Bayesian games and their use in auctions

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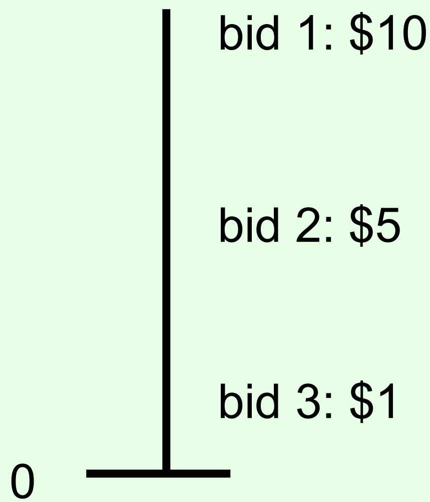
What is mechanism design?

- In mechanism design, we get to design the game (or mechanism)
  - e.g. the rules of the auction, marketplace, election, …
- Goal is to obtain good outcomes when agents behave strategically (game-theoretically)
- Mechanism design often considered part of game theory
- 2007 Nobel Prize in Economics!
  - 2012, 2020 Prizes also related
- Before we get to mechanism design, first we need to know how to evaluate mechanisms

Example: (single-item) auctions

- Sealed-bid auction: every bidder submits bid in a sealed envelope
- First-price sealed-bid auction: highest bid wins, pays amount of own bid
- Second-price sealed-bid auction: highest bid wins, pays amount of second-highest bid

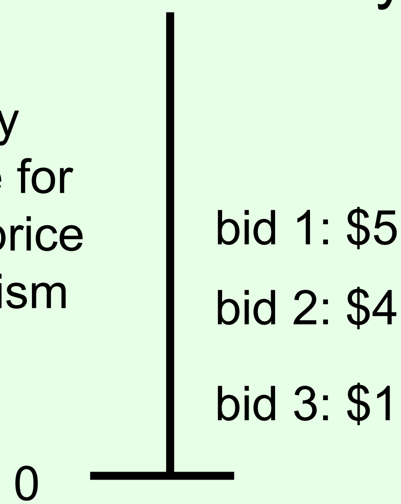


first-price: bid 1 wins, pays $10
second-price: bid 1 wins, pays $5

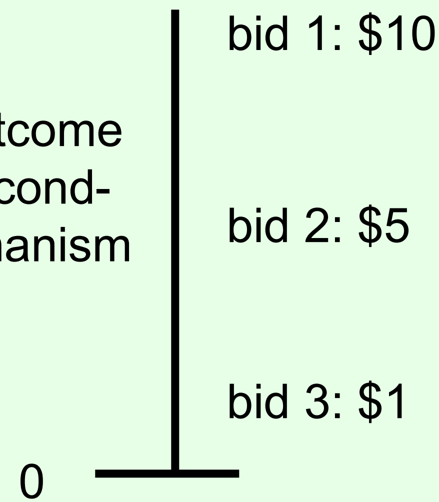
Which auction generates more revenue?

- Each bid depends on
  - bidder's true valuation for the item (utility = valuation - payment),
  - bidder's beliefs over what others will bid (→ game theory),
  - and... the auction mechanism used
- In a first-price auction, it does not make sense to bid your true valuation
  - Even if you win, your utility will be 0…
- In a second-price auction, (we will see next that) it always makes sense to bid your true valuation

a likely outcome for the first-price mechanism



a likely outcome for the second-price mechanism



Are there other auctions that perform better? How do we know when we have found the best one?

Bidding truthfully is optimal in the Vickrey auction!

- What should a bidder with value v bid?

b = highest bid among other bidders

0

Option 1: Win the item at price b, get utility v - b

Option 2: Lose the item, get utility 0

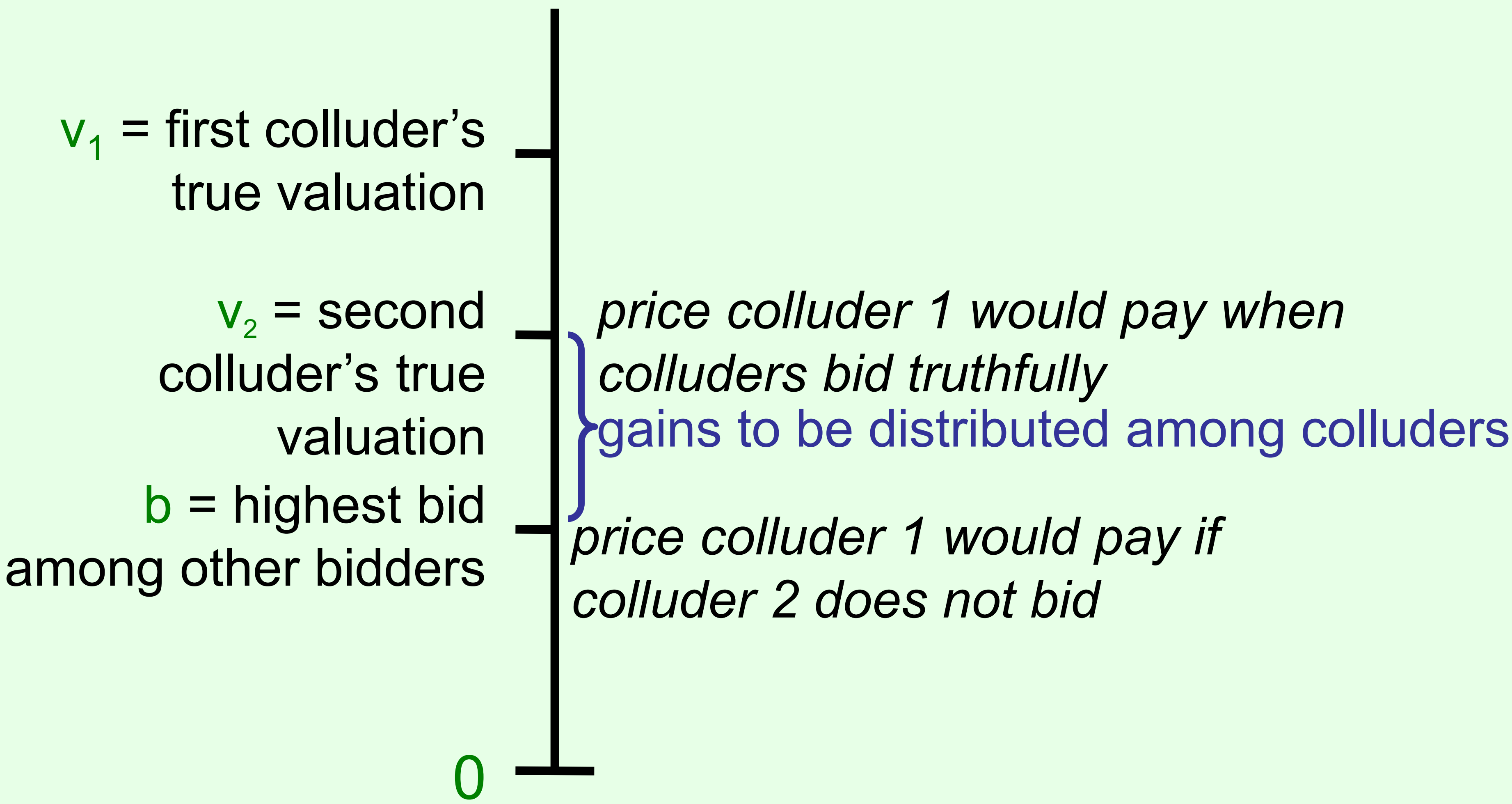
Would like to win if and only if v - b > 0

– but bidding truthfully accomplishes this!

We say the Vickrey auction is strategy-proof

Collusion in the Vickrey auction

- Example: two colluding bidders



Attempt #1 at using game theory to predict auction outcome

- First-price sealed-bid (or Dutch) auction
- Bidder 1 has valuation 4, bidder 2 has val. 2
- Discretized version, random tie-breaking

	0	1	2	3	4
0	2, 1	0, 1	0, 0	0, -1	0, -2
1	3, 0	1.5, .5	0, 0	0, -1	0, -2
2	2, 0	2, 0	1, 0	0, -1	0, -2
3	1, 0	1, 0	1, 0	.5, -.5	0, -2
4	0, 0	0, 0	0, 0	0, 0	0, -1

- What aspect(s) of auctions is this missing?

Bayesian games

- In a Bayesian game a player's utility depends on that player's type as well as the actions taken in the game
  - Notation: θ_i is player i's type, drawn according to some distribution from set of types Θ_i
  - Each player knows/learns its own type, not those of the others, before choosing action
    - Pure strategy s_i is a mapping from Θ_i to A_i (where A_i is i's set of actions)
  - In general players can also receive signals about other players' utilities; we will not go into this

row player type 1 (prob. 0.5)

	L	R
U	4	6
D	2	4

column player type 1 (prob. 0.5)

	L	R
U	4	6
D	4	6

row player type 2 (prob. 0.5)

	L	R
U	2	4
D	4	2

column player type 2 (prob. 0.5)

	L	R
U	2	2
D	4	2

Converting Bayesian games to normal form

row player type 1 (prob. 0.5)

	L	R
U	4	6
D	2	4

column player type 1 (prob. 0.5)

	L	R
U	4	6
D	4	6

row player type 2 (prob. 0.5)

	L	R
U	2	4
D	4	2

column player type 2 (prob. 0.5)

	L	R
U	2	2
D	4	2

	type 1: L type 2: L	type 1: L type 2: R	type 1: R type 2: L	type 1: R type 2: R
type 1: U type 2: U	3, 3	4, 3	4, 4	5, 4
type 1: U type 2: D	4, 3.5	4, 3	4, 4.5	4, 4
type 1: D type 2: U	2, 3.5	3, 3	3, 4.5	4, 4
type 1: D type 2: D	3, 4	3, 3	3, 5	3, 4

exponential blowup in size

Bayes-Nash equilibrium

- A profile of strategies is a Bayes-Nash equilibrium if it is a Nash equilibrium for the normal form of the game
  - Minor caveat: each type should have >0 probability
- Alternative definition: for every i, for every type θ_i, for every alternative action a_i, we must have:

$$\Sigma_{\theta_{-i}} P(\theta_{-i})\ u_i(\theta_i, \sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})) \geq$$
$$\Sigma_{\theta_{-i}} P(\theta_{-i})\ u_i(\theta_i, a_i, \sigma_{-i}(\theta_{-i}))$$

First-price sealed-bid auction BNE

- Suppose every bidder (independently) draws a valuation from [0, 1]
- What is a Bayes-Nash equilibrium for this?
- Say a bidder with value v_i bids $v_i(n-1)/n$
- Claim: this is an equilibrium!
- Proof: suppose all others use this strategy
- For a bid $b < (n-1)/n$, the probability of winning is $(bn/(n-1))^{n-1}$, so the expected value is $(v_i-b)(bn/(n-1))^{n-1}$
- Derivative w.r.t. b is $- (bn/(n-1))^{n-1} + (v_i-b)(n-1)b^{n-2}(n/(n-1))^{n-1}$ which should equal zero
- Implies $-b + (v_i-b)(n-1) = 0$, which solves to $b = v_i(n-1)/n$

Analyzing the expected revenue of the first-price and second-price (Vickrey) auctions

- First-price auction: probability of there not being a bid higher than b is $(bn/(n-1))^n$ (for $b < (n-1)/n$)
  - This is the cumulative density function of the highest bid
- Probability density function is the derivative, that is, it is $nb^{n-1}(n/(n-1))^n$
- Expected value of highest bid is $n(n/(n-1))^n \int^{(n-1)/n} b^n db = (n-1)/(n+1)$
- Second-price auction: probability of there not being two bids higher than b is $b^n + nb^{n-1}(1-b)$
  - This is the cumulative density function of the second-highest bid
- Probability density function is the derivative, that is, it is $nb^{n-1} + n(n-1)b^{n-2}(1-b) - nb^{n-1} = n(n-1)(b^{n-2} - b^{n-1})$
- Expected value is $(n-1) - n(n-1)/(n+1) = (n-1)/(n+1)$

Revenue equivalence theorem

- Suppose valuations for the single item are drawn i.i.d. from a continuous distribution over [L, H] (with no “gaps”), and agents are risk-neutral
- Then, any two auction mechanisms that
  - in equilibrium always allocate the item to the bidder with the highest valuation, and
  - give an agent with valuation L an expected utility of 0,

  will lead to the same expected revenue for the auctioneer

(As an aside) what if bidders are not risk-neutral?

- Behavior in second-price/English/Japanese does not change, but behavior in first-price/Dutch does
- Risk averse: first price/Dutch will get higher expected revenue than second price/Japanese/English
- Risk seeking: second price/Japanese/English will get higher expected revenue than first price/Dutch

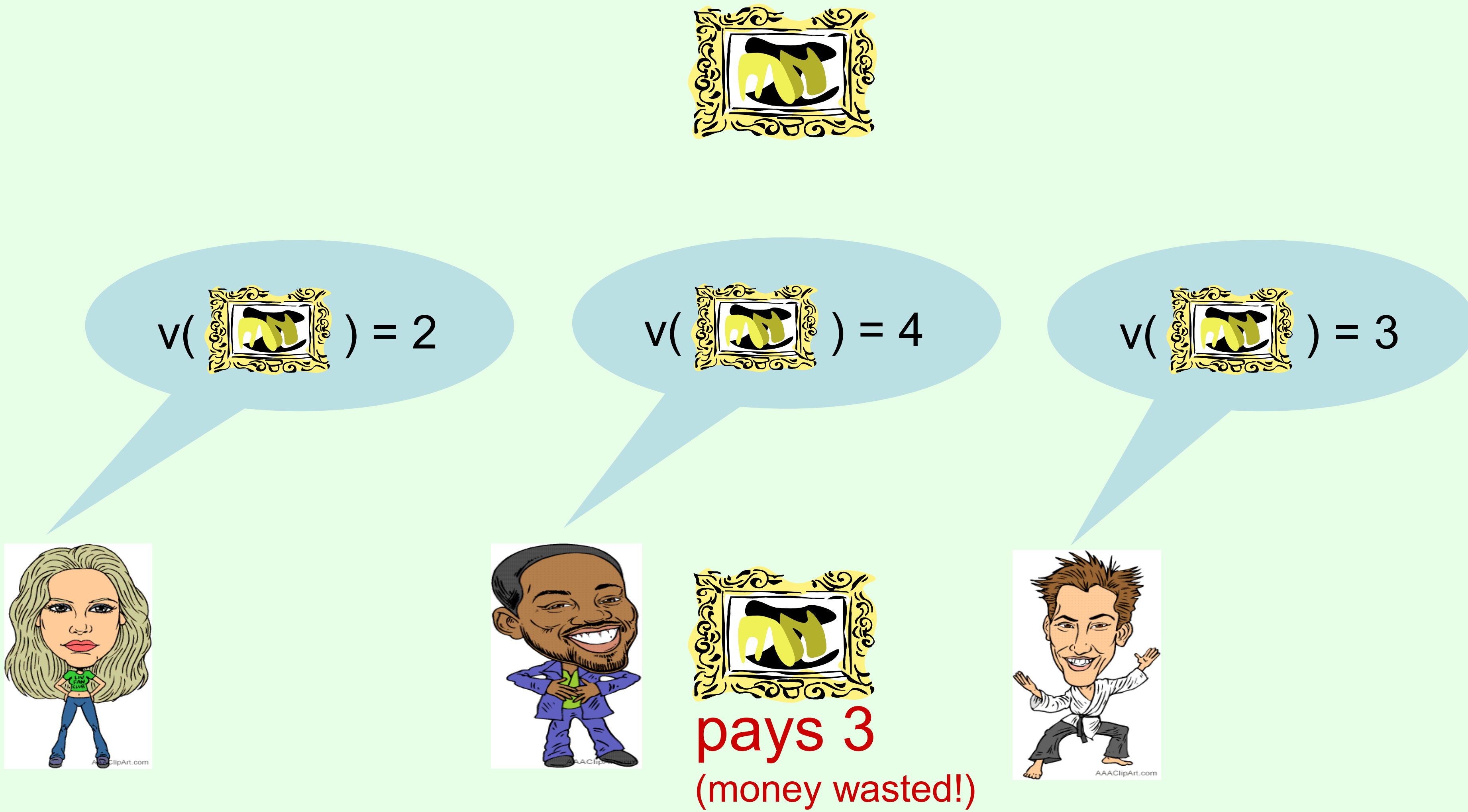
(As an aside) interdependent valuations

- E.g. bidding on drilling rights for an oil field
- Each bidder i has its own geologists who do tests, based on which the bidder assesses an expected value v_i of the field
- If you win, it is probably because the other bidders' geologists' tests turned out worse, and the oil field is not actually worth as much as you thought
  - The so-called winner's curse
- Hence, bidding v_i is no longer a dominant strategy in the second-price auction
- In English and Japanese auctions, you can update your valuation based on other agents' bids, so no longer equivalent to second-price
- In these settings, English (or Japanese) > second-price > first-price/Dutch in terms of revenue

Expected-revenue maximizing ("optimal") auctions [Myerson 81]

- Vickrey auction does not maximize expected revenue
  - E.g. with only one bidder, better off making a take-it-or-leave-it offer (or equivalently setting a reserve price)
- Suppose agent i draws valuation from probability density function f_i (cumulative density F_i)
- Bidder's virtual valuation $\psi(v_i) = v_i - (1 - F_i(v_i))/f_i(v_i)$
  - Under certain conditions, this is increasing; assume this
- The bidder with the highest virtual valuation (according to his reported valuation) wins (unless all virtual valuations are below 0, in which case nobody wins)
- Winner pays value of lowest bid that would have made him win
- E.g. if all bidders draw uniformly from [0, 1], Myerson auction = second-price auction with reserve price ½

Vickrey auction without a seller



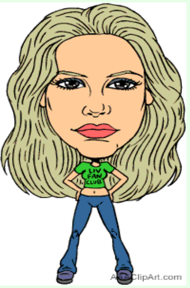
Can we redistribute the payment?

Idea: give everyone 1/n of the payment

v() = 2

v() = 4

v() = 3



receives 1

pays 3

receives 1

receives 1

not strategy-proof

Bidding higher can increase your redistribution payment

Incentive compatible redistribution

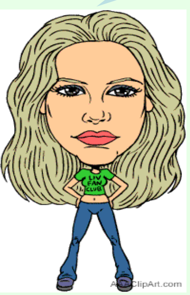
[Bailey 97, Porter et al. 04, Cavallo 06]

Idea: give everyone 1/n of second-highest **other** bid

v( ) = 2

v( ) = 4

v( ) = 3



receives 1

pays 3

receives 2/3

receives 2/3

2/3 wasted (22%)

Strategy-proof

Your redistribution does not depend on your bid;

incentives are the same as in Vickrey

Bailey-Cavallo mechanism…

- Bids: $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n \geq 0$
- First run Vickrey auction
- Payment is V_2
- First two bidders receive V_3/n
- Remaining bidders receive V_2/n
- Total redistributed: $2V_3/n+(n-2)V_2/n$

$R_1 = V_3/n$

$R_2 = V_3/n$

$R_3 = V_2/n$

$R_4 = V_2/n$

...

$R_{n-1} = V_2/n$

$R_n = V_2/n$

Is this the best possible?

Another redistribution mechanism

- Bids: $V_1 \geq V_2 \geq V_3 \geq V_4 \geq \ldots \geq V_n \geq 0$
- First run Vickrey
- Redistribution:

  Receive 1/(n-2) * second-highest **other** bid,

  - 2/[(n-2)(n-3)] third-highest **other** bid
- Total redistributed:

  $V_2 - 6V_4/[(n-2)(n-3)]$

$R_1 = V_3/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_2 = V_3/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_3 = V_2/(n-2) - 2/[(n-2)(n-3)]V_4$

$R_4 = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

...

$R_{n-1} = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

$R_n = V_2/(n-2) - 2/[(n-2)(n-3)]V_3$

Idea pursued further in Guo & Conitzer 07 / Moulin 07