Learning in games

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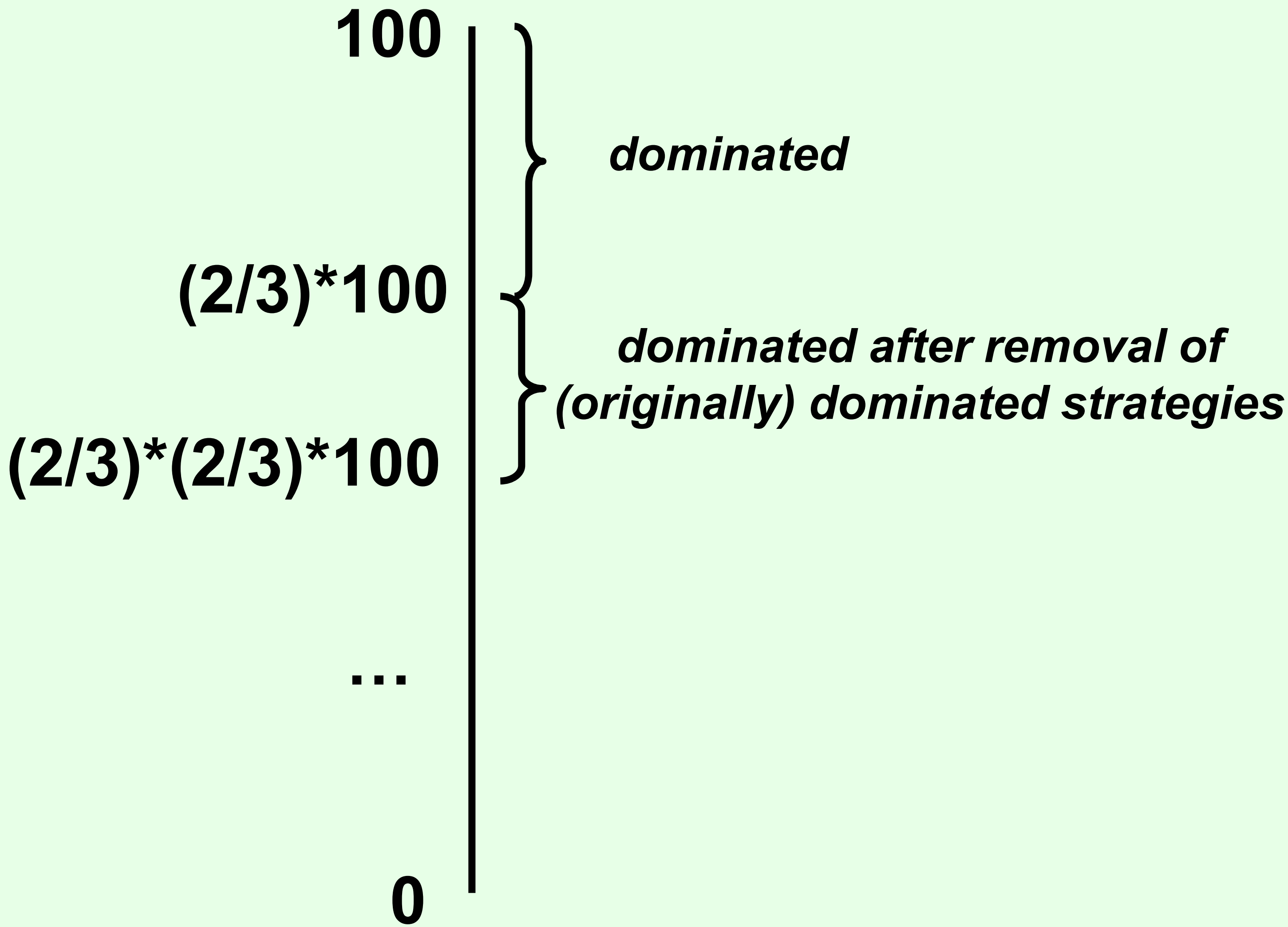
“2/3 of the average” game

- Everyone writes down a number between 0 and 100
- Person closest to 2/3 of the average wins
- Example:
  - A says 50
  - B says 10
  - C says 90
  - Average(50, 10, 90) = 50
  - 2/3 of average = 33.33
  - A is closest (|50-33.33| = 16.67), so A wins

“2/3 of the average” game revisited



Learning in (normal-form) games

- Approach we have taken so far when playing a game: just compute an optimal/equilibrium strategy
- Another approach: learn how to play a game by
  - playing it many times, and
  - updating your strategy based on experience
- Why?
  - Some of the game's utilities (especially the other players') may be unknown to you
  - The other players may not be playing an equilibrium strategy
  - Computing an optimal strategy can be hard
  - Learning is what humans typically do
  - …
- Learning strategies ~ strategies for the repeated game
- Does learning converge to equilibrium?

Iterated best response

- In the first round, play something arbitrary
- In each following round, play a best response against what the other players played in the previous round
- If all players play this, it can converge (i.e., we reach an equilibrium) or cycle

0, 0	-1, 1	1, -1
1, -1	0, 0	-1, 1
-1, 1	1, -1	0, 0

rock-paper-scissors

-1, -1	0, 0
0, 0	-1, -1

a simple congestion game

- Alternating best response: players alternatingly change strategies: one player best-responds each odd round, the other best-responds each even round

Fictitious play [Brown 1951]

- In the first round, play something arbitrary
- In each following round, play a best response against the empirical distribution of the other players' play
  - I.e., as if other player randomly selects from his past actions
- Again, if this converges, we have a Nash equilibrium
- Can still fail to converge…

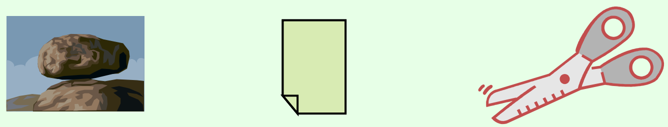
0, 0	-1, 1	1, -1
1, -1	0, 0	-1, 1
-1, 1	1, -1	0, 0

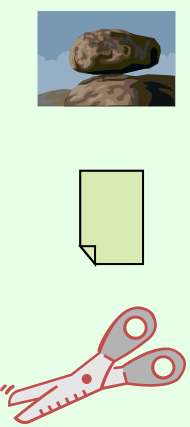
rock-paper-scissors

-1, -1	0, 0
0, 0	-1, -1

a simple congestion game

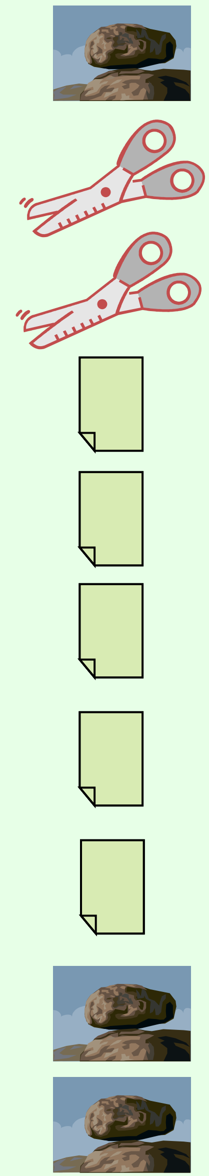
Fictitious play on rock-paper-scissors





0, 0	-1, 1	1, -1
1, -1	0, 0	-1, 1
-1, 1	1, -1	0, 0

Row



30% R, 50% P, 20% S

Column

30% R, 20% P, 50% S

Does the empirical distribution of play converge to equilibrium?

- … for iterated best response?
- … for fictitious play?

3, 0	1, 2
1, 2	2, 1

Fictitious play is guaranteed to converge in…

- Two-player zero-sum games [Robinson 1951]
- Generic 2x2 games [Miyasawa 1961]
- Games solvable by iterated strict dominance [Nachbar 1990]
- Weighted potential games [Monderer & Shapley 1996]
- **Not** in general [Shapley 1964]
- But, fictitious play always converges to the set of ½-approximate equilibria [Conitzer 2009; more detailed analysis by Goldberg, Savani, Sørensen, Ventre 2011]

Shapley's game on which fictitious play does not converge

- starting with (U, M):

0, 0	0, 1	1, 0
1, 0	0, 0	0, 1
0, 1	1, 0	0, 0

"Teaching"

- Suppose you are playing against a player that uses a strategy that eventually learns to best-respond
- Also suppose you are very patient, i.e., you only care about what happens in the long run
- How will you (the row player) play in the following repeated games?

4, 4	3, 5
5, 3	0, 0

1, 0	3, 1
2, 1	4, 0

- Note relationship to optimal strategies to commit to
- There is some work on learning strategies that are in equilibrium with each other [Brafman & Tennenholtz AIJ04]

Evolutionary game theory

- Given: a symmetric game

	dove	hawk
dove	1, 1	0, 2
hawk	2, 0	-1, -1

Nash equilibria: (d, h), (h, d), ((.5, .5), (.5, .5))

- A large population of players plays this game, players are randomly matched to play with each other
- Each player plays a pure strategy
  - Fraction of players playing strategy s = p_s
  - p is vector of all fractions p_s (the state)
- Utility for playing s is $u(s, p) = \Sigma_{s'} p_{s'} u(s, s')$
- Players reproduce at a rate that is proportional to their utility, their offspring play the same strategy
  - Replicator dynamic
- $dp_s(t)/dt = p_s(t)(u(s, p(t)) - \Sigma_{s'} p_{s'} u(s', p(t)))$
- What are the steady states of this?

Stability

	dove	hawk
dove	1, 1	0, 2
hawk	2, 0	-1, -1

- A steady state is stable if slightly perturbing the state will not cause us to move far away from the state
- E.g. everyone playing dove is not stable, because if a few hawks are added their percentage will grow
- What about the mixed steady state?
- Proposition: every stable steady state is a Nash equilibrium of the symmetric game
- Slightly stronger criterion: a state is asymptotically stable if it is stable, and after slightly perturbing this state, we will (in the limit) return to this state

Evolutionarily stable strategies

- Now suppose players play mixed strategies
- A (single) mixed strategy σ is evolutionarily stable if the following is true:
  - Suppose all players play σ
  - Then, whenever a very small number of invaders enters that play a different strategy σ',
  - the players playing σ must get strictly higher utility than those playing σ' (i.e., σ must be able to repel invaders)
- σ will be evolutionarily stable if and only if for all σ'
  - $u(\sigma, \sigma) > u(\sigma', \sigma)$, or:
  - $u(\sigma, \sigma) = u(\sigma', \sigma)$ and $u(\sigma, \sigma') > u(\sigma', \sigma')$
- Proposition: every evolutionarily stable strategy is asymptotically stable under the replicator dynamic

Invasion (1/2)

	Dove	Hawk
Dove	1, 1	0, 2
Hawk	2, 0	-1, -1

- Given: population P_1 that plays σ = 40% Dove, 60% Hawk
- Tiny population P_2 that plays σ' = 70% Dove, 30% Hawk invades
- $u(\sigma, \sigma) = .16*1 + .24*2 + .36*(-1) = .28$ but
  $u(\sigma', \sigma) = .28*1 + .12*2 + .18*(-1) = .34$
- σ' (initially) grows in the population; invasion is successful

Invasion (2/2)

	Dove	Hawk
Dove	1, 1	0, 2
Hawk	2, 0	-1, -1

- Now P_1 plays σ = 50% Dove, 50% Hawk
- Tiny population P_2 that plays σ' = 70% Dove, 30% Hawk invades
- $u(\sigma, \sigma) = u(\sigma', \sigma) = .5$, so second-order effect:
- $u(\sigma, \sigma') = .35*1 + .35*2 + .15*(-1) = .9$ but
  $u(\sigma', \sigma') = .49*1 + .21*2 + .09*(-1) = .82$
- σ' shrinks in the population; invasion is repelled

Evolutionarily stable strategies

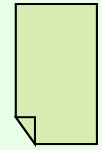
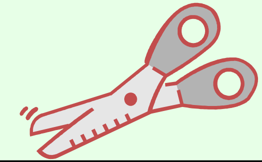
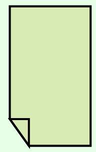
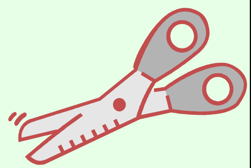
[Price and Smith, 1973]

- A strategy σ is evolutionarily stable if the following two conditions both hold:

  (1) For all σ', we have $u(\sigma, \sigma) \geq u(\sigma', \sigma)$ (i.e., symmetric Nash equilibrium)

  (2) For all σ' ($\neq \sigma$) with $u(\sigma, \sigma) = u(\sigma', \sigma)$, we have $u(\sigma, \sigma') > u(\sigma', \sigma')$

Rock-Paper-Scissors

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

- Only one Nash equilibrium (Uniform)
- u(Uniform, Rock) = u(Rock, Rock)
- No ESS

The standard Σ_2^P-complete problem

Input: *Boolean formula f over variables X_1 and X_2*

Q: *Does there exist an assignment of values to X_1 such that for every assignment of values to X_2 f is true?*

The ESS problem

Input: *symmetric 2-player normal-form game.*

Q: *Does it have an evolutionarily stable strategy?*

(Hawk-Dove: yes. Rock-Paper-Scissors: no. Safe-Left-Right: no.)