(d^{\mathrm{PE}}(\mathbf{x}(\mathbf{X}, t), e), \hat{d}) ds(\mathbf{X})$$
$$\approx \int_{\Gamma_C} \frac{1}{2} \sum_{e \in \mathcal{E} - I(\mathbf{X})} b(d^{\mathrm{PE}}(\mathbf{x}(\mathbf{X}, t), e), \hat{d}) ds(\mathbf{X})$$

$$\int_{\Gamma_C} \frac{1}{2} \sum_{e \in \mathcal{E} - I(\mathbf{X})} b(d^{\mathrm{PE}}(\mathbf{x}(\mathbf{X}, t), e), \hat{d}) ds(\mathbf{X})$$
$$\approx \sum_{\hat{a}} \frac{\|\mathbf{X}_{\hat{a}} - \mathbf{X}_{\hat{a}-1}\| + \|\mathbf{X}_{\hat{a}} - \mathbf{X}_{\hat{a}+1}\|}{4} \sum_{e \in \mathcal{E} - I(\mathbf{X}_{\hat{a}})} b(d^{\mathrm{PE}}(\mathbf{x}_{\hat{a}}, e), \hat{d})$$

Implementation

Boundary Elements for Contact

```
def find_boundary(e):
    # index all half-edges for fast query
    edge_set = set()
    for i in range(0, len(e)):
        for j in range(0, 3):
            edge_set.add((e[i][j], e[i][(j + 1) % 3]))

    # find boundary points and edges
    bp_set = set()
    be = []
    for eI in edge_set:
        if (eI[1], eI[0]) not in edge_set:
            # if the inverse edge of a half-edge does not
    exist,
            # then it is a boundary edge
            be.append([eI[0], eI[1]])
            bp_set.add(eI[0])
            bp_set.add(eI[1])
    return [list(bp_set), be]
```

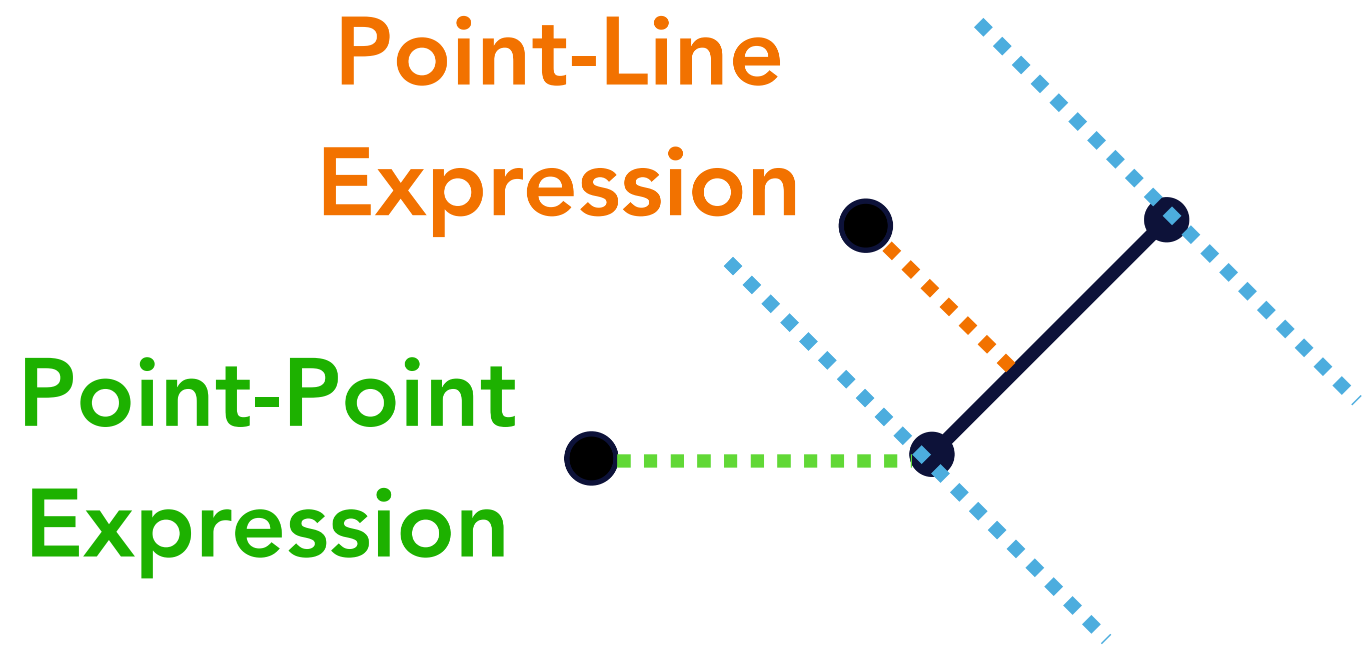
square_mesh.py

Implementation

Point-Edge Distance

$$d^{\mathrm{PE}}_{\mathrm{sq}}(\mathbf{p},\mathbf{e}_0,\mathbf{e}_1) = \min_{\lambda} \|\mathbf{p} - ((1-\lambda)\mathbf{e}_0 + \lambda \mathbf{e}_1)\|^2 \quad \text{s.t.} \quad \lambda \in [0,1]$$

$$d^{\mathrm{PE}}_{\mathrm{sq}}(\mathbf{p},\mathbf{e}_0,\mathbf{e}_1) = \begin{cases} \|\mathbf{p}-\mathbf{e}_0\|^2 & \text{if } (\mathbf{e}_1-\mathbf{e}_0)\cdot(\mathbf{p}-\mathbf{e}_0) < 0, \\ \|\mathbf{p}-\mathbf{e}_1\|^2 & \text{if } (\mathbf{e}_1-\mathbf{e}_0)\cdot(\mathbf{p}-\mathbf{e}_0) > \|\mathbf{e}_1-\mathbf{e}_0\|^2, \\ \frac{1}{\|\mathbf{e}_1-\mathbf{e}_0\|^2}\left(\det([\mathbf{p}-\mathbf{e}_0, \mathbf{e}_1-\mathbf{e}_0])\right)^2 & \text{otherwise}, \end{cases}$$



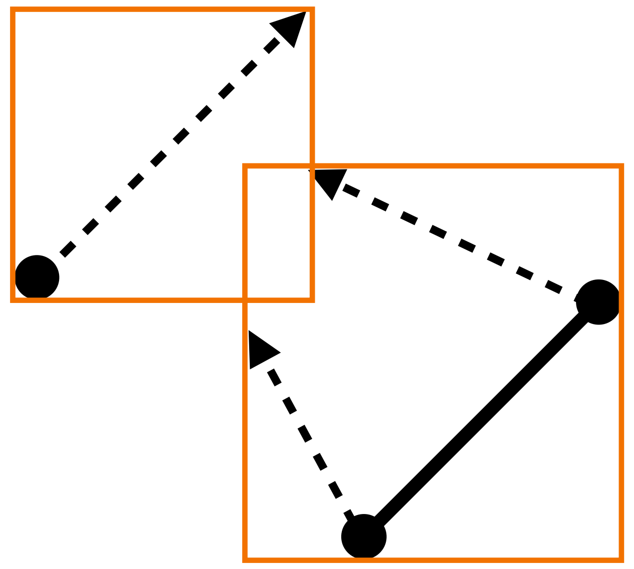

```python
import distance.PointPointDistance as PP
import distance.PointLineDistance as PL

def val(p, e0, e1):
    e = e1 - e0
    ratio = np.dot(e, p - e0) / np.dot(e, e)
    if ratio < 0:    # point(p)-point(e0) expression
        return PP.val(p, e0)
    elif ratio > 1:  # point(p)-point(e1) expression
        return PP.val(p, e1)
    else:            # point(p)-line(e0e1) expression
        return PL.val(p, e0, e1)
```

```python
def grad(p, e0, e1):
    e = e1 - e0
    ratio = np.dot(e, p - e0) / np.dot(e, e)
    if ratio < 0:    # point(p)-point(e0) expression
        g_PP = PP.grad(p, e0)
        return np.reshape([g_PP[0:2], g_PP[2:4], np.array
([0.0, 0.0])], (1, 6))[0]
    elif ratio > 1:  # point(p)-point(e1) expression
        g_PP = PP.grad(p, e1)
        return np.reshape([g_PP[0:2], np.array([0.0, 0.0]),
g_PP[2:4]], (1, 6))[0]
    else:            # point(p)-line(e0e1) expression
        return PL.grad(p, e0, e1)
```

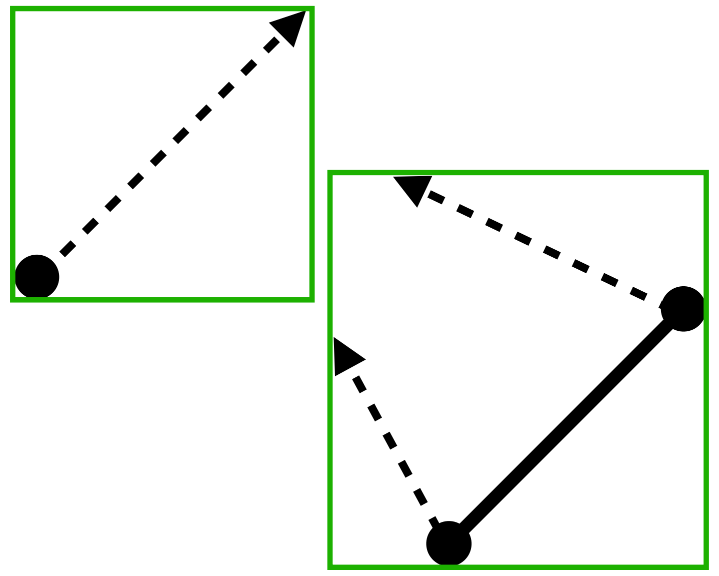
distance/PointEdgeDistance.py

Implementation

Broad Phase Continuous Collision Detection (CCD): Bounding Box Overlap



Case 1: needs narrow phase



Case 2: can skip

```python
    # self-contact
    for xI in bp:
        for eI in be:
            if xI != eI[0] and xI != eI[1]: # do not consider a point and its incident edge
                if CCD.bbox_overlap(x[xI], x[eI[0]], x[eI[1]], p[xI], p[eI[0]], p[eI[1]], alpha):
                    toc = CCD.narrow_phase_CCD(x[xI], x[eI[0]], x[eI[1]], p[xI], p[eI[0]], p[eI[1]], alpha)
                    if alpha > toc:
                        alpha = toc
```

BarrierEnergy.py

```python
from copy import deepcopy
import numpy as np
import math

import distance.PointEdgeDistance as PE

# check whether the bounding box of the trajectory of the point and the edge overlap
def bbox_overlap(p, e0, e1, dp, de0, de1, toc_upperbound):
    max_p = np.maximum(p, p + toc_upperbound * dp) # point trajectory bbox top-right
    min_p = np.minimum(p, p + toc_upperbound * dp) # point trajectory bbox bottom-left
    max_e = np.maximum(np.maximum(e0, e0 + toc_upperbound * de0), np.maximum(e1, e1 + toc_upperbound * de1)) # edge trajectory bbox top-right
    min_e = np.minimum(np.minimum(e0, e0 + toc_upperbound * de0), np.minimum(e1, e1 + toc_upperbound * de1)) # edge trajectory bbox bottom-left
    if np.any(np.greater(min_p, max_e)) or np.any(np.greater(min_e, max_p)):
        return False
    else:
        return True
```

distance/CCD.py

Implementation

Narrow Phase CCD: Additive CCD [Li et al. 2021]

Taking a point-edge pair as an example, the key insight of ACCD is that, given the current positions $\mathbf{p}$, $\mathbf{e}_0$, $\mathbf{e}_1$ and search directions $\mathbf{d}_p$, $\mathbf{d}_{e0}$, $\mathbf{d}_{e1}$, its TOI can be calculated as

$$\alpha_{\text{TOI}} = \frac{\|\mathbf{p} - ((1-\lambda)\mathbf{e}_0 + \lambda\mathbf{e}_1)\|}{\|\mathbf{d}_p - ((1-\lambda)\mathbf{d}_{e0} + \lambda\mathbf{d}_{e1})\|},$$

assuming $(1-\lambda)\mathbf{e}_0 + \lambda\mathbf{e}_1$ is the point on the edge that $\mathbf{p}$ will first collide with. The issue is that we do not a priori know λ. But we can derive a lower bound of α_{TOI} as

$$\begin{aligned}\alpha_{\text{TOI}} &\geq \frac{\min_{\lambda\in[0,1]} \|\mathbf{p} - ((1-\lambda)\mathbf{e}_0 + \lambda\mathbf{e}_1)\|}{\|\mathbf{d}_p\| + \|(1-\lambda)\mathbf{d}_{e0} + \lambda\mathbf{d}_{e1}\|} \\ &\geq \frac{d^{\text{PE}}(\mathbf{p}, \mathbf{e}_0, \mathbf{e}_1)}{\|\mathbf{d}_p\| + \max(\|\mathbf{d}_{e0}\|, \|\mathbf{d}_{e1}\|)} = \alpha_l\end{aligned}$$

Algorithm:

Make a local copy of x
$\alpha \leftarrow 0$
While distance not close enough
    Calculate lower bound α_l
    $x \leftarrow x + \alpha_l p$
    $\alpha \leftarrow \alpha + \alpha_l$
Return α

Only need to evaluate distances;
More robust than root-finding;
Generalize to higher-order primitives.

Implementation

Frictional Self-Contact: Discretization and Approximation

After temporal discretization:

$$\mathbf{T}_F^{n+1}(\mathbf{X}) \approx -\frac{\partial D^{n+1}(\mathbf{X})}{\partial \mathbf{x}^{n+1}(\mathbf{X})} = -\frac{\partial\big(\mu \|\mathbf{T}_C^n(\mathbf{X})\| f_0(\|\bar{\mathbf{V}}_F^{n+1}(\mathbf{X})\hat{h}\|)\big)}{\partial \mathbf{x}^{n+1}(\mathbf{X})}$$

Here $\bar{\mathbf{V}}_F^{n+1}(\mathbf{X}) = (\mathbf{I} - \mathbf{N}^n(\mathbf{X})\mathbf{N}^n(\mathbf{X})^T)(\mathbf{V}^{n+1}(\mathbf{X}) - \mathbf{V}^{n+1}(\mathbf{X}_2))$ is the approximate relative sliding velocity, where $\mathbf{N}^n$ and $\mathbf{X}_2$ are the normal direction and the point in contact with $\mathbf{X}$ in the last time step, $\hat{h}I = (\partial v/\partial x)^{-1}$, and

$$f_0(y) = \begin{cases} -\frac{y^3}{3\epsilon_v^2\hat{h}^2} + \frac{y^2}{\epsilon_v\hat{h}} + \frac{\epsilon_v\hat{h}}{3}, & y \in [0, \epsilon_v\hat{h}); \\ y, & y \geq \epsilon_v\hat{h}. \end{cases}$$

Therefore, considering self-contact, the approximate friction potential over the entire boundary can be written as

$$\int_{\Gamma_C} \frac{1}{2}\mu \|\mathbf{T}_C^n(\mathbf{X})\| f_0(\|\bar{\mathbf{V}}_F^{n+1}(\mathbf{X})\hat{h}\|) ds(\mathbf{X}),$$

Triangulation and smooth approximation to $\max()$:

$$\int_{\Gamma_C} \sum_{e \in \mathcal{E} - I(\mathbf{X})} \frac{1}{2}\mu\Big(-\frac{\partial b(d^{\mathrm{PE}}(\mathbf{x}^n(\mathbf{X}), e), \hat{d})}{\partial d}\Big) f_0(\|\bar{\mathbf{V}}_F^{n+1}(\mathbf{X}, e)\hat{h}\|) ds(\mathbf{X})$$

Implementation

Frictional Self-Contact: Discretization and Approximation (Cont.)

$$\int_{\Gamma_C} \sum_{e \in \mathcal{E} - I(\mathbf{X})} \frac{1}{2}\mu\Big(-\frac{\partial b(d^{\mathrm{PE}}(\mathbf{x}^n(\mathbf{X}), e), \hat{d})}{\partial d}\Big) f_0(\|\bar{\mathbf{V}}_F^{n+1}(\mathbf{X}, e)\hat{h}\|) ds(\mathbf{X})$$

$$\begin{aligned} P_f(x) &= \sum_{\hat{a}} A_{\hat{a}} \sum_{e \in \mathcal{E} - I(\mathbf{X}_{\hat{a}})} \frac{1}{2}\mu\Big(-\frac{\partial b(d^{\mathrm{PE}}(\mathbf{x}_{\hat{a}}^n, e), \hat{d})}{\partial d}\Big) f_0(\|\bar{\mathbf{V}}_F^{n+1}(\mathbf{X}_{\hat{a}}, e)\hat{h}\|) \\ &= \sum_{k \in \{(\hat{a}, e)\}} \mu \lambda_k^n f_0(\|\bar{\mathbf{v}}_k \hat{h}\|) \end{aligned}$$

where $A_{\hat{a}} = \frac{\|\mathbf{X}_{\hat{a}} - \mathbf{X}_{\hat{a}-1}\| + \|\mathbf{X}_{\hat{a}} - \mathbf{X}_{\hat{a}+1}\|}{2}$ is the integration weight. As we denote $\bar{\mathbf{v}}_k = \bar{\mathbf{V}}_F^{n+1}(\mathbf{X}_{\hat{a}}, e)$ and $\lambda_k^n = \frac{1}{2} A_{\hat{a}} (-\frac{\partial b(d^{\mathrm{PE}}(\mathbf{x}_{\hat{a}}^n, e), \hat{d})}{\partial d})$

Implementation

Frictional Self-Contact: Precomputing Normal Force Magnitude

$$\lambda_{\hat{a},e}^{n} = \frac{1}{2}A_{\hat{a}}\left(-\frac{\partial b(d_{\text{sq}}^{\text{PE}}(\mathbf{x}_{\hat{a}}^{n}, e), \hat{d}^2)}{\partial d^{\text{PE}}}\right) = \frac{1}{2}A_{\hat{a}}\left(-\frac{\partial b(d_{\text{sq}}^{\text{PE}}(\mathbf{x}_{\hat{a}}^{n}, e), \hat{d}^2)}{\partial d_{\text{sq}}^{\text{PE}}}\frac{\partial d_{\text{sq}}^{\text{PE}}}{\partial d^{\text{PE}}}\right)$$

$$= \frac{1}{2}A_{\hat{a}}\left(-\frac{\partial b(d_{\text{sq}}^{\text{PE}}(\mathbf{x}_{\hat{a}}^{n}, e), \hat{d}^2)}{\partial d_{\text{sq}}^{\text{PE}}}\right)2d^{\text{PE}}.$$

where $$\frac{\partial b(d_{\text{sq}}, \hat{d}^2)}{\partial d_{\text{sq}}} = \begin{cases} \frac{\kappa}{8}\hat{d}\left(\frac{1}{\hat{d}^2}\ln\frac{d_{\text{sq}}}{\hat{d}^2} + \frac{1}{d_{\text{sq}}}\left(\frac{d_{\text{sq}}}{\hat{d}^2} - 1\right)\right) & d < \hat{d}; \\ 0 & d \geq \hat{d}. \end{cases}$$

The set of boundary element pairs for semi-implicit friction are those with $d^{\text{PE}}(\mathbf{x}_{\hat{a}}^{n}, e) < \hat{d}$.

This set does not change per time step.

Implementation

Frictional Self-Contact: Gradient and Hessian Computation

$$\mathbf{v}_k = (\mathbf{I} - \mathbf{n}\mathbf{n}^T)\big(\mathbf{v}_p - ((1-r)\mathbf{v}_{e_0} + r\mathbf{v}_{e_1})\big)$$

$$\bar{\mathbf{v}}_k = (\mathbf{I} - \mathbf{n}^n(\mathbf{n}^n)^T)\big(\mathbf{v}_p - ((1-r^n)\mathbf{v}_{e_0} + r^n\mathbf{v}_{e_1})\big)$$

denote $\hat{\mathbf{v}}_k = \mathbf{v}_p - ((1 - r^n)\mathbf{v}_{e_0} + r^n\mathbf{v}_{e_1})$ **as the local relative velocity**

$$\frac{\partial \bar{\mathbf{v}}_k}{\partial \hat{\mathbf{v}}_k} = (\mathbf{I} - \mathbf{n}^n(\mathbf{n}^n)^T) \quad \text{and} \quad \frac{\partial \hat{\mathbf{v}}_k}{\partial[\mathbf{x}_p^T, \mathbf{x}_{e_0}^T, \mathbf{x}_{e_1}^T]^T} = \frac{1}{\hat{h}}\begin{bmatrix}\mathbf{I} & (r^n - 1)\mathbf{I} & -r^n\mathbf{I}\end{bmatrix}$$

$$\nabla P_f(x) = \sum_k \Big(\frac{\partial \hat{\mathbf{v}}_k}{\partial x}\Big)^T \boxed{\frac{\partial D_k(x)}{\partial \hat{\mathbf{v}}_k}}, \quad \nabla^2 P_f(x) = \sum_k \Big(\frac{\partial \hat{\mathbf{v}}_k}{\partial x}\Big)^T \boxed{\frac{\partial^2 D_k(x)}{\partial \hat{\mathbf{v}}_k^2}} \frac{\partial \hat{\mathbf{v}}_k}{\partial x}$$

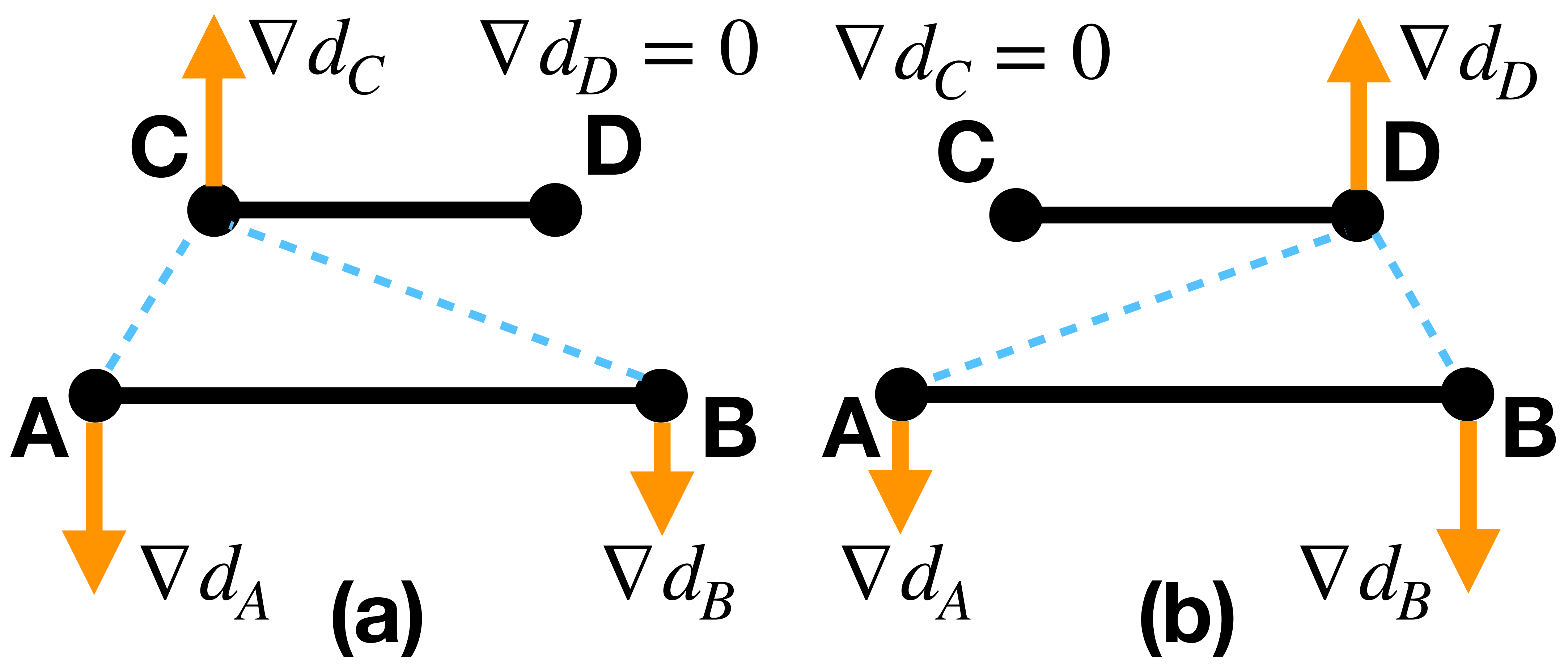
we've implemented them for (non-sliding)obstacle-solid friction

Demo!

github.com/liminchen/solid-sim-tutorial /7_self_contact, /8_self_friction

3D Frictional Self-Contact

- Point-Edge distance -> Point-Triangle distance
- Edge-edge quadratures necessary for low resolution
- Edge-edge distance is only C^0-continuous: needs smooth approximation
- Spatial data structures:
  - Spatial Hash
  - Bounding Box Hierarchy (BVH)
  - …



Milestone

- Mass-spring solids with boundary effects
- Finished introducing the simulation of **full-order elastic bodies**
  - Strain and Stress
  - Inversion-free elastodynamics
  - Strong form -> weak form -> discretization
  - Frictional self-contact

Next Lecture: Reduced-Order Modeling

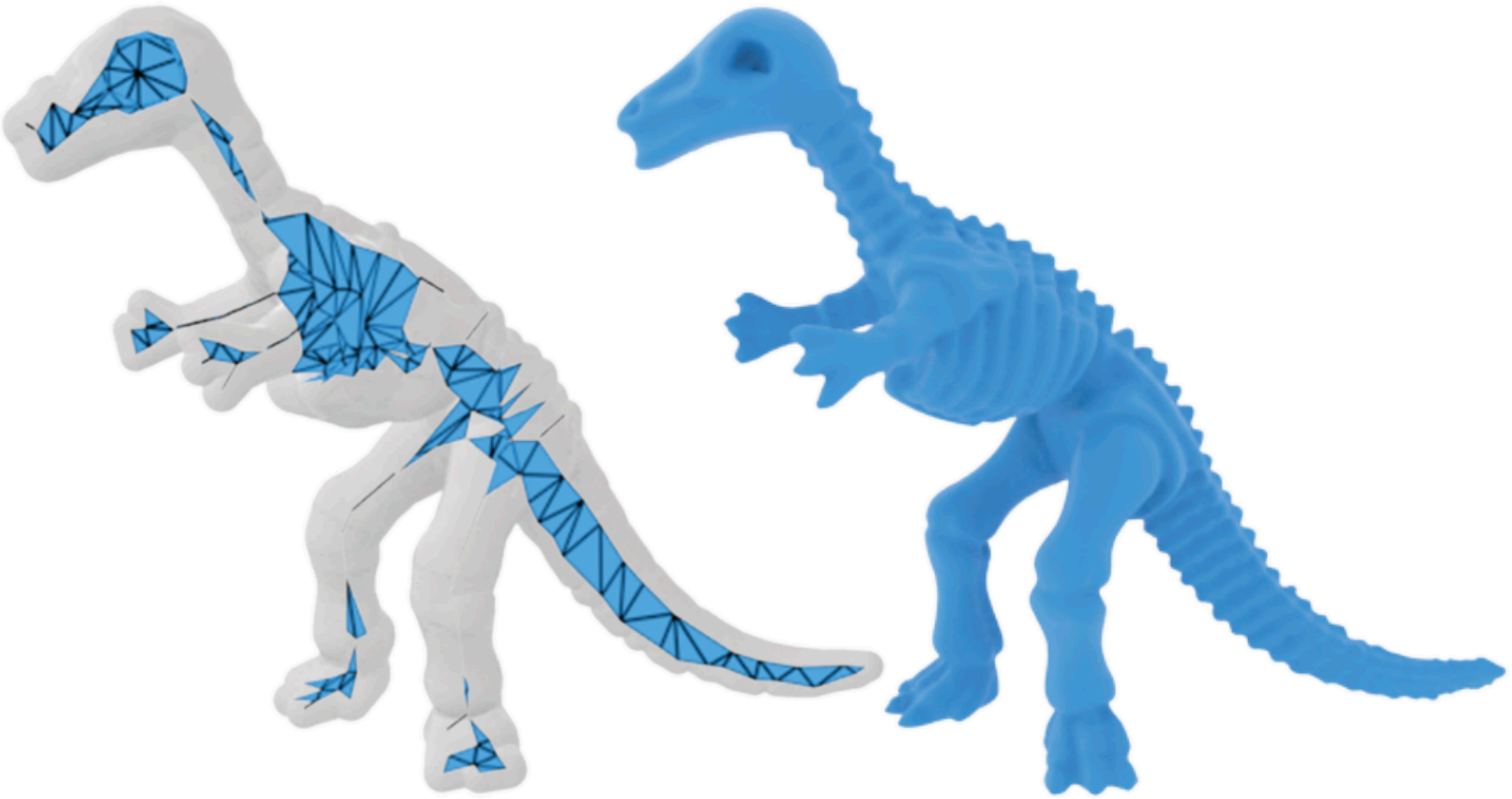


Image Sources