Normal-form games

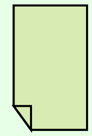
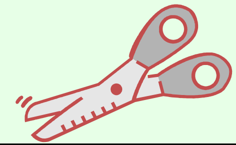
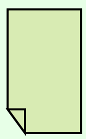
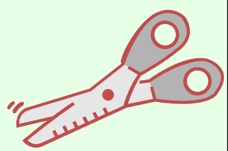
Vincent Conitzer

“2/3 of the average” game

- Everyone writes down a number between 0 and 100
- Person closest to 2/3 of the average wins
- Example:
  - A says 50
  - B says 10
  - C says 90
  - Average(50, 10, 90) = 50
  - 2/3 of average = 33.33
  - A is closest (|50-33.33| = 16.67), so A wins

Rock-paper-scissors

Column player aka. player 2 (simultaneously) chooses a column

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

Row player aka. player 1 chooses a row

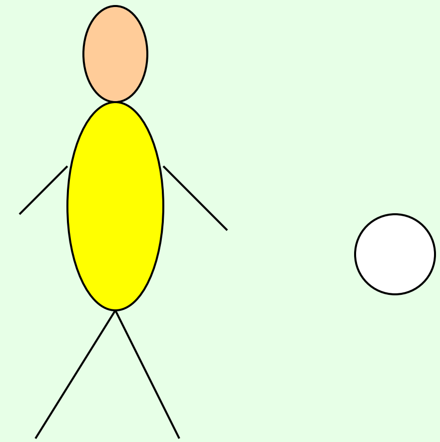
A row or column is called an action or (pure) strategy

Row player's utility is always listed first, column player's second

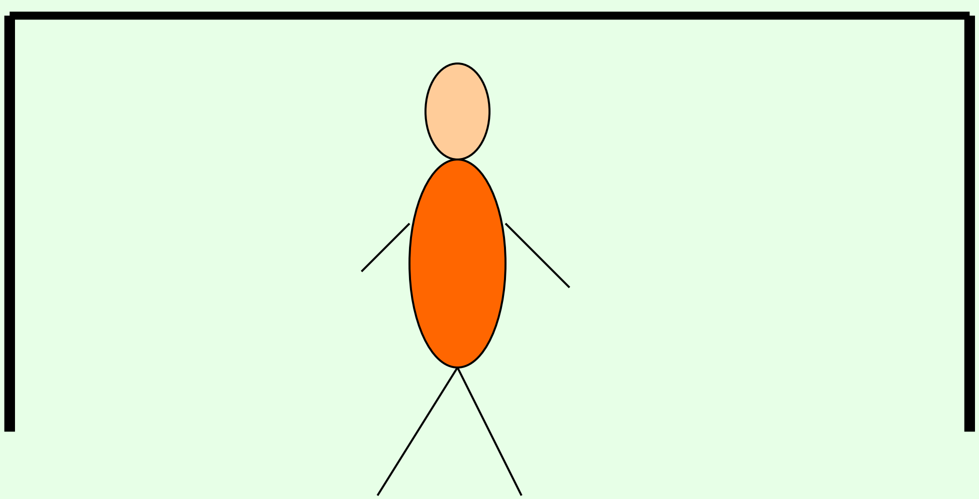
Zero-sum game: the utilities in each entry sum to 0 (or a constant)

Three-player game would be a 3D table with 3 utilities per entry, etc.

Matching pennies (~penalty kick)

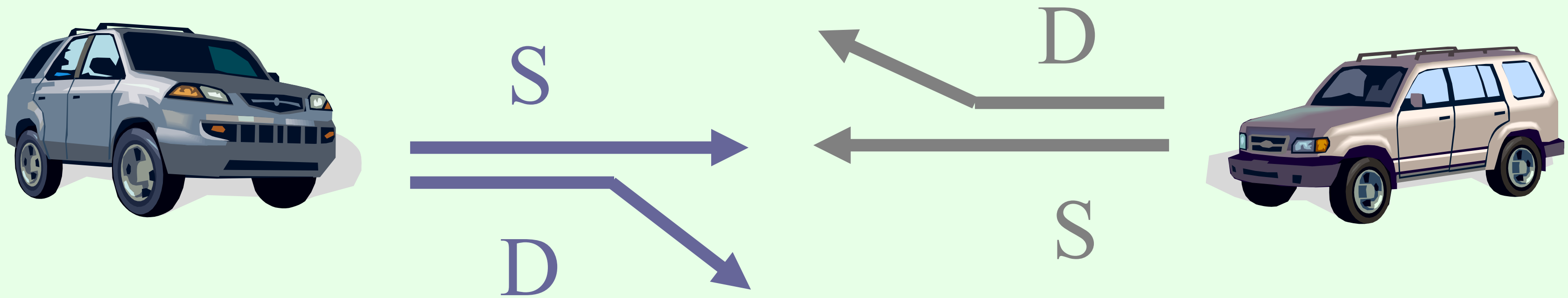


	L	R
L	1, -1	-1, 1
R	-1, 1	1, -1



"Chicken"

- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

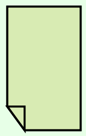


	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

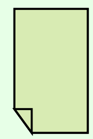
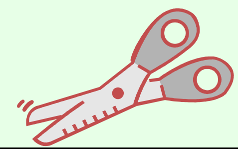
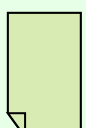
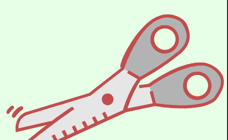
not zero-sum

Rock-paper-scissors – Seinfeld variant





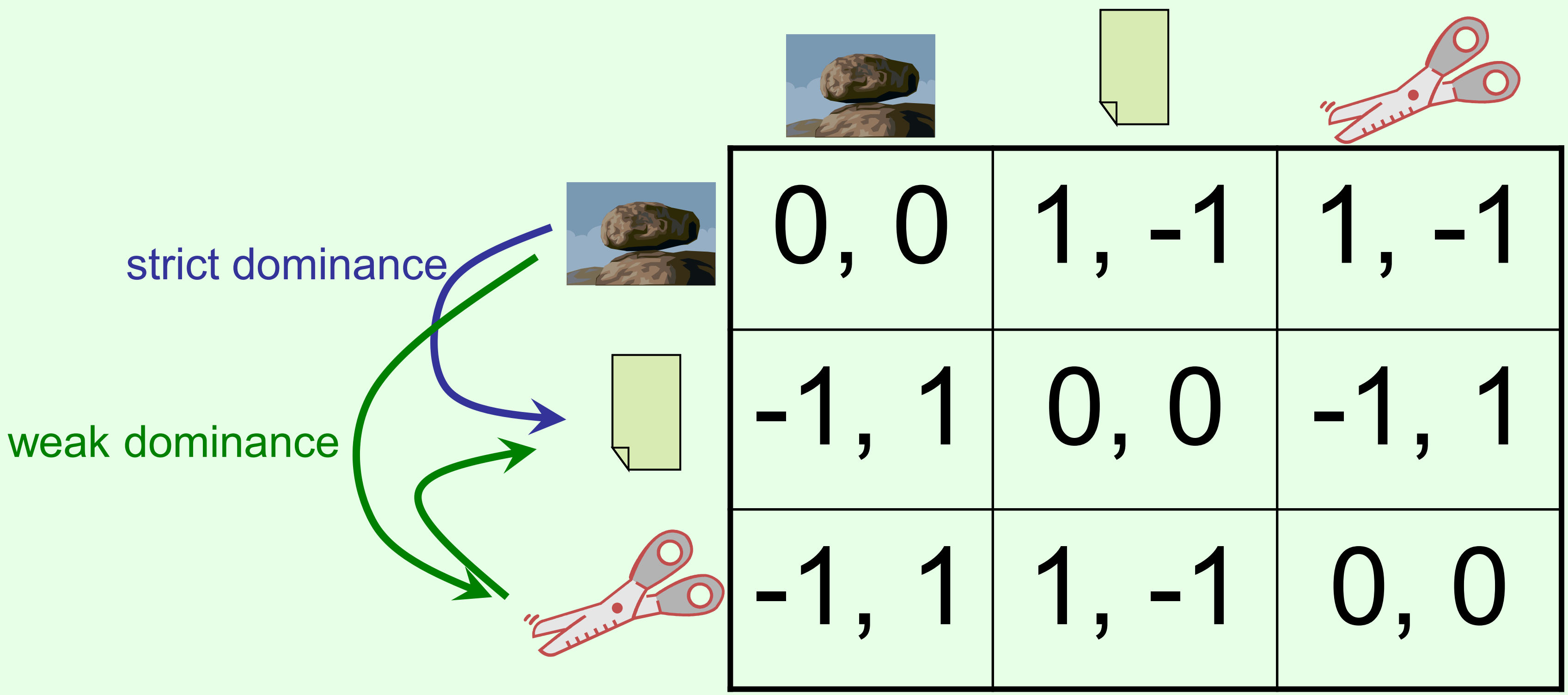
MICKEY: All right, rock beats paper!
(Mickey smacks Kramer's hand for losing)
KRAMER: I thought paper covered rock.
MICKEY: Nah, rock flies right through paper.
KRAMER: What beats rock?
MICKEY: (looks at hand) Nothing beats rock.

			
	0, 0	1, -1	1, -1
	-1, 1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

Dominance

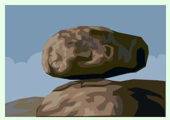
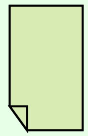
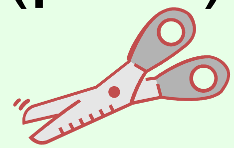
- Player i's strategy s_i **strictly dominates** s_i' if
  - for any s_{-i}, $u_i(s_i , s_{-i}) > u_i(s_i', s_{-i})$
- s_i **weakly dominates** s_i' if
  - for any s_{-i}, $u_i(s_i , s_{-i}) \geq u_i(s_i', s_{-i})$; and
  - for some s_{-i}, $u_i(s_i , s_{-i}) > u_i(s_i', s_{-i})$

$-i$ = "the player(s) other than i"



0, 0	1, -1	1, -1
-1, 1	0, 0	-1, 1
-1, 1	1, -1	0, 0

Mixed strategies

- Mixed strategy for player i = probability distribution over player i's (pure) strategies
- E.g.,1/3  , 1/3  , 1/3 
- Example of dominance by a mixed strategy:

1/2	3, 0	0, 0
1/2	0, 0	3, 0
	1, 0	1, 0

Usage:
σ_i denotes a mixed strategy, s_i denotes a pure strategy

Checking for dominance by mixed strategies

- Linear program for checking whether strategy s_i^* is strictly dominated by a mixed strategy:
- maximize ε
- such that:
  - for any s_{-i}, $\Sigma_{s_i} \mathbf{p}_{\mathbf{s}_i} u_i(s_i, s_{-i}) \geq u_i(s_i^*, s_{-i}) + \varepsilon$
  - $\Sigma_{s_i} \mathbf{p}_{\mathbf{s}_i} = 1$

- Linear program for checking whether strategy s_i^* is weakly dominated by a mixed strategy:
- maximize $\Sigma_{s_{-i}}[(\Sigma_{s_i} \mathbf{p}_{\mathbf{s}_i} u_i(s_i, s_{-i})) - u_i(s_i^*, s_{-i})]$
- such that:
  - for any s_{-i}, $\Sigma_{s_i} \mathbf{p}_{\mathbf{s}_i} u_i(s_i, s_{-i}) \geq u_i(s_i^*, s_{-i})$
  - $\Sigma_{s_i} \mathbf{p}_{\mathbf{s}_i} = 1$

Iterated dominance

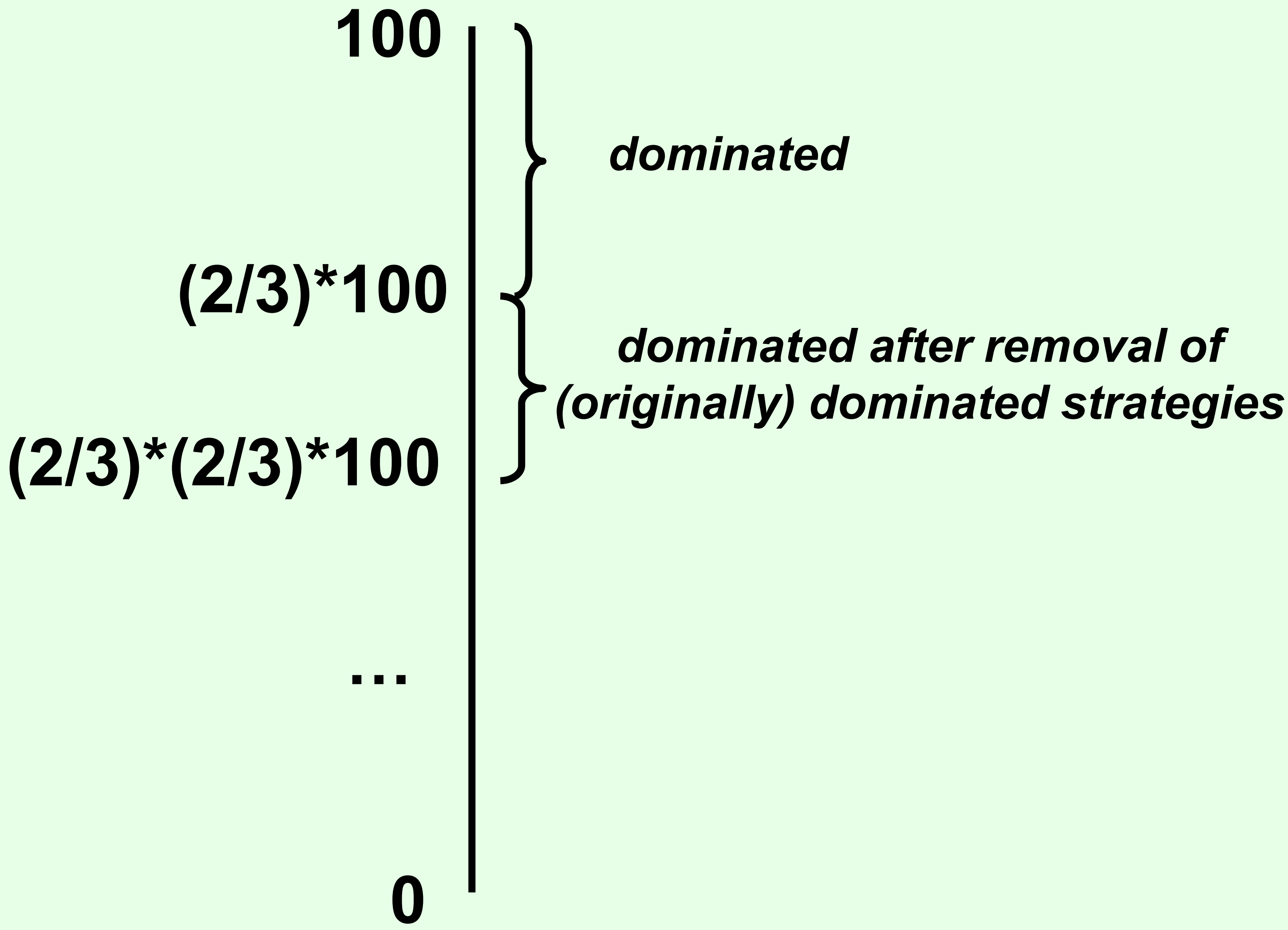
- Iterated dominance: remove (strictly/weakly) dominated strategy, repeat
- Iterated strict dominance on Seinfeld's RPS:

0, 0	1, -1	1, -1
-1, 1	0, 0	-1, 1
-1, 1	1, -1	0, 0

→

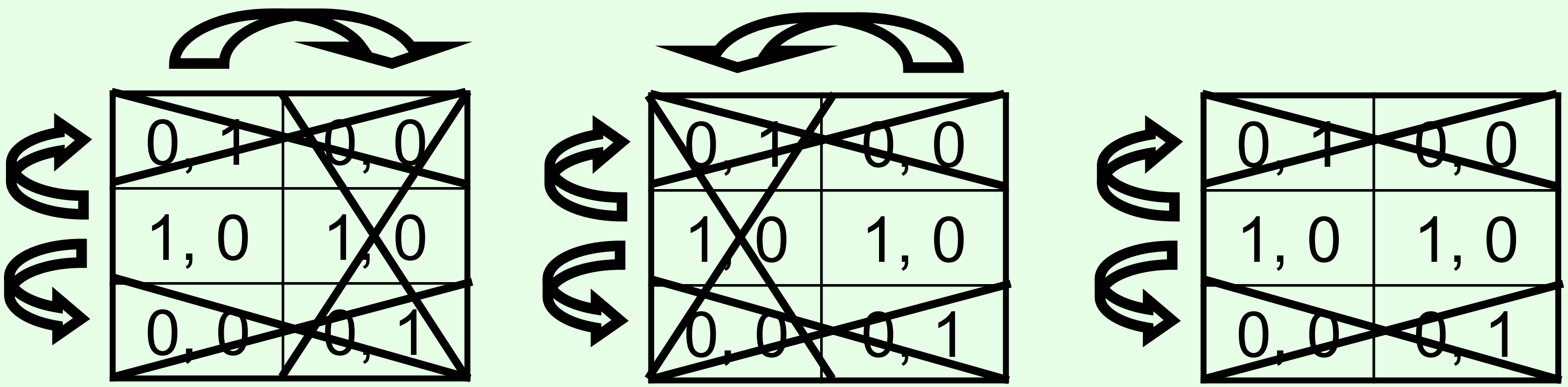
0, 0	1, -1
-1, 1	0, 0

“2/3 of the average” game revisited



Iterated dominance: path (in)dependence

Iterated weak dominance is path-dependent: sequence of eliminations may determine which solution we get (if any)
(whether or not dominance by mixed strategies allowed)



Iterated strict dominance is path-independent: elimination process will always terminate at the same point
(whether or not dominance by mixed strategies allowed)

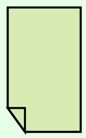
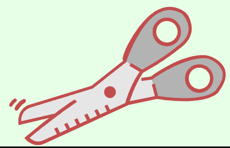
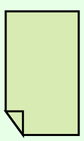
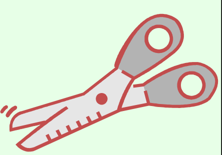
Two computational questions for iterated dominance

- 1. Can a given strategy be eliminated using iterated dominance?
- 2. Is there some path of elimination by iterated dominance such that only one strategy per player remains?

- For strict dominance (with or without dominance by mixed strategies), both can be solved in polynomial time due to path-independence:
  - Check if any strategy is dominated, remove it, repeat
- For weak dominance, both questions are NP-hard (even when all utilities are 0 or 1), with or without dominance by mixed strategies [Conitzer, Sandholm 05]
  - Weaker version proved by [Gilboa, Kalai, Zemel 93]

Two-player zero-sum games revisited

- Recall: in a zero-sum game, payoffs in each entry sum to zero
  - … or to a constant: recall that we can subtract a constant from anyone’s utility function without affecting their behavior
- What the one player gains, the other player loses

			
	0, 0	-1, 1	1, -1
	1, -1	0, 0	-1, 1
	-1, 1	1, -1	0, 0

Note: a general-sum k-player game can be modeled as a zero-sum (k+1)-player game by adding a dummy player absorbing the remaining utility, so zero-sum games with 3 or more players have to deal with the difficulties of general-sum games; this is why we focus on 2-player zero-sum games here.

Best-response strategies

- Suppose you know your opponent's mixed strategy
  - E.g., your opponent plays rock 50% of the time and scissors 50%
- What is the best strategy for you to play?
- Rock gives .5*0 + .5*1 = .5
- Paper gives .5*1 + .5*(-1) = 0
- Scissors gives .5*(-1) + .5*0 = -.5
- So the best response to this opponent strategy is to (always) play rock
- There is always some pure strategy that is a best response
  - Suppose you have a mixed strategy that is a best response; then every one of the pure strategies that that mixed strategy places positive probability on must also be a best response

How to play matching pennies

Them

		L	R
Us	L	1, -1	-1, 1
	R	-1, 1	1, -1

- Assume opponent knows our **mixed** strategy
- If we play L 60%, R 40%...
- … opponent will play R…
- … we get .6*(-1) + .4*(1) = -.2
- What's optimal for us? What about rock-paper-scissors?

Matching pennies with a sensitive target

		Them L	*Them* R
Us	L	1, -1	-1, 1
Us	R	-2, 2	1, -1

- If we play 50% L, 50% R, opponent will attack L
  - We get .5*(1) + .5*(-2) = -.5
- What if we play 55% L, 45% R?
- Opponent has choice between
  - L: gives them .55*(-1) + .45*(2) = .35
  - R: gives them .55*(1) + .45*(-1) = .1
- We get -.35 > -.5

Matching pennies with a sensitive target

Them

Us	L	R
L	1, -1	-1, 1
R	-2, 2	1, -1

- What if we play 60% L, 40% R?
- Opponent has choice between
  - L: gives them .6*(-1) + .4*(2) = .2
  - R: gives them .6*(1) + .4*(-1) = .2
- We get -.2 either way
- This is the maximin strategy
  - Maximizes our minimum utility

Let's change roles

		Them L	R
Us	L	1, -1	-1, 1
	R	-2, 2	1, -1

- Suppose **we** know **their** strategy
- If they play 50% L, 50% R,
  - We play L, we get .5*(1)+.5*(-1) = 0
- If they play 40% L, 60% R,
  - If we play L, we get .4*(1)+.6*(-1) = -.2
  - If we play R, we get .4*(-2)+.6*(1) = -.2
- This is the minimax strategy

von Neumann's minimax theorem [1928]: maximin value = minimax value (~LP duality)

Minimax theorem [von Neumann 1928]

- Maximin utility: $\max_{\sigma_i} \min_{s_{-i}} u_i(\sigma_i, s_{-i})$
  $(= - \min_{\sigma_i} \max_{s_{-i}} u_{-i}(\sigma_i, s_{-i}))$
- Minimax utility: $\min_{\sigma_{-i}} \max_{s_i} u_i(s_i, \sigma_{-i})$
  $(= - \max_{\sigma_{-i}} \min_{s_i} u_{-i}(s_i, \sigma_{-i}))$

- Minimax theorem:
  $\max_{\sigma_i} \min_{s_{-i}} u_i(\sigma_i, s_{-i}) = \min_{\sigma_{-i}} \max_{s_i} u_i(s_i, \sigma_{-i})$

- Minimax theorem does not hold with pure strategies only (example?)

Practice games

20, -20	0, 0
0, 0	10, -10

20, -20	0, 0	10, -10
0, 0	10, -10	8, -8

Solving for minimax strategies using linear programming

- maximize $\mathbf{u_i}$
- subject to

for any s_{-i}, $\Sigma_{s_i} \mathbf{p_{s_i}} u_i(s_i, s_{-i}) \geq \mathbf{u_i}$

$\Sigma_{s_i} \mathbf{p_{s_i}} = 1$

Can also convert linear programs to two-player zero-sum games, so they are equivalent

General-sum games

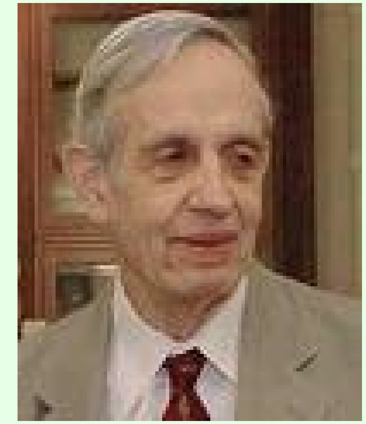
- You could still play a minimax strategy in general-sum games
  - I.e., pretend that the opponent is only trying to hurt you
- But this is not rational:

0, 0	3, 1
1, 0	2, 1

- If Column was trying to hurt Row, Column would play Left, so Row should play Down
- In reality, Column will play Right (strictly dominant), so Row should play Up
- Is there a better generalization of minimax strategies in zero-sum games to general-sum games?

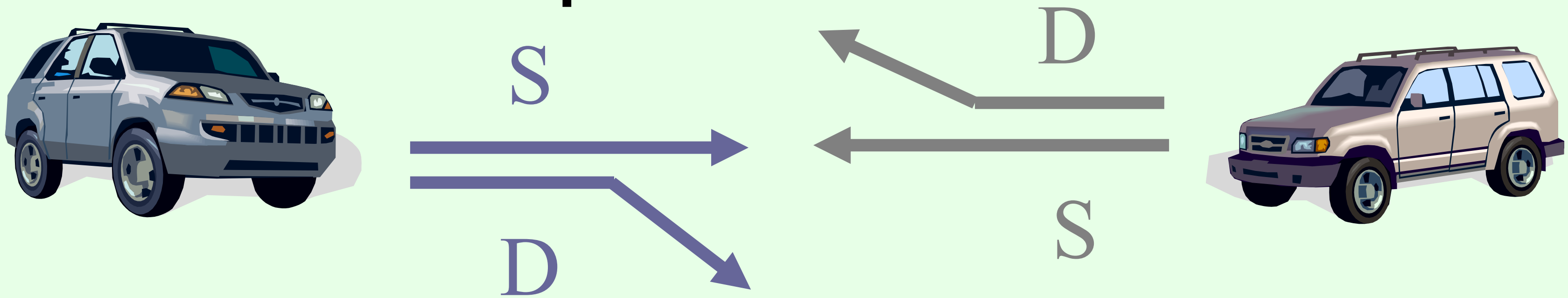
Nash equilibrium

[Nash 50]



- A vector of strategies (one for each player) is called a strategy profile
- A strategy profile $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a Nash equilibrium if each σ_i is a best response to σ_{-i}
  - That is, for any i, for any σ_i', $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$
- Note that this does not say anything about multiple agents changing their strategies at the same time
- In any (finite) game, at least one Nash equilibrium (possibly using mixed strategies) exists [Nash 50]
- (Note - singular: equilibrium, plural: equilibria)

Nash equilibria of "chicken"



	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- (D, S) and (S, D) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing your strategy will make you strictly worse off
- No other pure-strategy Nash equilibria

Nash equilibria of "chicken"…

	D	S
D	0, 0	-1, 1
S	1, -1	-5, -5

- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- Recall: if a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses
- So we need to make player 1 indifferent between D and S
- Player 1's utility for playing D = $-p^c_S$
- Player 1's utility for playing S = $p^c_D - 5p^c_S = 1 - 6p^c_S$
- So we need $-p^c_S = 1 - 6p^c_S$ which means $p^c_S = 1/5$
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
  - People may die! Expected utility -1/5 for each player

The presentation game

		Presenter	
		Put effort into presentation (E)	*Do not put effort into presentation (NE)*
Audience	*Pay attention (A)*	4, 4	-16, -14
	Do not pay attention (NA)	0, -2	0, 0

- Pure-strategy Nash equilibria: (A, E), (NA, NE)
- Mixed-strategy Nash equilibrium:

  ((1/10 A, 9/10 NA), (4/5 E, 1/5 NE))
  - Utility 0 for audience, -14/10 for presenter
  - Can see that some equilibria are strictly better for both players than other equilibria, i.e., some equilibria Pareto-dominate other equilibria

The “equilibrium selection problem”

- You are about to play a game that you have never played before with a person that you have never met
- According to which equilibrium should you play?
- Possible answers:
  - Equilibrium that maximizes the sum of utilities (social welfare)
  - Or, at least not a Pareto-dominated equilibrium
  - So-called focal equilibria
    - “Meet in Paris” game - you and a friend were supposed to meet in Paris at noon on Sunday, but you forgot to discuss where and you cannot communicate. All you care about is meeting your friend. Where will you go?
  - Equilibrium that is the convergence point of some learning process
  - An equilibrium that is easy to compute
  - …
- Equilibrium selection is a difficult problem

Some properties of Nash equilibria

- If you can eliminate a strategy using strict dominance or even iterated strict dominance, it will not occur in any (i.e., it will be played with probability 0 in every) Nash equilibrium
  - Weakly dominated strategies may still be played in some Nash equilibrium
- In 2-player zero-sum games, a profile is a Nash equilibrium if and only if both players play minimax strategies
  - Hence, in such games, if (σ_1, σ_2) and (σ_1', σ_2') are Nash equilibria, then so are (σ_1, σ_2') and (σ_1', σ_2)
    - No equilibrium selection problem here!

How hard is it to compute *one* (any) Nash equilibrium?

- Complexity was open for a long time
  - [Papadimitriou STOC01]: “together with factoring […] the most important concrete open question on the boundary of P today”
- Recent sequence of papers shows that computing one (any) Nash equilibrium is PPAD-complete (even in 2-player games) [Daskalakis, Goldberg, Papadimitriou 2006; Chen, Deng 2006]
- All known algorithms require exponential time (in the worst case)

What if we want to compute a Nash equilibrium with a specific property?

- For example:
  - An equilibrium that is not Pareto-dominated
  - An equilibrium that maximizes the expected social welfare (= the sum of the agents' utilities)
  - An equilibrium that maximizes the expected utility of a given player
  - An equilibrium that maximizes the expected utility of the worst-off player
  - An equilibrium in which a given pure strategy is played with positive probability
  - An equilibrium in which a given pure strategy is played with zero probability
  - …
- All of these are NP-hard (and the optimization questions are inapproximable assuming P ≠ NP), even in 2-player games [Gilboa, Zemel 89; Conitzer & Sandholm IJCAI-03/GEB-08]

Search-based approaches (for 2 players)

- Suppose we know the support X_i of each player i's mixed strategy in equilibrium
  - That is, which pure strategies receive positive probability
- Then, we have a linear feasibility problem:
  - for both i, for any $s_i \in S_i - X_i$, $\mathbf{p_i(s_i)} = 0$
  - for both i, for any $s_i \in X_i$, $\Sigma \mathbf{p_{-i}(s_{-i})} u_i(s_i, s_{-i}) = \mathbf{u_i}$
  - for both i, for any $s_i \in S_i - X_i$, $\Sigma \mathbf{p_{-i}(s_{-i})} u_i(s_i, s_{-i}) \leq \mathbf{u_i}$
- Thus, we can search over possible supports
  - This is the basic idea underlying methods in [Dickhaut & Kaplan 91; Porter, Nudelman, Shoham AAAI04/GEB08]
- Dominated strategies can be eliminated

Solving for a Nash equilibrium using MIP (2 players)

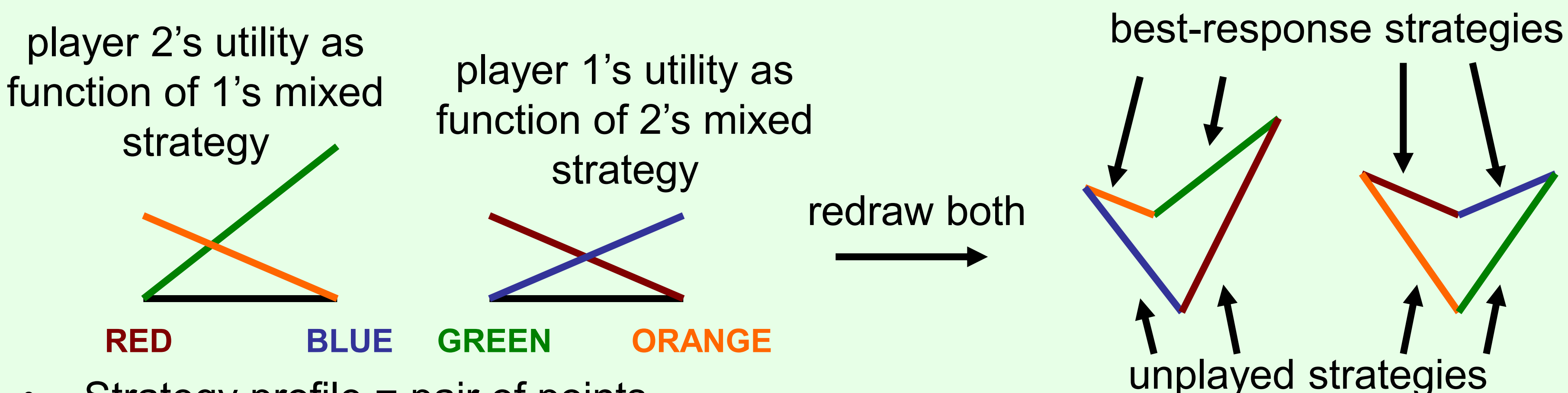
[Sandholm, Gilpin, Conitzer AAAI05]

- maximize *whatever you like (e.g., social welfare)*
- subject to
  - for both i, for any s_i, $\Sigma_{s_{-i}} \mathbf{p}_{\mathbf{s}_{-i}} u_i(s_i, s_{-i}) = \mathbf{u}_{\mathbf{s}_i}$
  - for both i, for any s_i, $\mathbf{u}_\mathbf{i} \geq \mathbf{u}_{\mathbf{s}_i}$
  - for both i, for any s_i, $\mathbf{p}_{\mathbf{s}_i} \leq \mathbf{b}_{\mathbf{s}_i}$
  - for both i, for any s_i, $\mathbf{u}_\mathbf{i} - \mathbf{u}_{\mathbf{s}_i} \leq M(1 - \mathbf{b}_{\mathbf{s}_i})$
  - for both i, $\Sigma_{s_i} \mathbf{p}_{\mathbf{s}_i} = 1$

- $\mathbf{b}_{\mathbf{s}_i}$ is a binary variable indicating whether s_i is in the support, M is a large number

Lemke-Howson algorithm (1-slide sketch!)

	GREEN	ORANGE
RED	1, 0	0, 1
BLUE	0, 2	1, 0



- Strategy profile = pair of points
- Profile is an equilibrium iff every pure strategy is either a best response or unplayed
- I.e. equilibrium = pair of points that includes all the colors
  - … except, pair of bottom points doesn't count (the "artificial equilibrium")
- Walk in some direction from the artificial equilibrium; at each step, throw out the color used twice

Correlated equilibrium [Aumann 74]

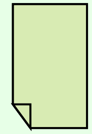
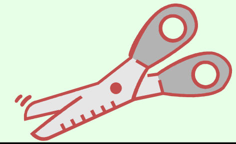
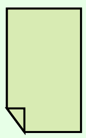
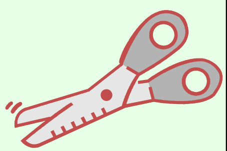
- Suppose there is a trustworthy mediator who has offered to help out the players in the game
- The mediator chooses a profile of pure strategies, perhaps randomly, then tells each player what her strategy is in the profile (but not what the other players' strategies are)
- A correlated equilibrium is a distribution over pure-strategy profiles so that every player wants to follow the recommendation of the mediator (if she assumes that the others do so as well)
- Every Nash equilibrium is also a correlated equilibrium
  - Corresponds to mediator choosing players' recommendations independently
- … but not vice versa
- *(Note: there are more general definitions of correlated equilibrium, but it can be shown that they do not allow you to do anything more than this definition.)*

A correlated equilibrium for “chicken”

	D	S
D	0, 0 20%	-1, 1 40%
S	1, -1 40%	-5, -5 0%

- Why is this a correlated equilibrium?
- Suppose the mediator tells the row player to Dodge
- From Row’s perspective, the conditional probability that Column was told to Dodge is 20% / (20% + 40%) = 1/3
- So the expected utility of Dodging is (2/3)*(-1) = -2/3
- But the expected utility of Straight is (1/3)*1 + (2/3)*(-5) = -3
- So Row wants to follow the recommendation
- If Row is told to go Straight, he knows that Column was told to Dodge, so again Row wants to follow the recommendation
- Similar for Column

A nonzero-sum variant of rock-paper-scissors (Shapley's game [Shapley 64])

			
	0, 0 0	0, 1 1/6	1, 0 1/6
	1, 0 1/6	0, 0 0	0, 1 1/6
	0, 1 1/6	1, 0 1/6	0, 0 0

- If both choose the same pure strategy, both lose
- These probabilities give a correlated equilibrium:
- E.g. suppose Row is told to play Rock
- Row knows Column is playing either paper or scissors (50-50)
  - Playing Rock will give ½; playing Paper will give 0; playing Scissors will give ½
- So Rock is optimal (not uniquely)

Solving for a correlated equilibrium using linear programming (n players!)

- Variables are now $\mathbf{p_s}$ where s is a profile of pure strategies

- maximize *whatever you like (e.g., social welfare)*
- subject to
  - for any i, s_i, s_i', $\Sigma_{s_{-i}}\, \mathbf{p_{(s_i,\, s_{-i})}}\, u_i(s_i, s_{-i}) \geq \Sigma_{s_{-i}}\, \mathbf{p_{(s_i,\, s_{-i})}}\, u_i(s_i', s_{-i})$
  - $\Sigma_s\, \mathbf{p_s} = 1$

Commitment

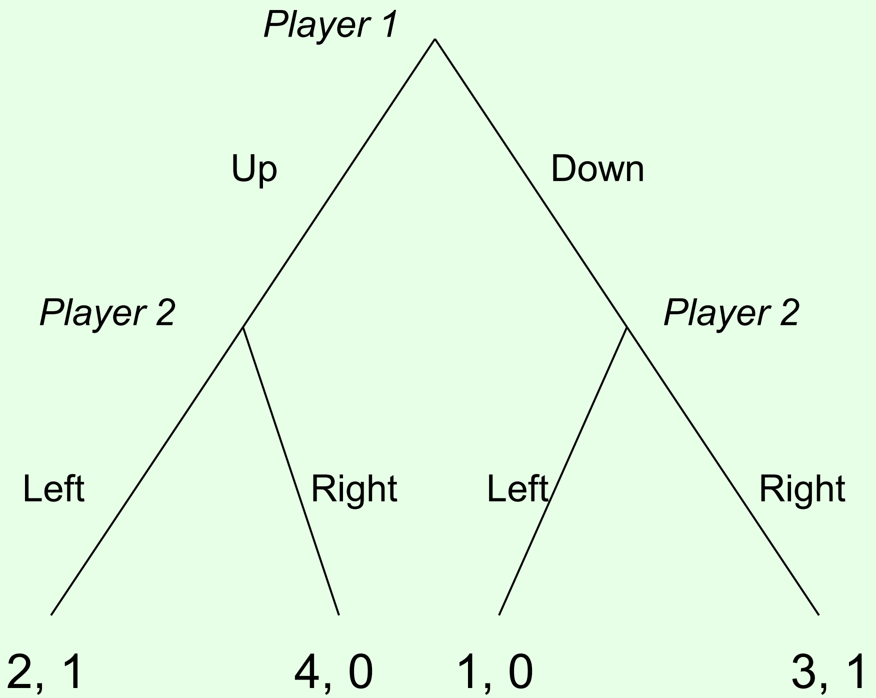
- Consider the following (normal-form) game:

2, 1	4, 0
1, 0	3, 1

- How should this game be played?
- Now suppose the game is played as follows:
  - Player 1 commits to playing one of the rows,
  - Player 2 observes the commitment and then chooses a column
- What is the optimal strategy for player 1?
- What if 1 can commit to a mixed strategy?

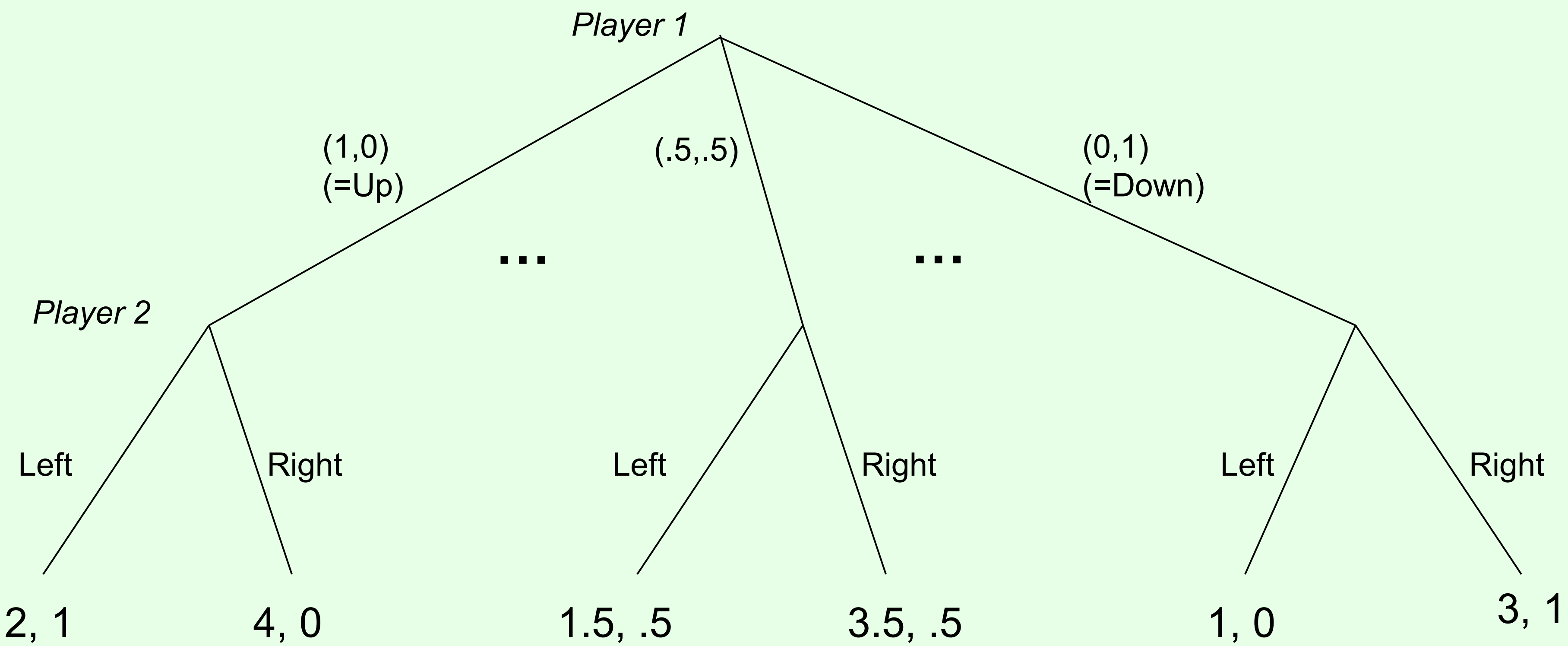
Commitment as an extensive-form game

- For the case of committing to a pure strategy:



Commitment as an extensive-form game

- For the case of committing to a mixed strategy:



- Infinite-size game; computationally impractical to reason with the extensive form here

Solving for the optimal mixed strategy to commit to

[Conitzer & Sandholm 2006, von Stengel & Zamir 2010]

- For every column t separately, we will solve separately for the best mixed row strategy (defined by $\mathbf{p_s}$) that induces player 2 to play t
- maximize $\Sigma_s \mathbf{p_s} u_1(s, t)$
- subject to

  for any t', $\Sigma_s \mathbf{p_s} u_2(s, t) \geq \Sigma_s \mathbf{p_s} u_2(s, t')$

  $\Sigma_s \mathbf{p_s} = 1$
- (May be infeasible, e.g., if t is strictly dominated)
- Pick the t that is best for player 1

Visualization

	L	C	R
U	0,1	1,0	0,0
M	4,0	0,1	0,0
D	0,0	1,0	1,1