15896/15483 Truth, Justice, and Algorithms
Fall 2018 Procaccia and Psomas HW 1

Due October 1, 2018, at 11:59pm

1. **Cake Cutting** (30 points: 20/10)

   Consider the cake cutting problem with n players and valuation functions $v_1, \ldots, v_n$ satisfying additivity, normalization, and divisibility. Denote the *social welfare* of an allocation $\boldsymbol{A}$ by $\mathrm{sw}(\boldsymbol{A}) = \sum_{i=1}^{n} v_i(A_i)$.

   (a) Show that for all valuation functions $v_1, \ldots, v_n$,
   $$\frac{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is an allocation of the cake}\}}{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is a \textit{proportional} allocation of the cake}\}} = O(\sqrt{n}).$$

   (b) Give a family of examples of $v_1, \ldots, v_n$ (one example for each value of n) such that
   $$\frac{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is an allocation of the cake}\}}{\sup\{\mathrm{sw}(\boldsymbol{A}) : \boldsymbol{A} \text{ is a \textit{proportional} allocation of the cake}\}} = \Omega(\sqrt{n}).$$

2. **The Partial Nash Algorithm** (25 points: 5/10/10)

   Consider a setting with a set M of m divisible goods and a set N of n players. Define an allocation $x \in \mathbb{R}^{n \times m}$ as an $n \times m$ matrix in which x_{ij} denotes the fraction of good j allocated to player i. Let $\mathscr{F} = \{x \,|\, x_{ij} \geq 0 \text{ and } \sum_i x_{ij} \leq 1\}$ denote the set of feasible allocations. Lastly, assume that each player i has a homogeneous valuation function $v_i : \mathbb{R}^{n\times m} \to \mathbb{R}$; i.e., each player i's valuation for the allocation $x' = c \cdot x$ satisfies $v_i(x') = c \cdot v_i(x)$ for any $c \geq 0$.

   We define *Nash Fairness (NF)* as follows. An allocation x^* is Nash fair if, for any other allocation x', the total proportional change in valuations is not positive; i.e.,
   $$\sum_{i \in N} \frac{v_i(x') - v_i(x^*)}{v_i(x^*)} \leq 0.$$

   It is known that an NF allocation exists, and, in fact, it is the unique allocation that maximizes the Nash product $\prod_{i \in N} v_i(x)$; you may rely on this fact in your solution.

   The Partial Nash (PN) algorithm first computes the NF allocation x^*, and then assigns each player i a fraction of x_i^* that depends on the extent to which the presence of i inconveniences the other players (i.e., decreases the value of other players).

Algorithm 1 Partial Nash (PN) algorithm

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1: Compute the NF allocation $x^*$ based on the reported bids.
2: For each player $i$, remove her and compute the NF allocation $x^*_{-i}$ that would occur in her absence.
3: Allocate to each player $i$ a fraction $f_i$ of everything she receives according to $x^*$, where
       $f_i = \frac{\prod_{i' \neq i} v_{i'}(x^*)}{\prod_{i' \neq i} v_{i'}(x^*_{-i})}$.
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(a) Show that the allocation produced by the PN algorithm is feasible.

(b) Prove that the PN algorithm is strategyproof; that is, no player can benefit by reporting untruthfully.

(c) Prove that the PN algorithm always yields an allocation such that, for every player i, $v_i(x) \geq \frac{1}{e} \cdot v_i(x^*)$; i.e., it provides a $1/e$ approximation of the optimal allocation.

Hint. *Given a sequence of n real numbers $d_1, \ldots, d_n$ such that $\sum_{i=1}^{n} d_i \leq 1$, $\prod_{i=1}^{n}(1+d_i) \leq (1+1/n)^n$.*

3. $1/2$-**EFX** (30 points: 20/10)

Consider a setting with n players and a set G of m indivisible goods. Let $\vec{v} = (v_1, \ldots, v_n)$ represent the additive valuation functions of the n players and assume that all players have positive valuations for all items.

As mentioned in class, it is an open problem whether EFX allocations exist in settings with more than two players; therefore, in this problem, we consider a relaxation of EFX. Define an allocation to be $1/2$-EFX if, for any two players i and j, i's value for j's bundle minus any good is at most twice i's value for her own bundle.

Definition 1. *An allocation $\boldsymbol{A}$ is $1/2$-EFX if, for all i and j, $\forall g \in A_j$, $v_i(A_i) \geq (1/2) \cdot v_i(A_j \setminus \{g\})$.*

Consider the following algorithm for finding a $1/2$-EFX allocation for n players.

Algorithm 2 $1/2$-EFX Allocation

```
Require: $n, G, (v_1, \ldots, v_n)$                          ▷ Input: players, goods, and valuation functions
 1: $P \leftarrow G$                                           ▷ Initialize: all goods in pool
 2: for $i \in [n]$ do
 3:     $A_i \leftarrow \emptyset$                              ▷ Initialize: all players start with no goods
 4: end for
 5: while $P \neq \emptyset$ do                                ▷ Repeat while pool not empty
 6:     $g^* \leftarrow \text{pop}(P)$                          ▷ Remove an arbitrary good from the pool
 7:     $j \leftarrow \text{FindUnenviedPlayer}(\boldsymbol{A})$ ▷ and give it to an unenvied player
 8:     $A_j \leftarrow A_j \cup \{g^*\}$
 9:     if $\exists i \in [n], g \in A_j$ such that $v_i(A_i) < \frac{1}{2}v_i(A_j \setminus \{g\})$ then   ▷ if this breaks 1/2-EFX
10:         $P \leftarrow P \cup A_i$                           ▷ Return $i$'s old allocation to the pool
11:         $A_j \leftarrow A_j \setminus \{g^*\}$
12:         $A_i \leftarrow \{g^*\}$                            ▷ and give $i$ $\{g^*\}$
13:     end if
14:     $A \leftarrow \text{RemoveEnvyCycles}(\boldsymbol{A})$  ▷ Ensure the envy graph is acyclic
15: end while
```

You may assume that FindUnenviedPlayer always returns an unenvied player (if one exists). Furthermore, given a $1/2$-EFX allocation with an envy graph that contains cycles, RemoveEnvyCycles returns a $1/2$-EFX allocation with an acyclic envy graph. Note that, in this algorithm, RemoveEnvyCycles ensures that when FindUnenviedPlayer is called, an unenvied player does exist.

(a) Prove that at the beginning of each iteration of the while loop, the partial allocation is $1/2$-EFX.

(b) Prove that the algorithm terminates.

Hint. *Consider what happens to the social welfare in each round.*

4. **Random EF Allocations** (20 points)

   Consider a setting with m indivisible goods and n players with additive valuations, where each player has value drawn i.i.d. and uniformly at random from $[0,1]$ for each good. Prove that for any constant number of players $n \geq 2$, the probability that an EF allocation exists goes to 1 as the number of goods m goes to infinity (that is, an EF allocation exists *with high probability*).

   Hint. ***k-th order statistics.*** *Given $X_1,\ldots,X_n$ i.i.d. random variables drawn uniformly at random from $[0,1]$, the expectation of the k-th smallest value is $\frac{k}{n+1}$.*

   Hint. ***Hoeffding's inequality.*** *Hoeffding's inequality provides an upper bound on the probability that the sum of (bounded) independent random variables deviates from its expected value by more than a prescribed amount. Let $X_1,\ldots,X_n$ be independent random variables where each X_i is bounded by $[a_i,b_i]$. Now, define the random variable X as the sum of all X_i's: $X = X_1 + \cdots + X_n$. Hoeffding's inequality states that*

$$\Pr\left[|X - \mathbb{E}[X]| \geq t\right] \leq 2\exp\left(\frac{-2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right).$$