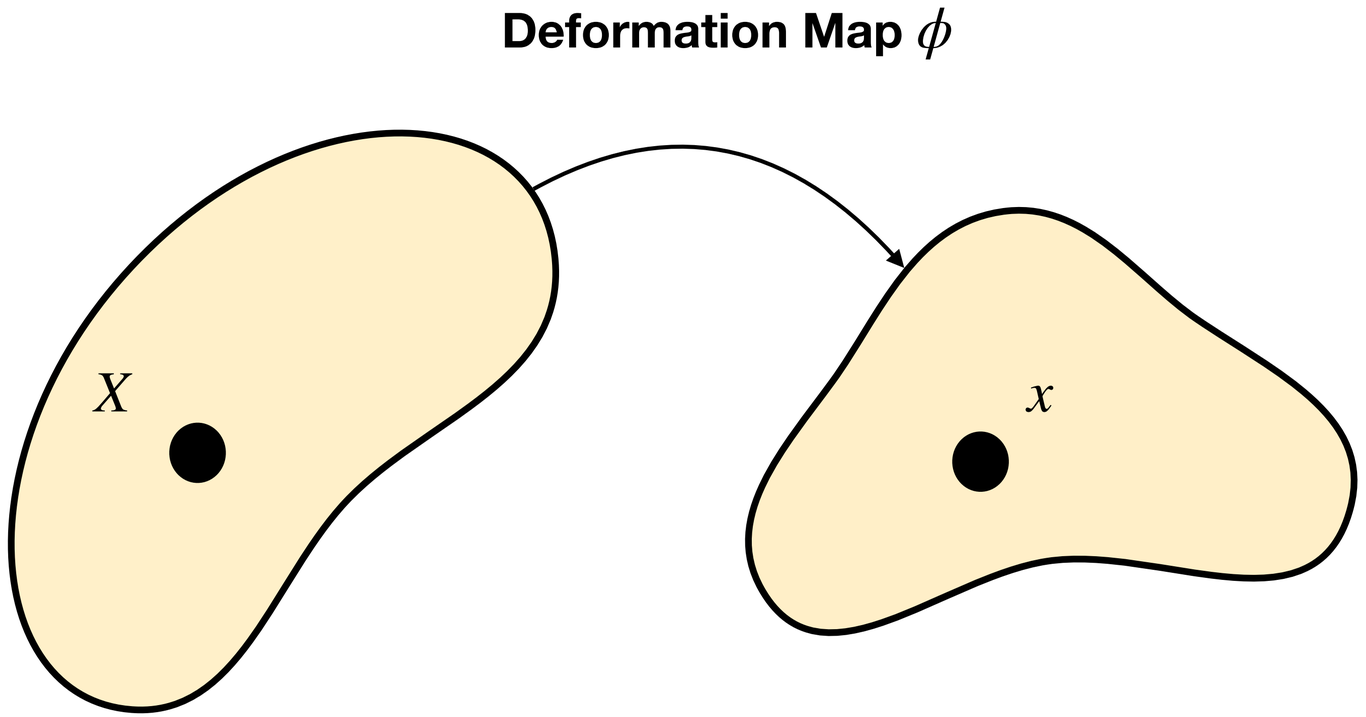
Instructor: Minchen Li



Lec 8: Strain Energy

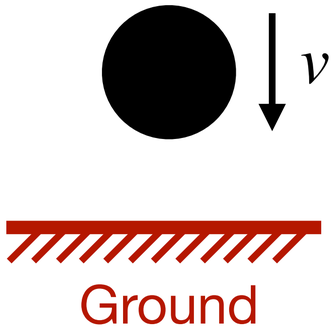
15-769: Physically-based Animation of Solids and Fluids (F23)

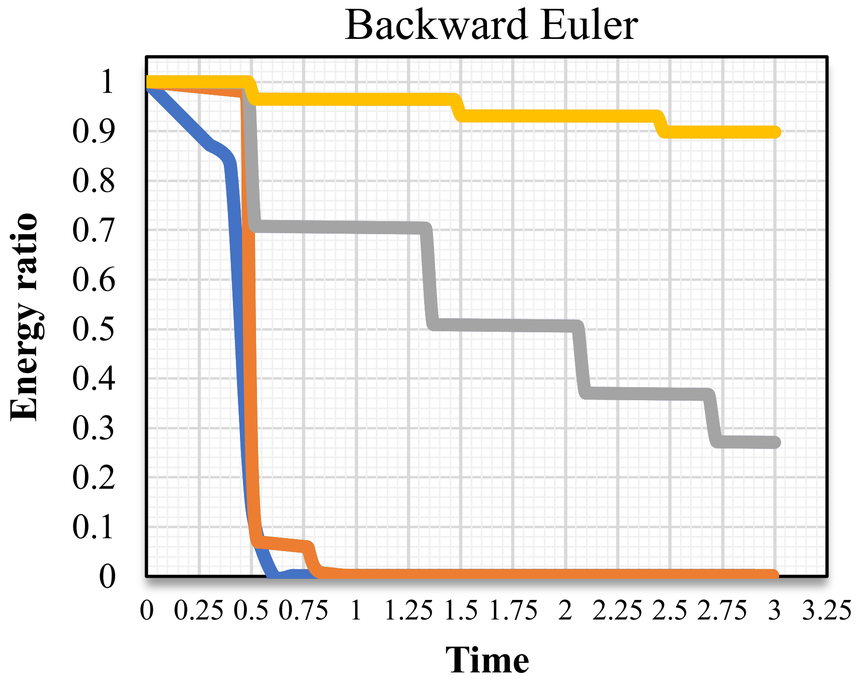
Recap: Restitution

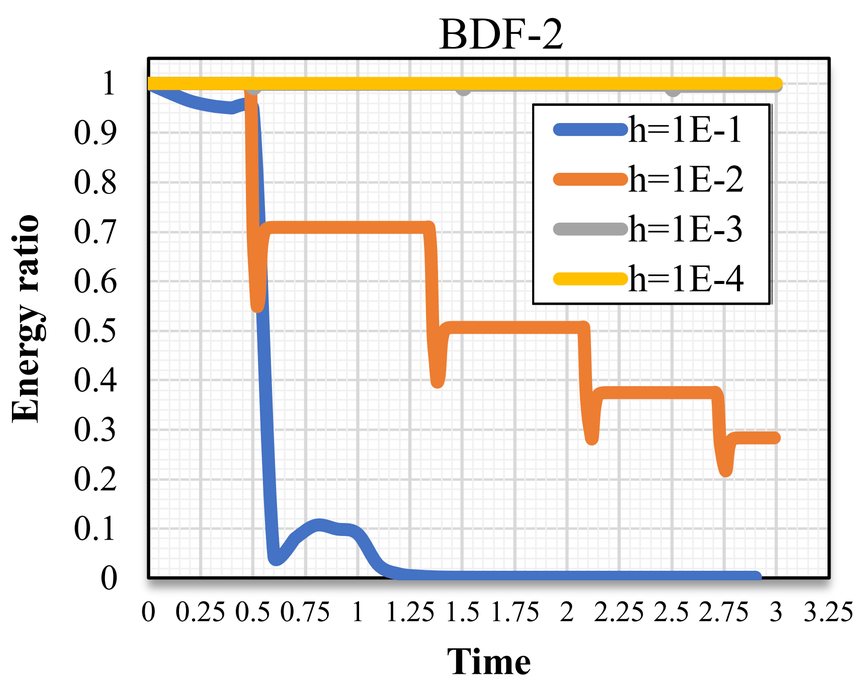
Approximating normal contact as a conservative force with potential energy $P_b(x)$,

Restitution ⇔ Energy Conservation of Time Stepping

e.g.







Use high-order rules, e.g. BDF-2: $\min_{x^{n+1}} \frac{1}{2}\|x^{n+1} - \tilde{x}^n_{BDF2}\|^2_M + \frac{4}{9}h^2 \sum P(x^{n+1})$

BDF-2 for $y' = f(t, y)$:

$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}hf(t_{n+2}, y_{n+2})$

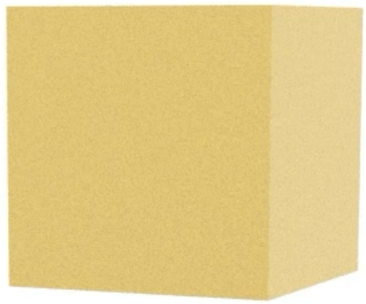
Yunuo Chen*, Minchen Li*, Lei Lan, Hao Su, Yin Yang, Chenfanfu Jiang. A Unified Newton Barrier Method for Multibody Dynamics. ACM ToG (SIGGRAPH), 2022.

Recap: Restitution

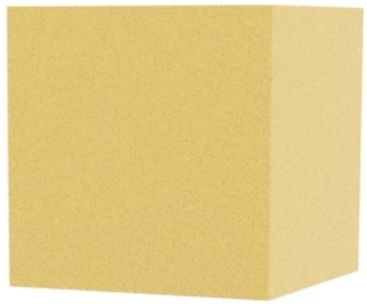
Use Damping energy to control restitution:

$$\frac{k_d}{2} v^T \frac{\partial^2 P_b}{\partial x^2}(x^n) v$$

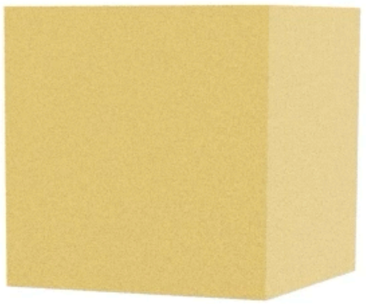
$$\min_{x^{n+1}} \frac{1}{2} \| x^{n+1} - \tilde{x}^n_{BDF2} \|^2_M + \frac{4}{9} h^2 \sum P(x^{n+1})$$



k_d=0



k_d=0.005



k_d=0.1

Yunuo Chen*, Minchen Li*, Lei Lan, Hao Su, Yin Yang, Chenfanfu Jiang. A Unified Newton Barrier Method for Multibody Dynamics. ACM ToG (SIGGRAPH), 2022.

Recap: Moving Boundary Conditions

$$\min_x \frac{1}{2}\|x-(x^n+hv^n)\|_M^2 + h^2\sum P(x) \quad \textbf{s.t.} \quad A^{n+1}x = b^{n+1}$$

$$\min_x \frac{1}{2}\|x-(x^n+hv^n)\|_M^2 + h^2\sum P(x) + \frac{\kappa_M}{2}\|A^{n+1}x-b^{n+1}\|_W^2$$

Algorithm:

Initialize κ_M

While solution not accurate enough

$\min_x E(x) + P_M(x,\kappa_M)$

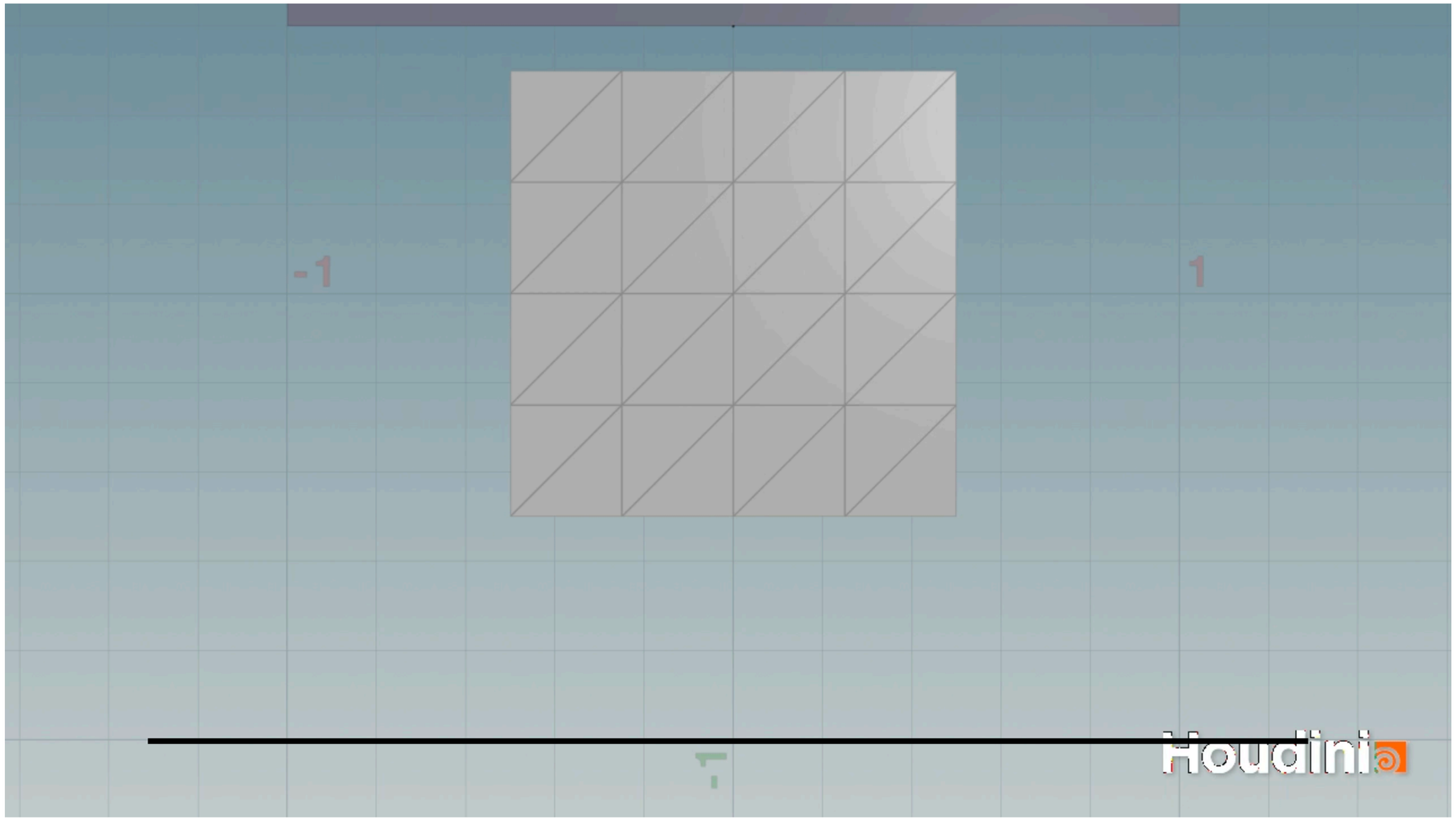
$\kappa_M \leftarrow 2\kappa_M$

use Augmented Lagrangian for high accuracy:

$$\max_\lambda \min_x E(x) + \frac{\kappa_M}{2}\|A^{n+1}x-b^{n+1}\|_W^2 + \lambda^T\sqrt{W}(A^{n+1}x-b^{n+1})$$

For both methods, can switch to DOF elimination when accurate enough.

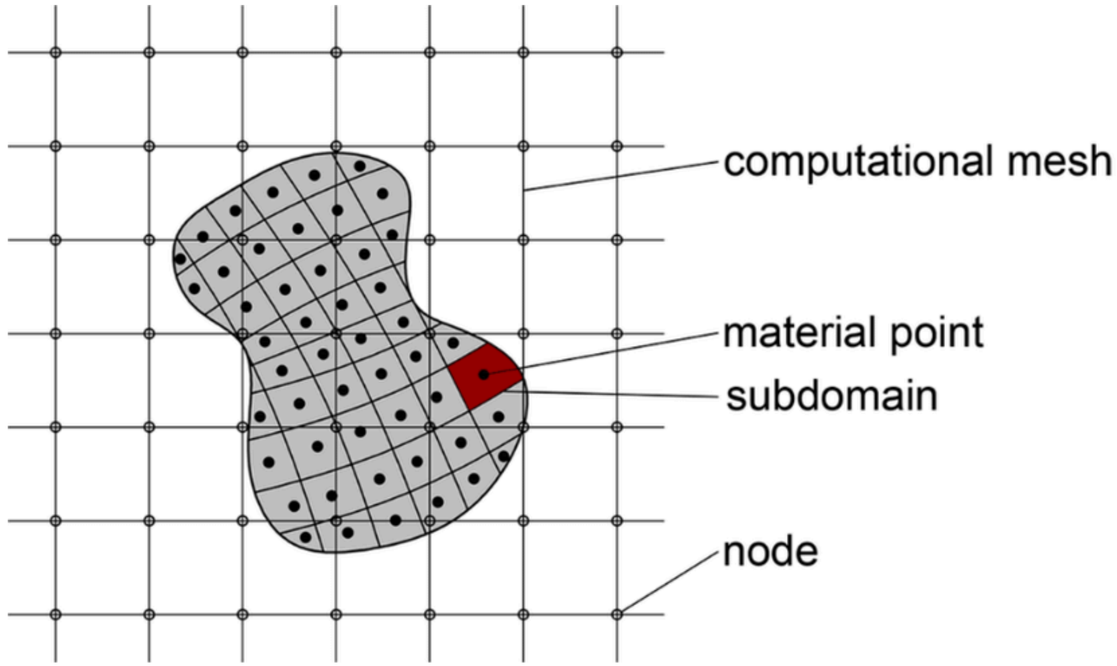
Inversion Issue



More rigorous elasticity formulation could help!

Continuum Assumption

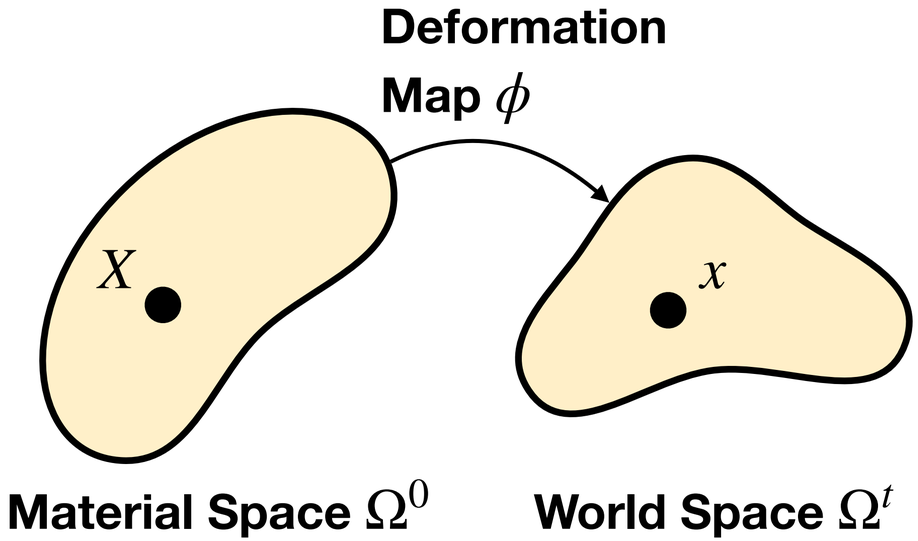
- Treating materials (solid, liquid, or gas) as **continuous pieces of matter**
  - Not modeling microscopic interactions between molecules and atoms
- Particles in MPM or SPH are not molecules or atoms
  - They are **markers** of a continuous piece of the material
- Define quantities (e.g. density $\rho(X,t)$, velocity $V(X,t)$, etc) as continuous functions of position $X \in \mathbb{R}^2$



Continuum Motion

- **Kinematics**: the study of motion occurred in continuum materials
- Our focus: change of shape, or **deformation**

Definition (Deformation/Flow Map)**.** We consider the motion of material to be determined by a mapping $\phi(\cdot, t) : \Omega^0 \to \Omega^t$ for $\Omega^0, \Omega^t \subset \mathbb{R}^d$ where $d = 2$ or 3 is the dimension of the simulated problem (or domain). The mapping ϕ is sometimes called the flow map or the deformation map.



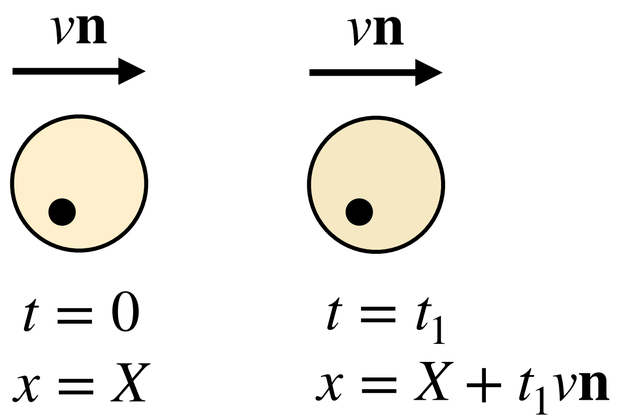
$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) = \phi(\mathbf{X}, t)$$

$$x(X,\, 0) = X$$

Continuum Motion — Example

Example If our object is moving with a constant speed v along direction $\mathbf{n}$, then we have

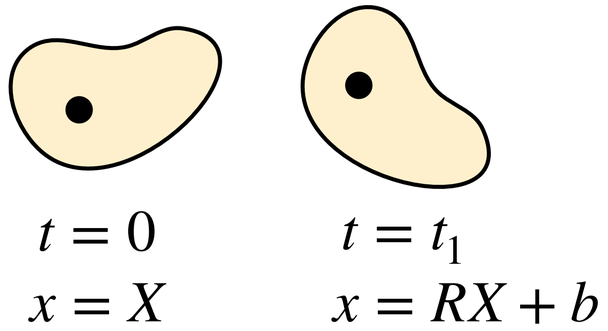
$$\mathbf{x} = \mathbf{X} + tv\mathbf{n}.$$



If an object went through some rigid motion after time t (compared to time 0), we will have

$$\mathbf{x} = \mathbf{R}\mathbf{X} + \boldsymbol{b},$$

where $\mathbf{R}$ is a rotation matrix, $\boldsymbol{b}$ is some translation. $\mathbf{R}$ and $\boldsymbol{b}$ will probably be some function with respect to time t and initial position $\mathbf{X}$, depending on the actual motion. □



Velocity and Acceleration

Lagrangian View v.s. Eulerian View

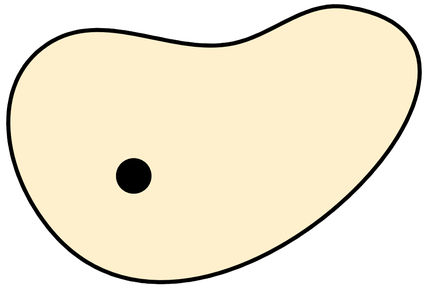
$$\mathbf{V}(\mathbf{X}, t) = \frac{\partial \phi}{\partial t}(\mathbf{X}, t)$$

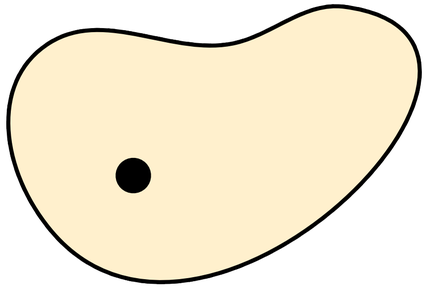
$$\mathbf{A}(\mathbf{X}, t) = \frac{\partial^2 \phi}{\partial t^2}(\mathbf{X}, t) = \frac{\partial \mathbf{V}}{\partial t}(\mathbf{X}, t)$$

$$\mathbf{V}(\cdot, t) : \Omega^0 \to \mathbb{R}^d \text{ and } \mathbf{A}(\cdot, t) : \Omega^0 \to \mathbb{R}^d$$

Remark The velocity $\mathbf{V}$ and acceleration $\mathbf{A}$ defined above are based on the "Lagrangian view", where they are functions of the material configuration $\mathbf{X}$ and time t. Physically, this means we are measuring them on a specific particle. This particle has its mass and occupies some volume since the beginning. □

Lagrangian view:
Quantity measured
at a point on the solid





Eulerian view:
Quantity measured
at a point in space

Deformation Gradient

Definition (Deformation Gradient)**.** The Jacobian of the deformation map ϕ is called the deformation gradient. The physics of elasticity is naturally described in terms of this Jacobian. It is standard notation to use $\mathbf{F}$ to refer to the Jacobian of the deformation mapping

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial \phi}{\partial \mathbf{X}}(\mathbf{X}, t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}(\mathbf{X}, t).$$

Discretely it is often a small 2×2 or 3×3 matrix. One special case is for a cloth/thin shell in 3D, $\mathbf{F}$ is 3×2 because the material space is really just 2D. It can be thought of as $\mathbf{F}(\cdot, t) : \Omega^0 \to \mathbb{R}^{d \times d}$. In other words, for every material point $\mathbf{X}$, $\mathbf{F}(\mathbf{X}, t)$ is the $\mathbb{R}^{d \times d}$ matrix describing the deformation Jacobian of the material at time t. We can also use the index notation

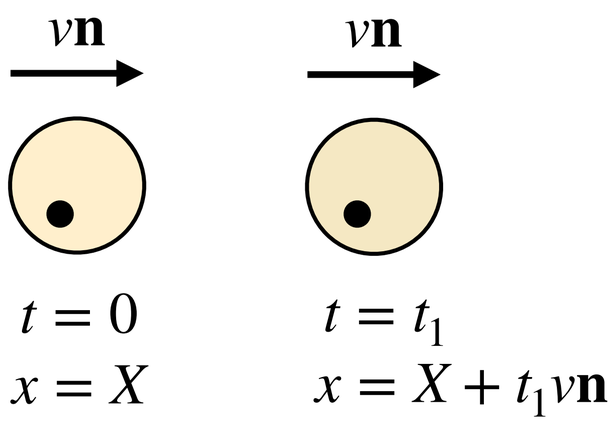
$$F_{ij} = \frac{\partial \phi_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j}, \quad i, j = 1, \ldots, d.$$

Deformation Gradient — Example

Example If our object is moving with a constant speed v along direction $\mathbf{n}$, then we have

$$\mathbf{x} = \mathbf{X} + tv\mathbf{n}.$$

$F = I$

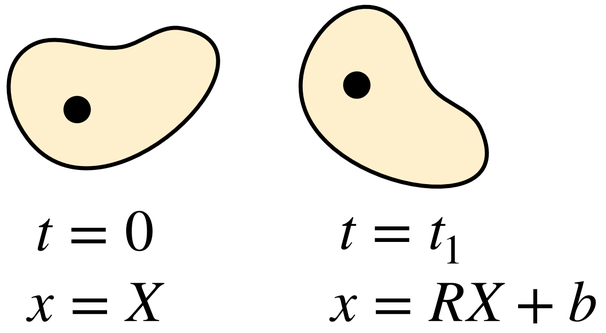


If an object went through some rigid motion after time t (compared to time 0), we will have

$$\mathbf{x} = \mathbf{R}\mathbf{X} + \boldsymbol{b},$$

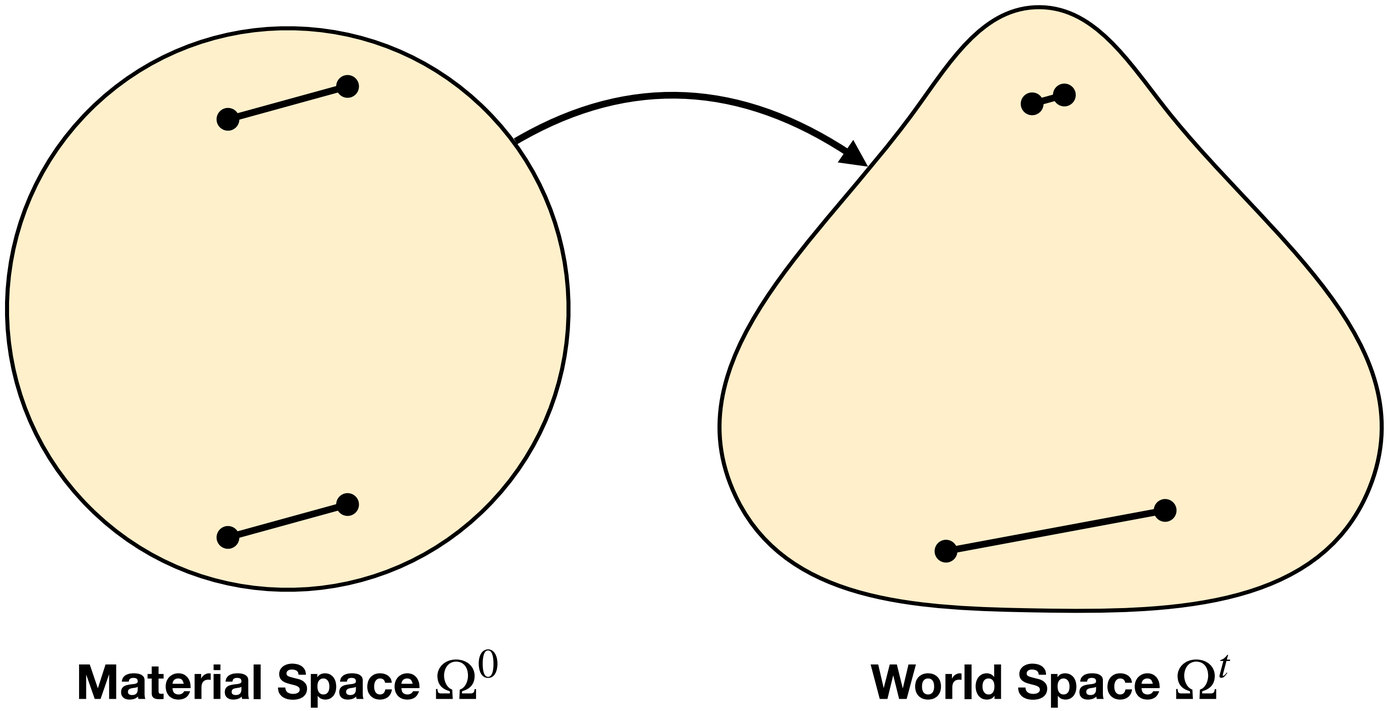
$F = R$

where $\mathbf{R}$ is a rotation matrix, $\boldsymbol{b}$ is some translation. $\mathbf{R}$ and $\boldsymbol{b}$ will probably be some function with respect to time t and initial position $\mathbf{X}$, depending on the actual motion. □



Deformation Gradient — Example

- the deformation gradient represents how deformed a material is **locally**



Volume Change

- $J = \det(F)$ characterizes infinitesimal volume change
  - the ratio between the infinitesimal volumes of material in Ω^t and Ω^0
- F is a **rotation matrix** iff $F^{-1} = F^T$ and $J = 1$.
- J > 1 means volume increase, J < 1 means volume decrease
- J = 0 means the volume becomes zero: **degenerate** to a line or a plane (3D)
- J < 0 means the material is **inverted**

Strain Energy

- Measure elastic potential locally for each point (based on F), and then integrate them over the whole domain
- F is also called **strain**
- The elastic potential P_e is also called **strain energy**

integrated from **strain energy density functions** $\Psi(\mathbf{F}) : \mathbb{R}^{d\times d} \to \mathbb{R}$ at each material point in the solid domain:

$$P_e = \int_{\Omega_0} \Psi(\mathbf{F}) d\mathbf{X}.$$

Strain Energy

Rigid Null Space

- Observation: for solids undergoing only translational and/or rotational motions, P_e remains 0

- — Any strain energy density functions $\Psi(F)$ have a **rigid null space:**

$$\Psi(\mathbf{F}) = 0 \quad \forall\, \mathbf{F} = \mathbf{R}$$

a square matrix $\mathbf{F}$ is a rotation matrix if and only if

$$\mathbf{F}^T = \mathbf{F}^{-1} \quad \text{and} \quad J \equiv \det(\mathbf{F}) = 1.$$

Strain Energy

A Straightforward Formulation

- Quadratically penalize any deviation of F from being a rotation matrix:

$$\Psi(\mathbf{F}) = \frac{\mu}{4}\|\mathbf{F}^T\mathbf{F} - \mathbf{I}\|_{\mathrm{F}}^2 + \frac{\lambda}{2}(J-1)^2$$

  - μ and λ are the stiffness parameters

a square matrix $\mathbf{F}$ is a rotation matrix if and only if

$$\mathbf{F}^T = \mathbf{F}^{-1} \quad \text{and} \quad J \equiv \det(\mathbf{F}) = 1.$$

Strain Energy

The neo-Hookean Model

$$\Psi_{\mathrm{NH}}(\mathbf{F}) = \frac{\mu}{2}\left(\mathrm{tr}(\mathbf{F}^T\mathbf{F}) - d\right) - \mu \ln(J) + \frac{\lambda}{2}\ln^2(J)$$

Taking the derivative of $\Psi_{\mathrm{NH}}(\mathbf{F})$ w.r.t. $\mathbf{F}$, we obtain

$$\frac{\partial \Psi}{\partial \mathbf{F}}(\mathbf{F}) = \mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda \ln(J)\mathbf{F}^{-T}.$$

- μ-term is minimized if $F = F^{-T}$, λ-term is minimized if J = 1
- Barrier term on J, so **inversion-free!**

Strain Energy

Lame Parameters

Definition (Lame Parameters)**.** In standard strain energy density functions, the stiffness parameters μ and λ are called Lame parameters. They are related to the **Young's modulus** E (a measure of resistance to stretch) and **Poisson ratio** ν (a measure of incompressibility) of the solid:

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

Material	Youngs Modulus /GPa
Mild Steel	210
Copper	120
Bone	18
Plastic	2
Rubber	0.02

Material	*Poisson's ratio*
rubber	≈0.5
gold	0.42
saturated clay	0.40–0.50
magnesium	0.35
titanium	0.34
copper	0.33
aluminium-alloy	0.33
clay	0.30–0.45

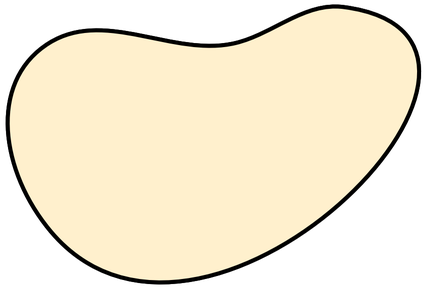
Strain Energy

Rotation Invariance

Definition (Rotation Invariance)**.** The energy density function of any nonlinear elastic models is rotation invariant. Mathematically speaking,

$$\Psi(\mathbf{F}) = \Psi(\mathbf{RF}) \quad \forall\, \mathbf{F} \in \mathbb{R}^{d\times d} \text{ and } d \times d \text{ rotation matrix } \mathbf{R}.$$

Intuitively, any rotations after deformation should not further change the strain energy density function value.

$\Psi($  $) = \Psi($ $)$

Strain Energy

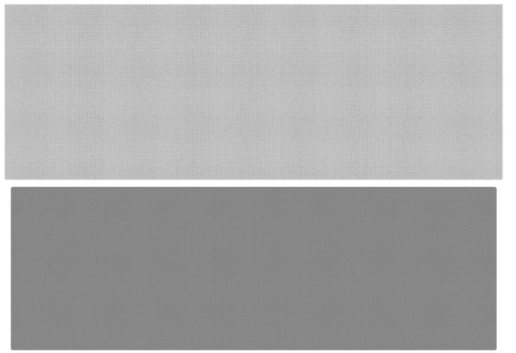
Linear Elasticity

Definition (Linear Elasticity)**.** Linear elasticity has the energy density function

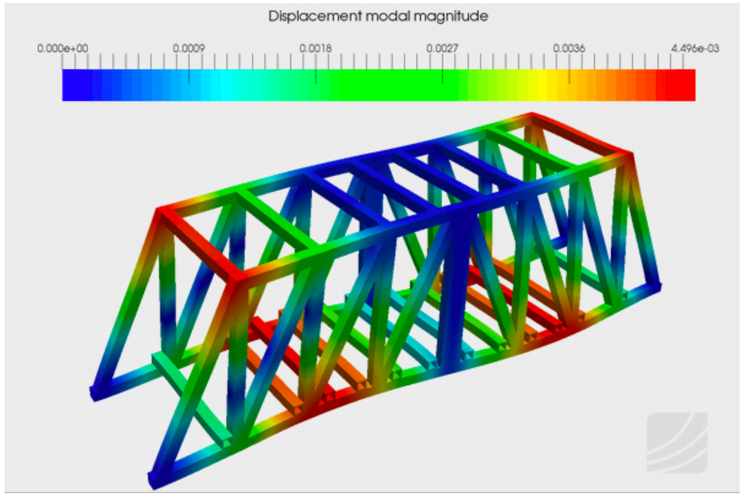
$$\Psi_{\text{lin}}(\mathbf{F}) = \mu\|\boldsymbol{\epsilon}\|_{\text{F}}^2 + \frac{\lambda}{2}\text{tr}^2(\boldsymbol{\epsilon}).$$

Here $\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T) - \mathbf{I}$ is the **small strain tensor**, and we see that $\Psi_{\text{lin}}(\mathbf{F})$ is a quadratic function of $\mathbf{F}$.

Topology Optimization



- No rigid null space, nor rotation invariant
- specially designed for infinitesimal strains
- calibrated to real-world experiments under small deformations



Structural Analysis

Strain Energy

Isotropic and Anisotropic Elasticity

Definition (Isotropic Elasticity)**.** The energy density function of isotropic elastic models satisfies

$$\Psi(\mathbf{F}) = \Psi(\mathbf{FR}) \quad \forall\, \mathbf{F} \in \mathbb{R}^{d\times d} \text{ and } d \times d \text{ rotation matrix } \mathbf{R}.$$

Intuitively, the same amount of stretch in any direction will result in the same energy change. There is no special directions that the material is harder or easier to be deformed than others.

Rotation Invariance:

$$\Psi(\mathbf{F}) = \Psi(\mathbf{RF})$$

- neo-Hookean and our intuitive model are both isotropic.
- linear elasticity is not, as it is not designed for rotational motions
- For anisotropic elastic models, the stretch resistance differs in different directions
  - e.g. cloth, bones, tissues, woods, etc

Polar Singular Value Decomposition

Unlike standard SVD that always keep $\mathbf{\Sigma}_{ii}$ non-negative by allowing $\det(\mathbf{U}) = -1$ and $\det(\mathbf{V}) = -1$, Polar SVD always keep $\det(\mathbf{U}) = 1$ and $\det(\mathbf{V}) = 1$ while allowing $\mathbf{\Sigma}_{ii}$ to be negative. It is called "Polar SVD" mainly because the Polar decomposition $\mathbf{F} = \mathbf{R}\boldsymbol{S}$ can be reconstructed via $\mathbf{R} = \mathbf{U}\mathbf{V}^T$ and $\boldsymbol{S} = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^T$, where $\mathbf{R}$ is the closest rotation to $\mathbf{F}$ and $\boldsymbol{S}$ is symmetric.

Algorithm 6: Polar SVD from Standard SVD

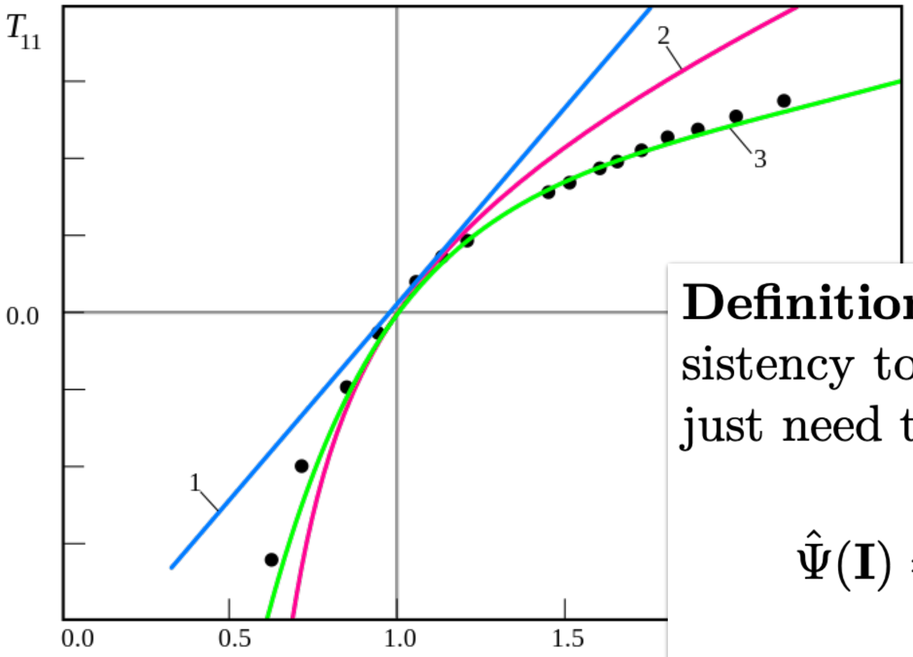
Result: $\mathbf{U}$, $\mathbf{\Sigma}$, $\mathbf{V}$

$(\mathbf{U}, \mathbf{\Sigma}\ \mathbf{V}) \leftarrow \text{StandardSVD}(\mathbf{F})$;
if $\det(\mathbf{U}) < 0$ **then**
    $\mathbf{U}(:, d) \leftarrow -\mathbf{U}(:, d)$;
    $\mathbf{\Sigma}_{dd} \leftarrow -\mathbf{\Sigma}_{dd}$;
if $\det(\mathbf{V}) < 0$ **then**
    $\mathbf{V}(:, d) \leftarrow -\mathbf{V}(:, d)$;
    $\mathbf{\Sigma}_{dd} \leftarrow -\mathbf{\Sigma}_{dd}$;

$$\Psi_{\text{NH}}(\mathbf{F}) = \hat{\Psi}_{\text{NH}}(\mathbf{\Sigma}) = \frac{\mu}{2}(\sum_{i}^{d} \sigma_i^2 - d) - \mu \ln(J) + \frac{\lambda}{2} \ln^2(J)$$

Consistency to Linear Elasticity

- Experimental results (dots) and predictions for Linear elasticity (1), neo-Hookean (2) and Mooney-Rivlin (3) models:



Definition (Consistency to Linear Elasticity). To verify the consistency to linear elasticity of a strain energy density function $\Psi(\mathbf{F})$, we just need to check whether the following relations all hold:

$$\hat{\Psi}(\mathbf{I}) = 0, \quad \frac{\partial \hat{\Psi}}{\partial \sigma_i}(\mathbf{I}) = 0, \quad \text{and} \quad \frac{\partial^2 \hat{\Psi}}{\partial \sigma_i \partial \sigma_j}(\mathbf{I}) = 2\mu\delta_{ij} + \lambda.$$

Here $1 \leq i, j \leq d$, and $\delta_{ij} = 1$ if $i = j$, otherwise it is 0.

Simplified Models

Corotated Linear Elasticity

Definition 13.7 (Corotated Linear Elasticity)**.** To make linear elasticity rotation-aware while keeping its simplicity, we can introduce a base rotation $\mathbf{R}^n$ and construct an energy density function

$$\Psi_{\mathrm{LC}}(\mathbf{F}) = \Psi_{\mathrm{lin}}((\mathbf{R}^n)^T\mathbf{F}), \tag{13.15}$$

penalizing any deviation between $\mathbf{F}$ and this fixed $\mathbf{R}^n$. This is called corotated linear elasticity. □

not rotation invariant, not suitable for large deformation (there can be artificially stiffened behaviors)

Calculating R:

$$\mathbf{R}^n = \arg\min_{\mathbf{R}} \|\mathbf{F}^n - \mathbf{R}\|_{\mathrm{F}}^2 \quad s.t. \quad \mathbf{R}^T = \mathbf{R}^{-1} \quad \text{and} \quad \det(\mathbf{R}) = 1. \tag{13.16}$$

As mentioned earlier, the solution is given by the Polar decomposition on $\mathbf{F}^n$, and with Polar SVD $\mathbf{F}^n = \mathbf{U}^n\mathbf{\Sigma}^n(\mathbf{V}^n)^T$, we have $\mathbf{R}^n = \mathbf{U}^n(\mathbf{V}^n)^T$.

Simplified Models

As-Rigid-As-Possible (ARAP)

$$\Psi_{\mathrm{ARAP}}(\mathbf{F}) = \mu \sum_{i}^{a} (\sigma_i - 1)^2$$

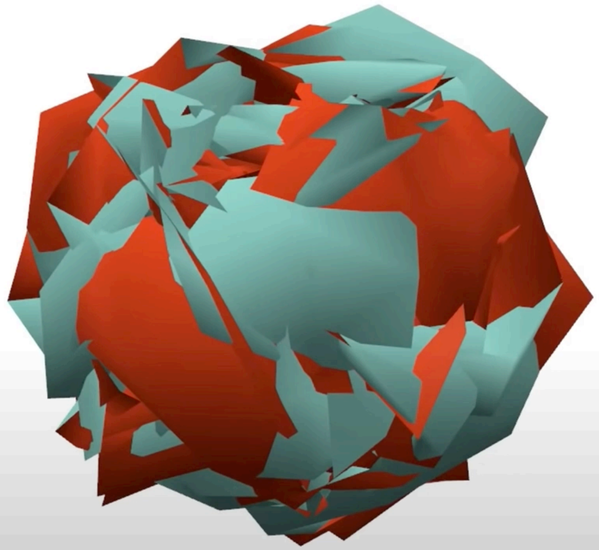
or $\mu \|F - R\|^2$

- Simple and efficient to calculate
- Often used in animation, shape modeling, surface parameterization

Strain Energy

Invertibility

- Definition: Strain energy density functions allowing $\det(F) \leq 0$
  - No line search filtering needed
  - Can deal with inverted configurations
- Usually smoother — easier to optimize
- e.g. Stable neo-Hookean
  - [Smith et al. 2018]



Bunny with randomized vertices

Next Lecture: Stress and Its Derivatives

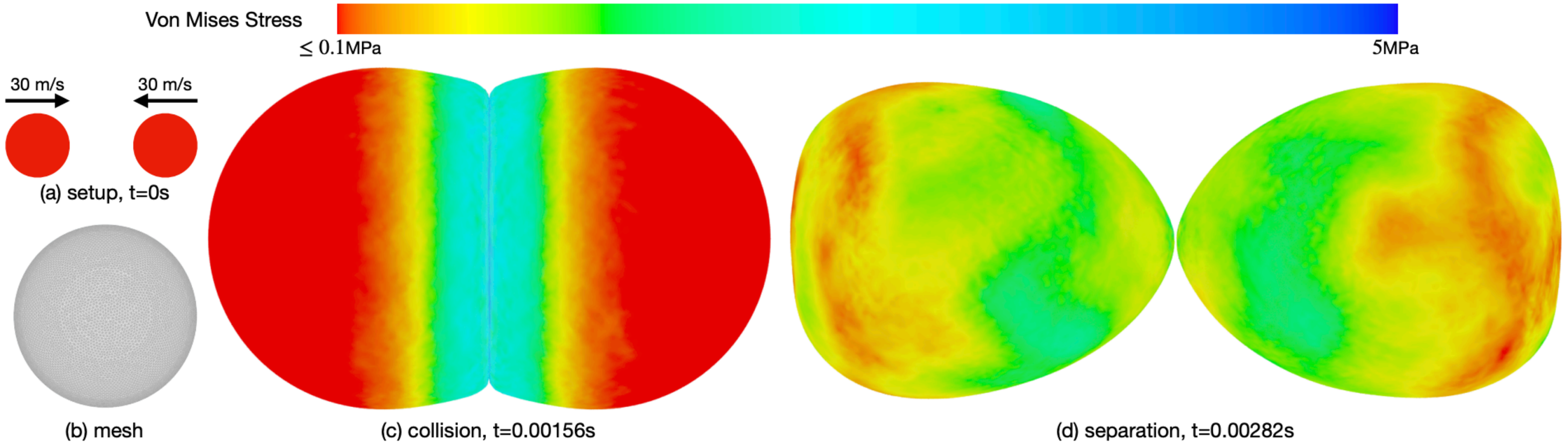


Image Sources

- https://www.math.ucla.edu/~cffjiang/research/mpmcourse/mpmcourse.pdf
- https://en.wikipedia.org/wiki/Neo-Hookean_solid
- https://www.researchgate.net/publication/334433486_Analysis_of_the_Slope_Response_to_an_Increase_in_Pore_Water_Pressure_Using_the_Material_Point_Method/figures?lo=1
- http://physicsnet.co.uk/a-level-physics-as-a2/materials/young-modulus/
- https://padeepz.net/ce6602-syllabus-structural-analysis-2-regulation-2013-anna-university/
- https://www.youtube.com/watch?v=WV-J7u9aoHk