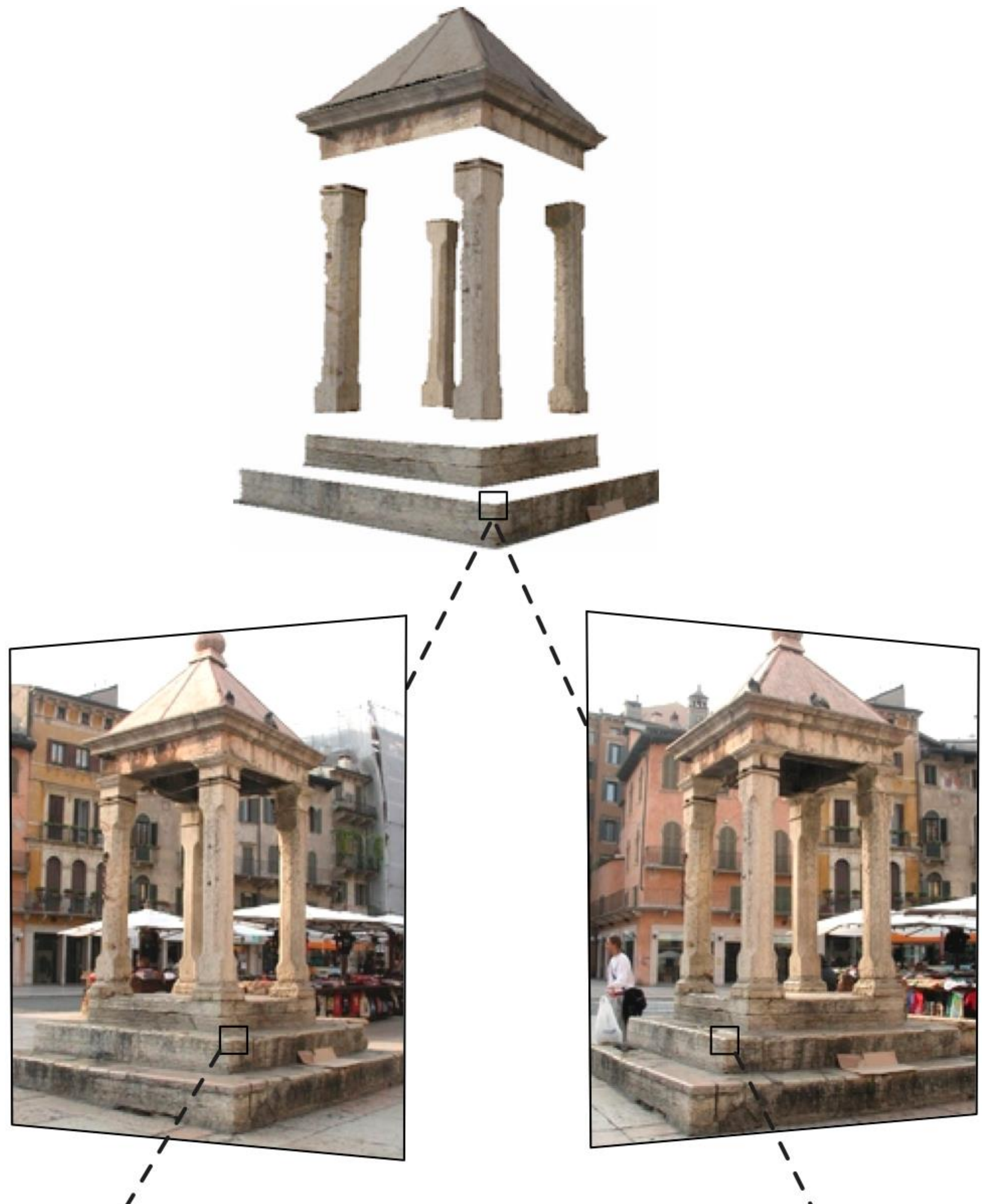
Two-view geometry



16-385 Computer Vision
Spring 2019, Lecture 10

http://www.cs.cmu.edu/~16385/

Course announcements

- Homework 2 is due on Wednesday.
  - How many of you have looked at/started/finished homework 2?

- Homework 3 will be released on Wednesday and will be due Friday, March 8th.
  - Do you prefer Sunday, March 10th?

- Yannis has extra office hours on Tuesday 3-5pm.

- The Hartley-Zisserman book is available online for free from CMU's library.

Overview of today’s lecture

- Leftover from previous lecture: Other types of cameras, calibration.
- Triangulation.
- Epipolar geometry.
- Essential matrix.
- Fundamental matrix.
- 8-point algorithm.

Slide credits

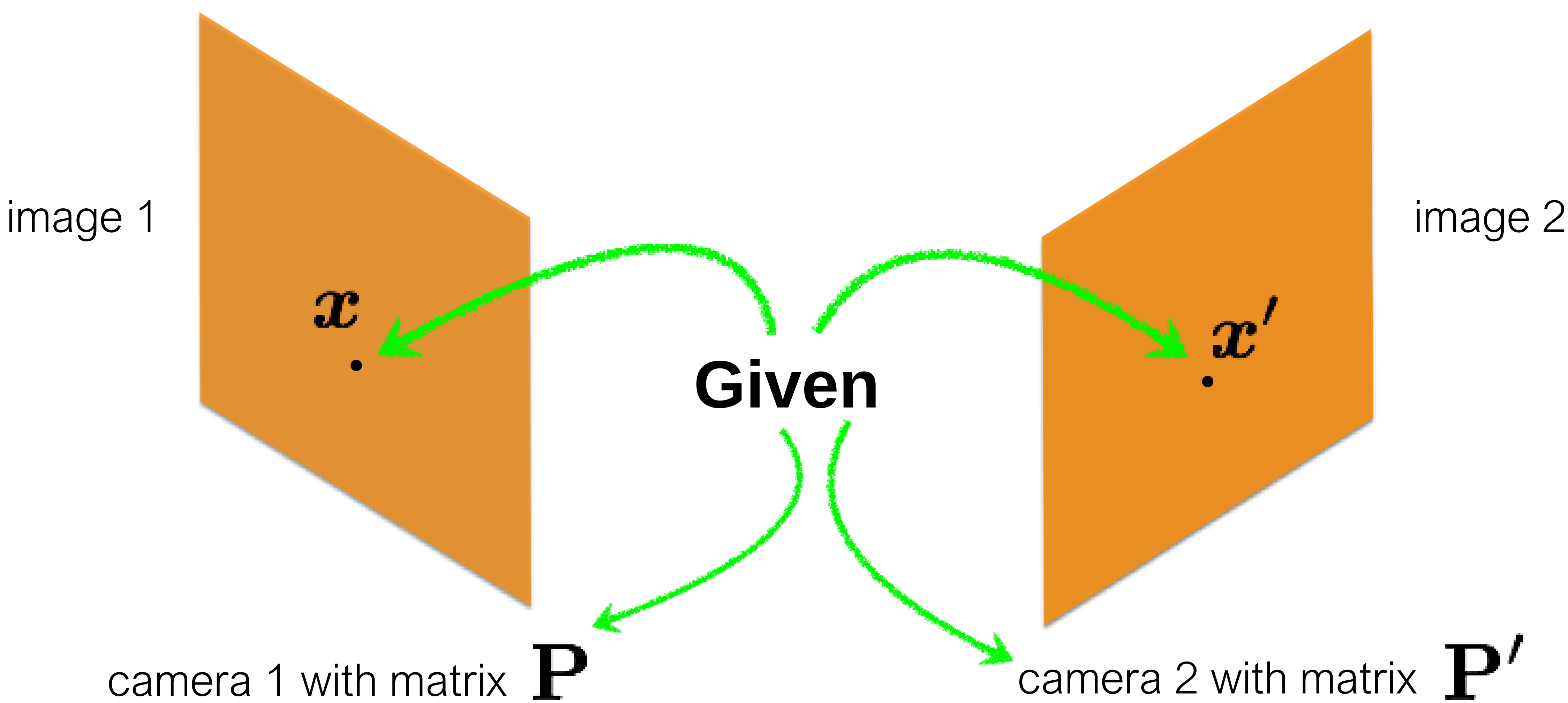
Most of these slides were adapted from:

- Kris Kitani (16-385, Spring 2017).

Triangulation

	Structure (scene geometry)	Motion (camera geometry)	Measurements
Pose Estimation	known	estimate	3D to 2D correspondences
Triangulation	estimate	known	2D to 2D coorespondences
Reconstruction	estimate	estimate	2D to 2D coorespondences

Triangulation



Triangulation

Which 3D points map to **x**?

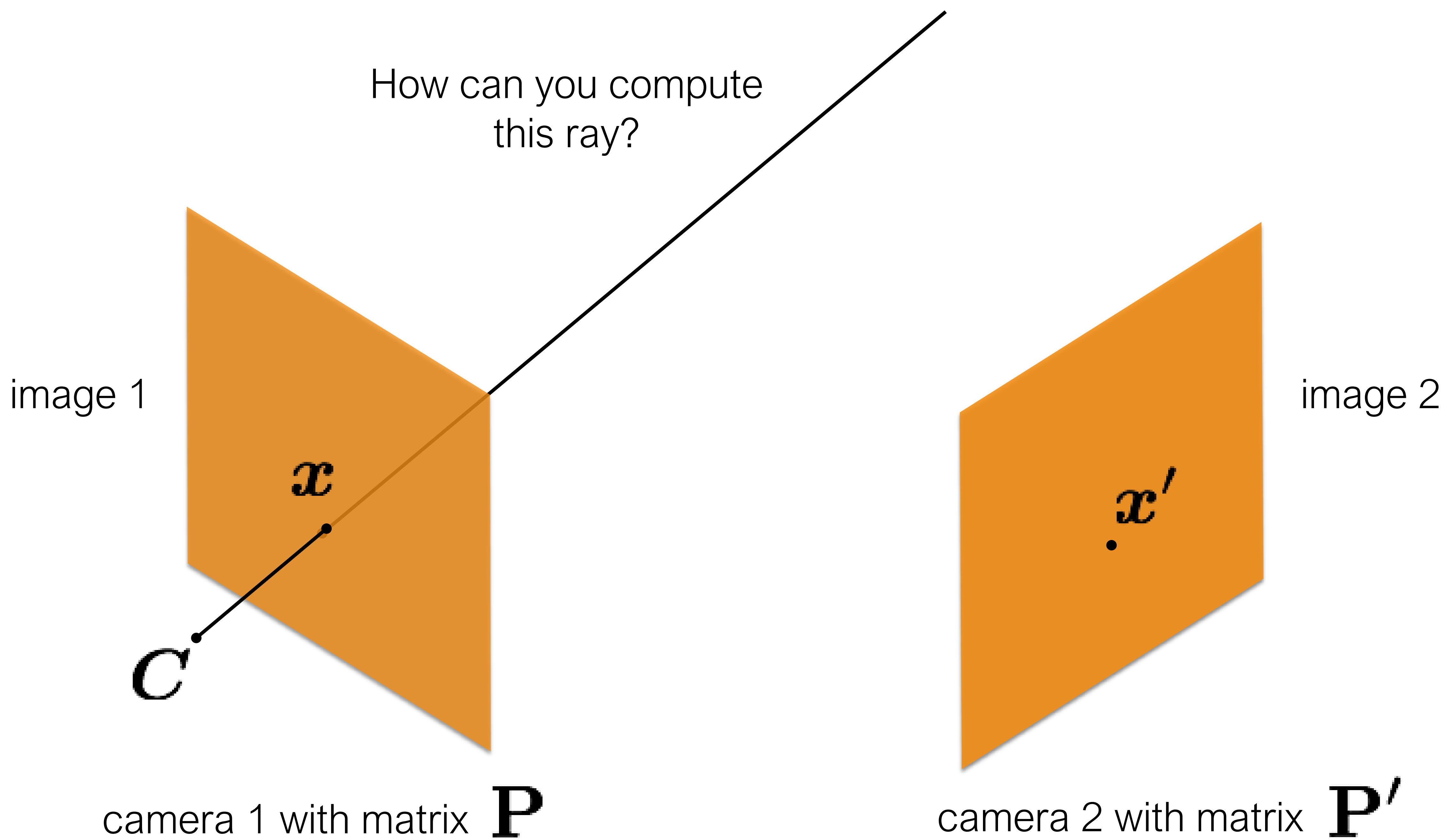
image 1

image 2

camera 1 with matrix $\mathbf{P}$

camera 2 with matrix $\mathbf{P}'$

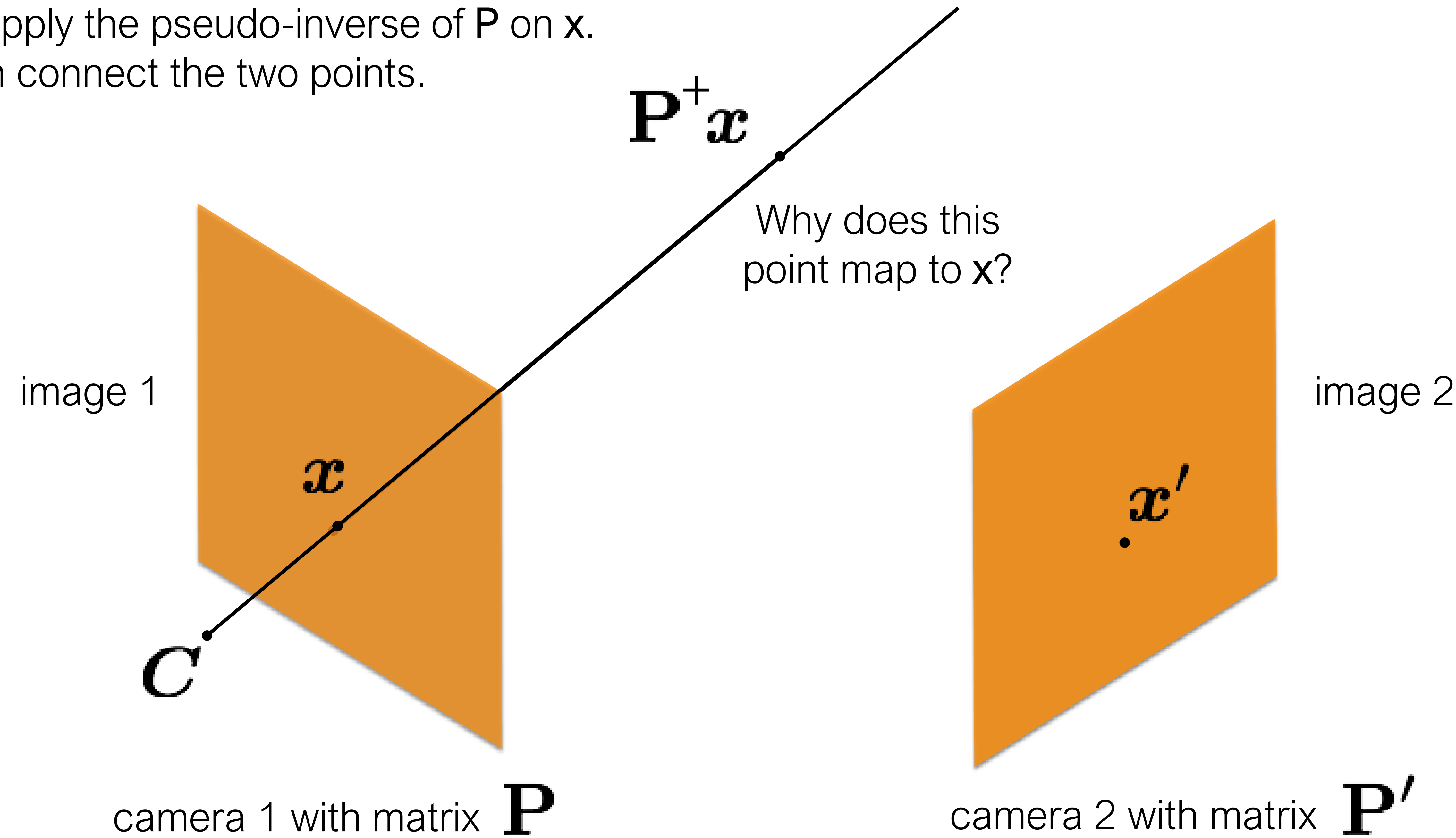
Triangulation



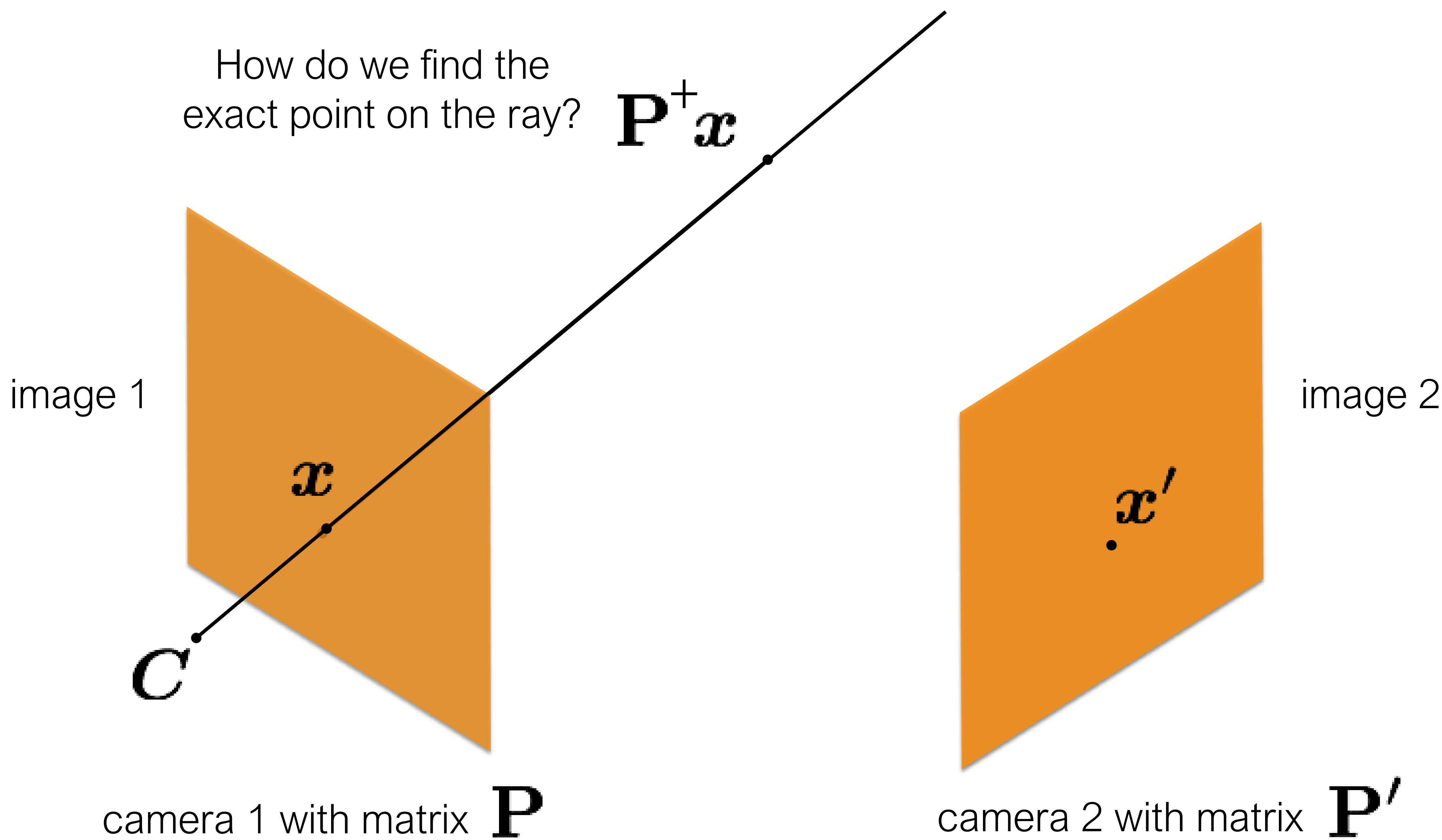
Triangulation

Create two points on the ray:
1) find the camera center; and
2) apply the pseudo-inverse of P on x.

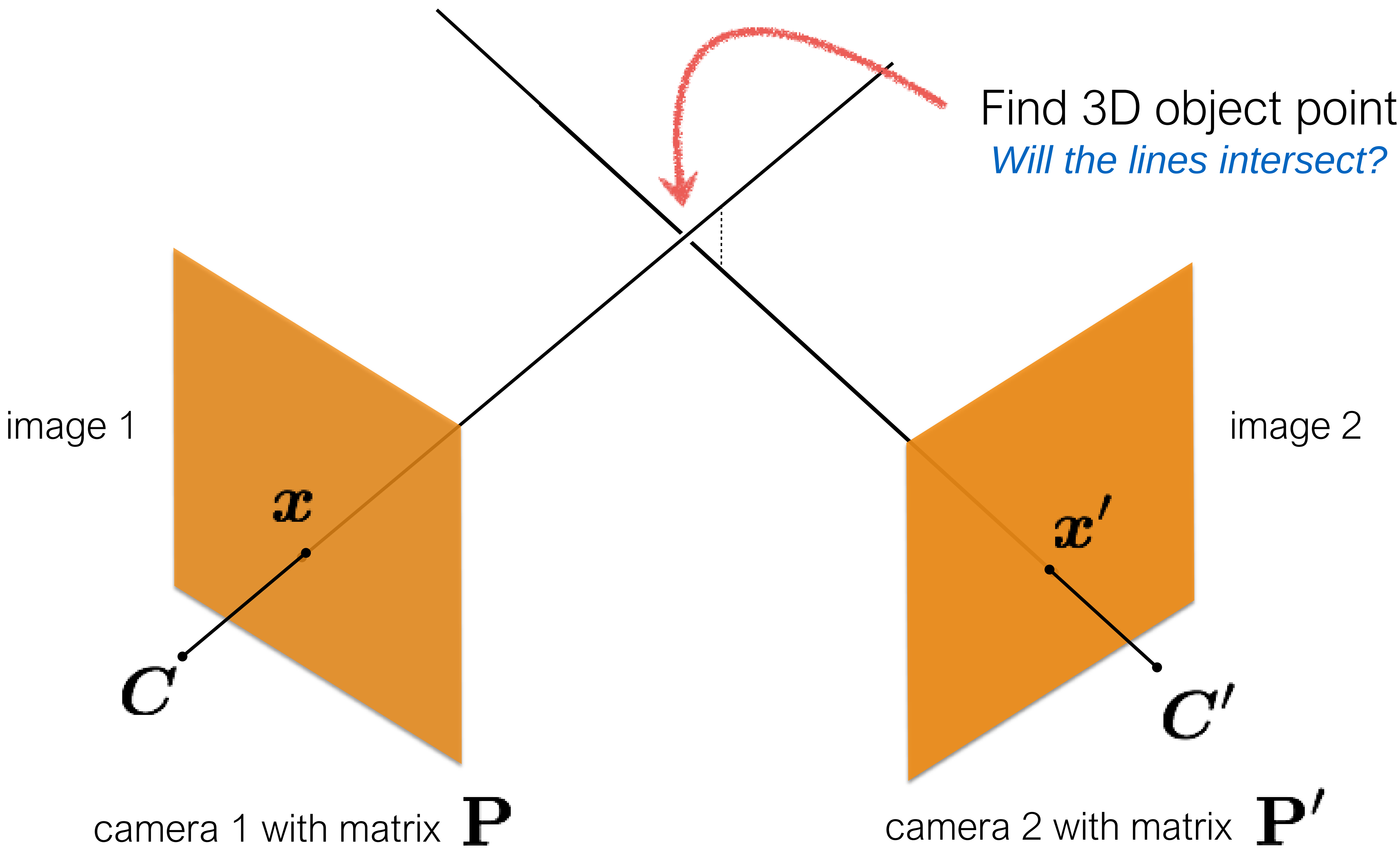
Then connect the two points.



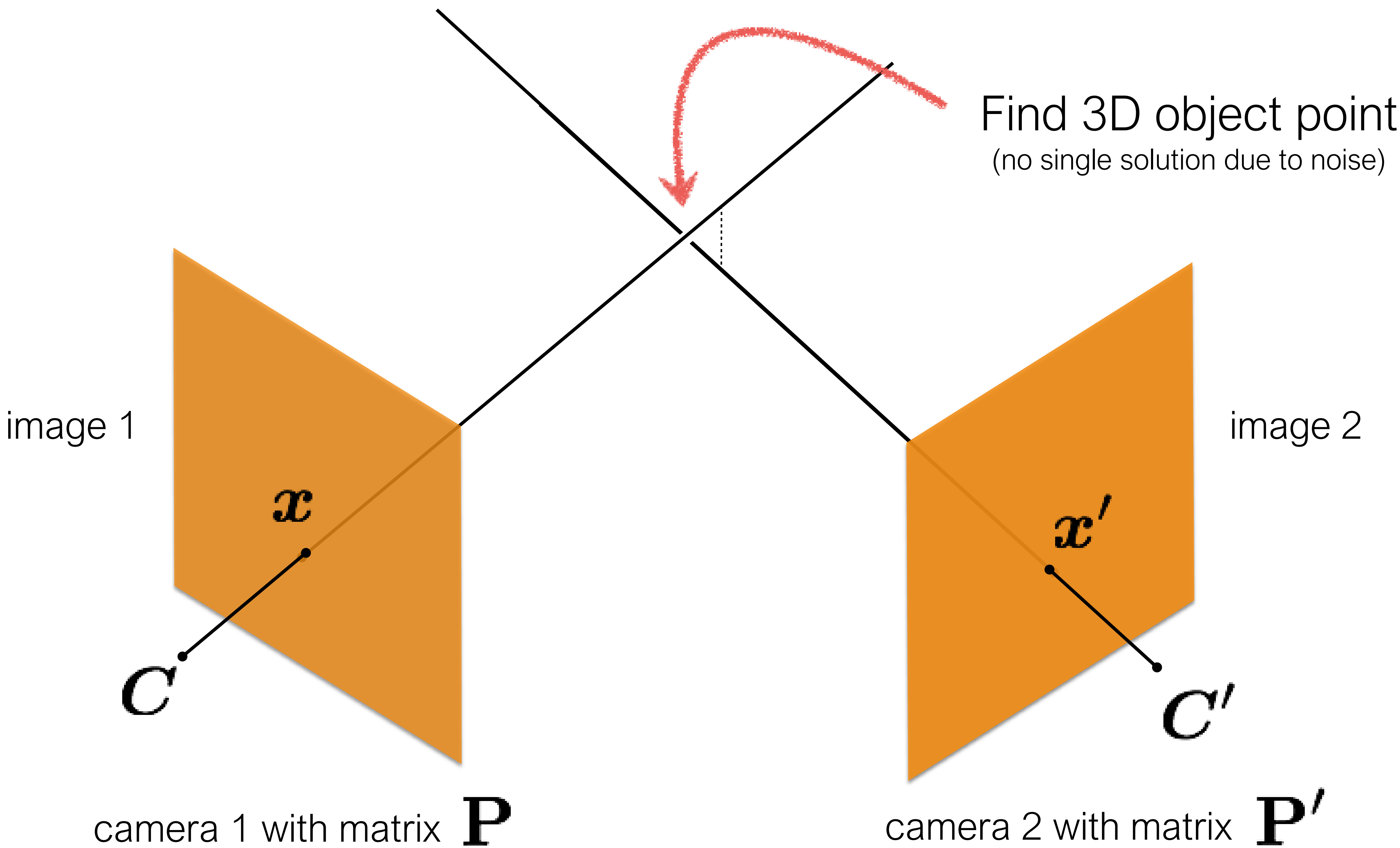
Triangulation



Triangulation



Triangulation



Triangulation

Given a set of (noisy) matched points

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

and camera matrices

$$\mathbf{P}, \mathbf{P}'$$

Estimate the 3D point

$$\mathbf{X}$$

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(homorogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How do we solve for unknowns in a similarity relation?

Remove scale factor, convert to linear system and solve with

$$\mathbf{x} = \mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Also, this is a similarity relation because it involves homogeneous coordinates

$$\mathbf{x} = \alpha\mathbf{P}\boldsymbol{X}$$

(homogeneous coordinate)

Same ray direction but differs by a scale factor

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

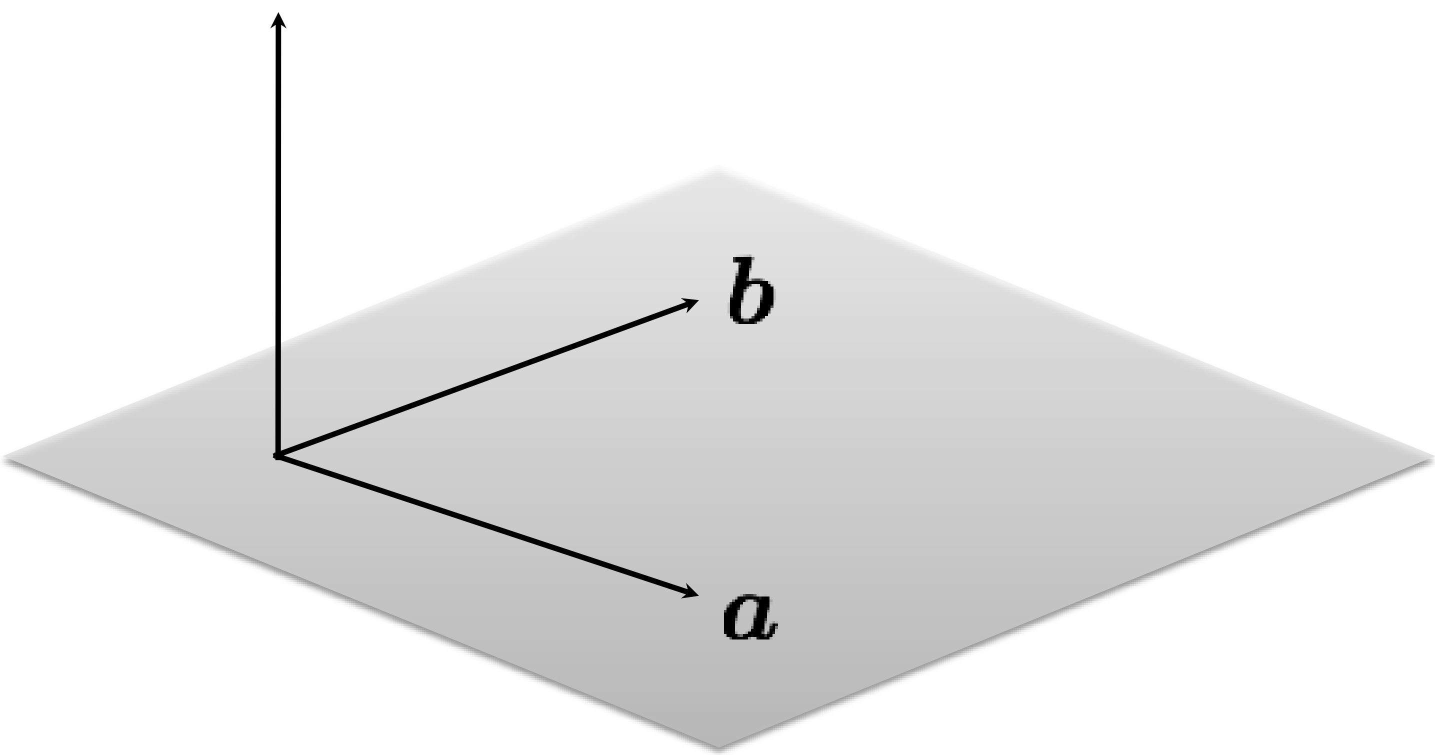
How do we solve for unknowns in a similarity relation?

Remove scale factor, convert to linear system and solve with SVD!

Recall: Cross Product

Vector (cross) product
takes two vectors and returns a vector perpendicular to both

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$



$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

cross product of two vectors in the same direction is zero

$$\boldsymbol{a} \times \boldsymbol{a} = 0$$

remember this!!!

$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

$$\mathbf{x} = \alpha \mathbf{P} \boldsymbol{X}$$

Same direction but differs by a scale factor

$$\mathbf{x} \times \mathbf{P} \boldsymbol{X} = \mathbf{0}$$

Cross product of two vectors of same direction is zero
(this equality removes the scale factor)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{——} & \boldsymbol{p}_1^\top & \text{——} \\ \text{——} & \boldsymbol{p}_2^\top & \text{——} \\ \text{——} & \boldsymbol{p}_3^\top & \text{——} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \text{---} & \boldsymbol{p}_1^\top & \text{---} \\ \text{---} & \boldsymbol{p}_2^\top & \text{---} \\ \text{---} & \boldsymbol{p}_3^\top & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you [redacted] equations

Using the fact that the cross product should be zero

$$\mathbf{x} \times \mathbf{P}\boldsymbol{X} = \mathbf{0}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \times \begin{bmatrix} \boldsymbol{p}_1^\top \boldsymbol{X} \\ \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \\ x\boldsymbol{p}_2^\top \boldsymbol{X} - y\boldsymbol{p}_1^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Third line is a linear combination of the first and second lines.
(x times the first line plus y times the second line)

One 2D to 3D point correspondence give you 2 equations

$$\begin{bmatrix} y\boldsymbol{p}_3^\top \boldsymbol{X} - \boldsymbol{p}_2^\top \boldsymbol{X} \\ \boldsymbol{p}_1^\top \boldsymbol{X} - x\boldsymbol{p}_3^\top \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}_i \boldsymbol{X} = \mathbf{0}$$

Now we can make a system of linear equations
(two lines for each 2D point correspondence)

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

sanity check! dimensions?

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

Concatenate the 2D points from both images

$$\begin{bmatrix} y\boldsymbol{p}_3^\top - \boldsymbol{p}_2^\top \\ \boldsymbol{p}_1^\top - x\boldsymbol{p}_3^\top \\ y'\boldsymbol{p}_3'^\top - \boldsymbol{p}_2'^\top \\ \boldsymbol{p}_1'^\top - x'\boldsymbol{p}_3'^\top \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{X} = \mathbf{0}$$

How do we solve homogeneous linear system?

S V D !

Recall: Total least squares

(**Warning:** change of notation. x is a vector of parameters!)

$$E_{\mathrm{TLS}} = \sum_i (\boldsymbol{a}_i \boldsymbol{x})^2$$

$$= \|\mathbf{A}\boldsymbol{x}\|^2$$ (matrix form)

$$\|\boldsymbol{x}\|^2 = 1$$ constraint

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

→ minimize $\dfrac{\|\mathbf{A}\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2}$

(Rayleigh quotient)

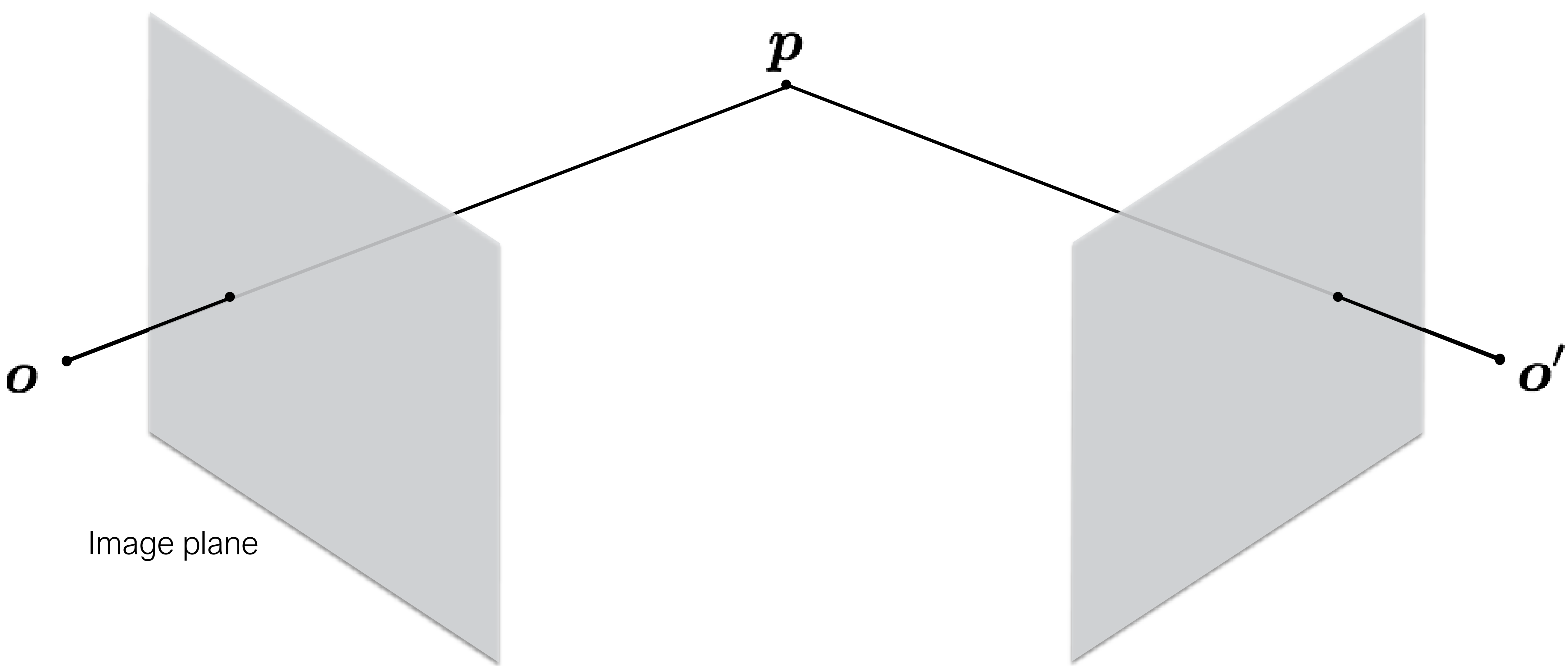
Solution is the eigenvector corresponding to smallest eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

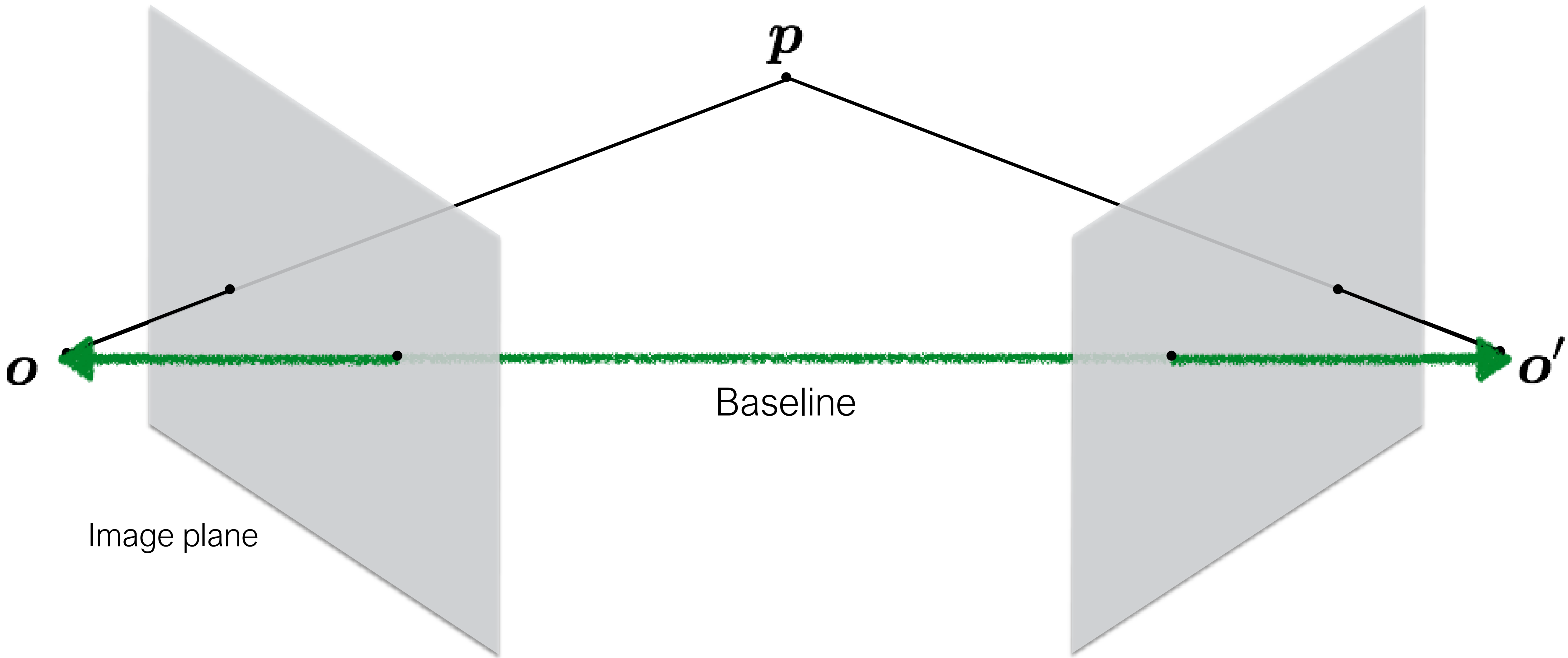
	Structure (scene geometry)	Motion (camera geometry)	Measurements
Pose Estimation	known	estimate	3D to 2D correspondences
Triangulation	estimate	known	2D to 2D coorespondences
Reconstruction	estimate	estimate	2D to 2D coorespondences

Epipolar geometry

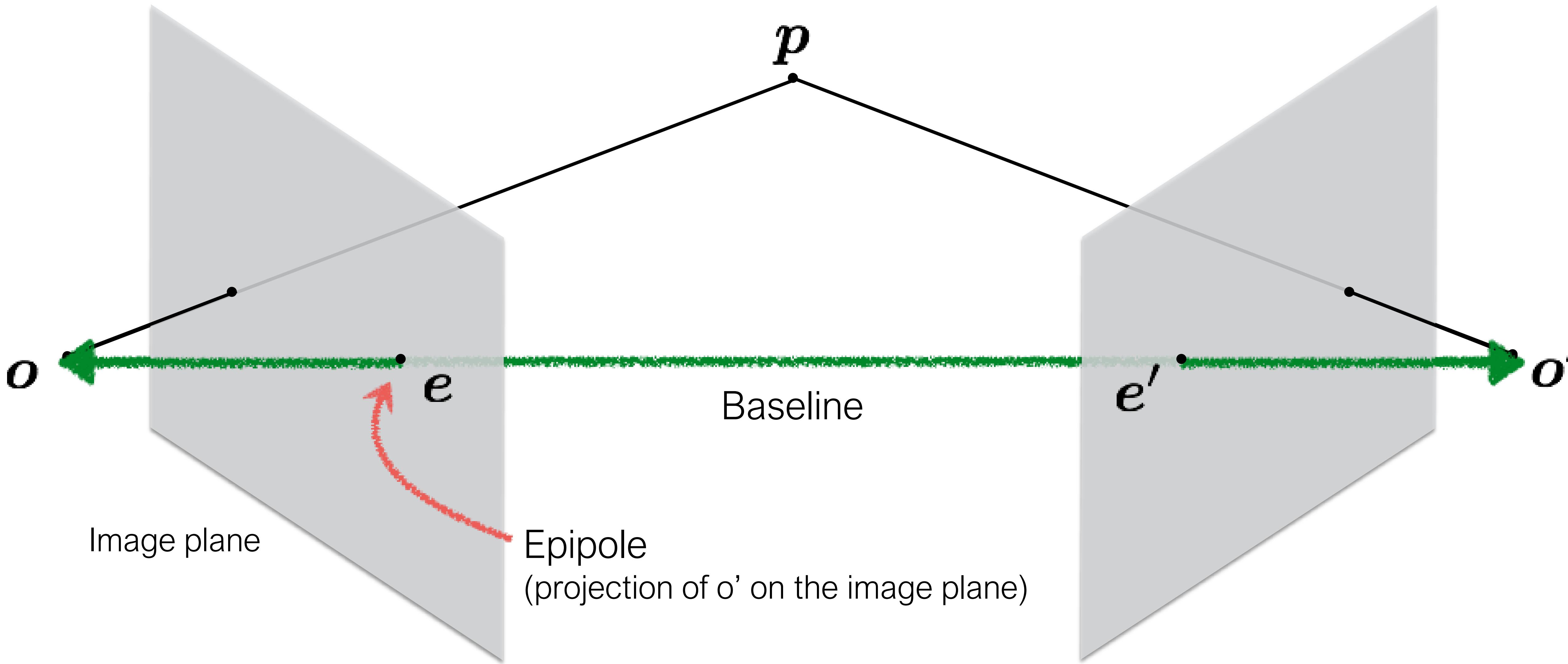
Epipolar geometry



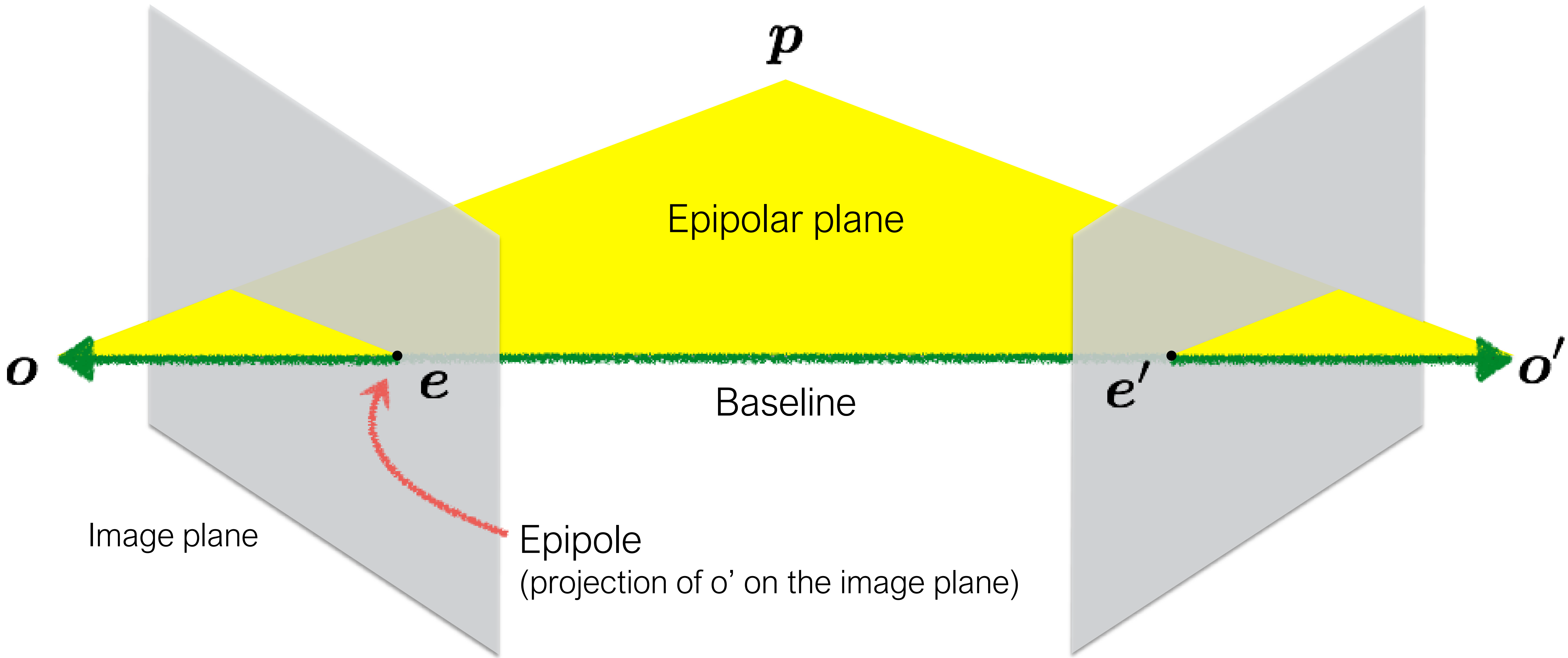
Epipolar geometry



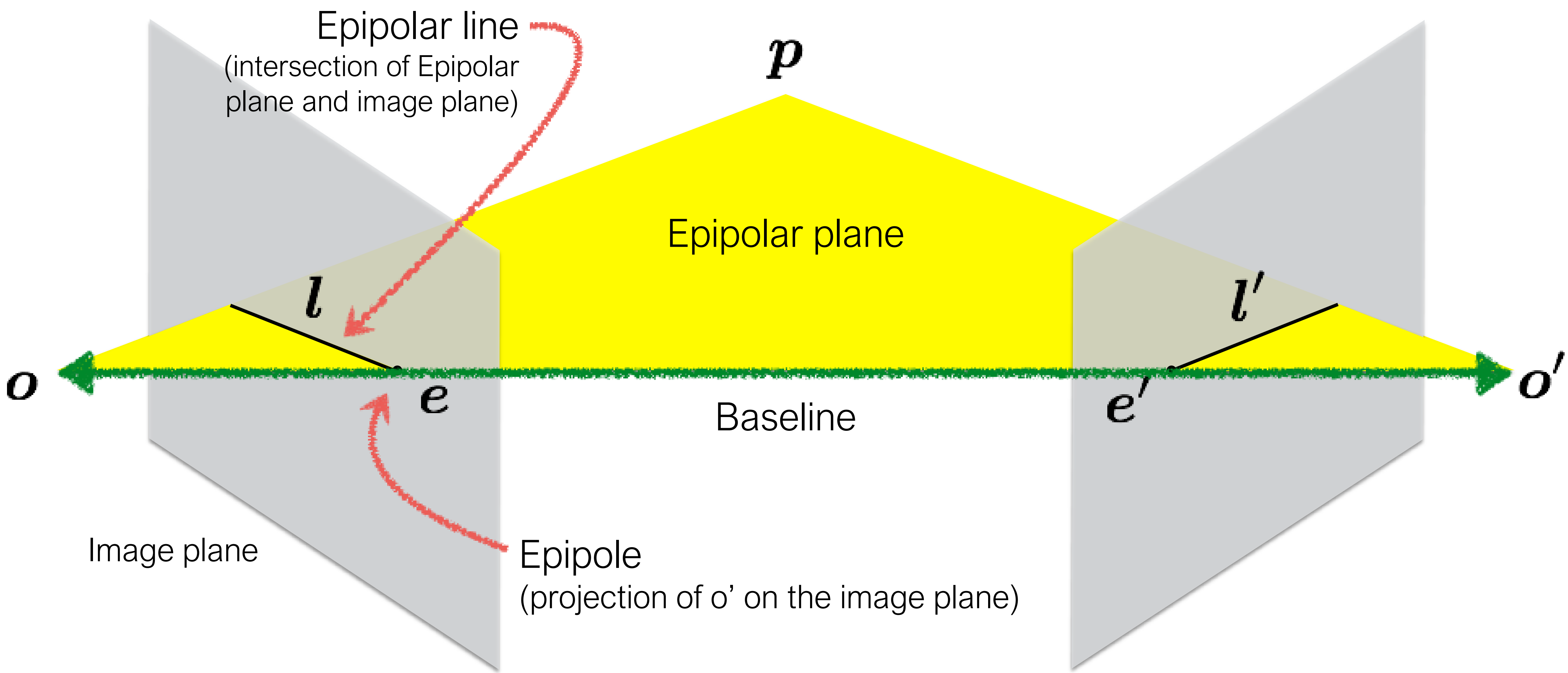
Epipolar geometry



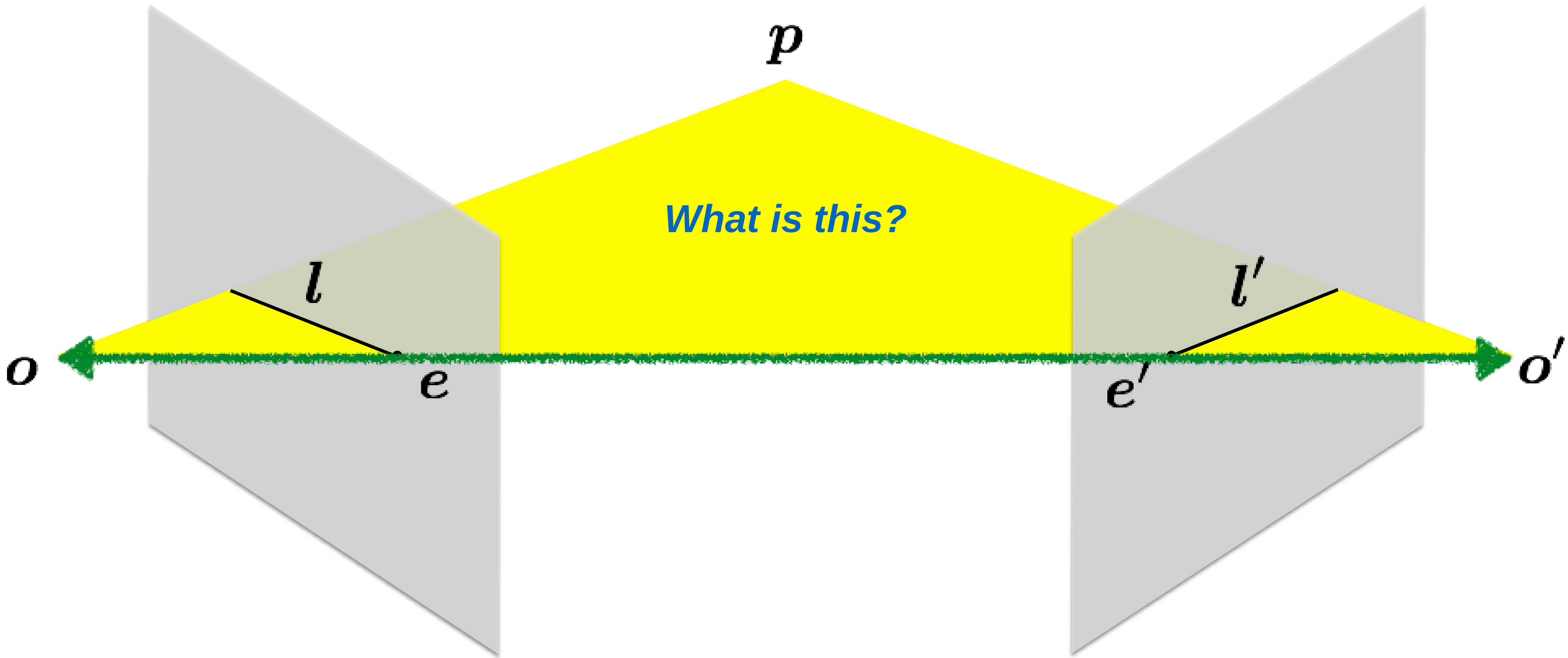
Epipolar geometry



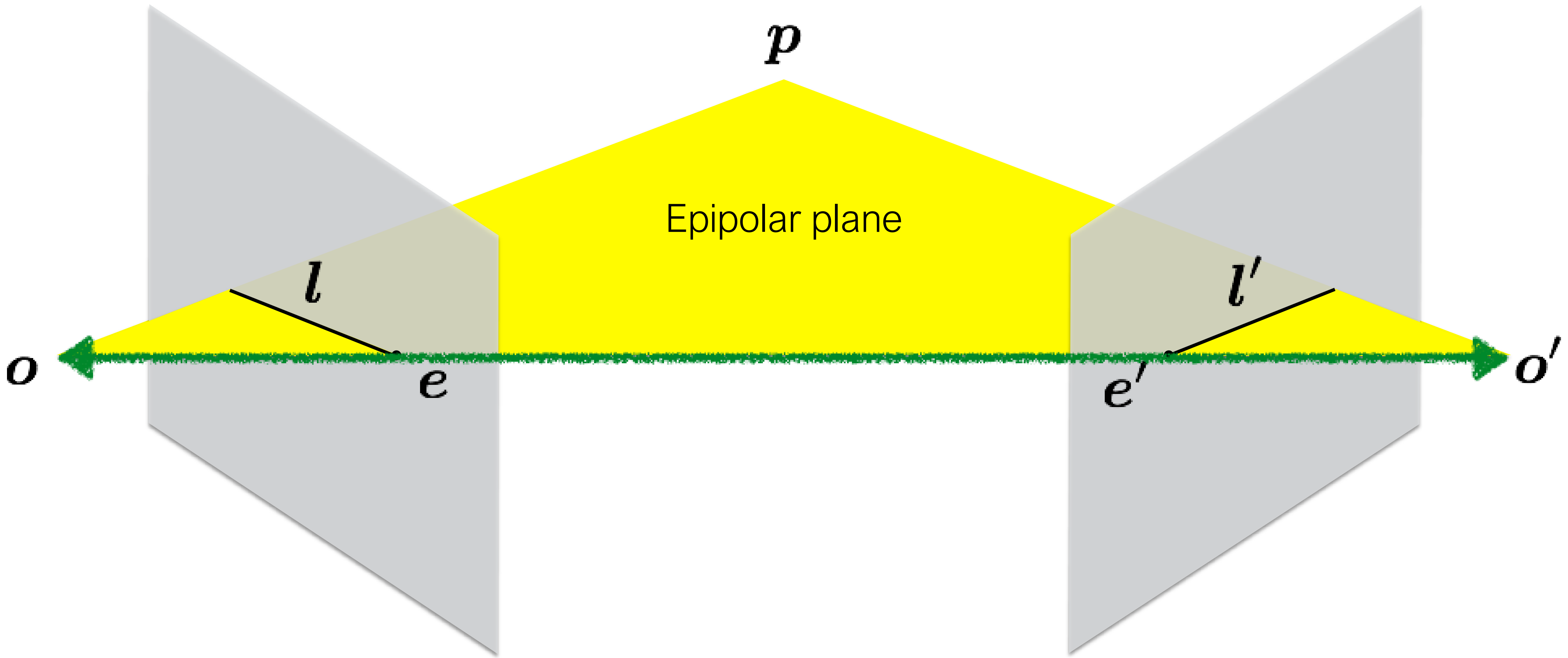
Epipolar geometry



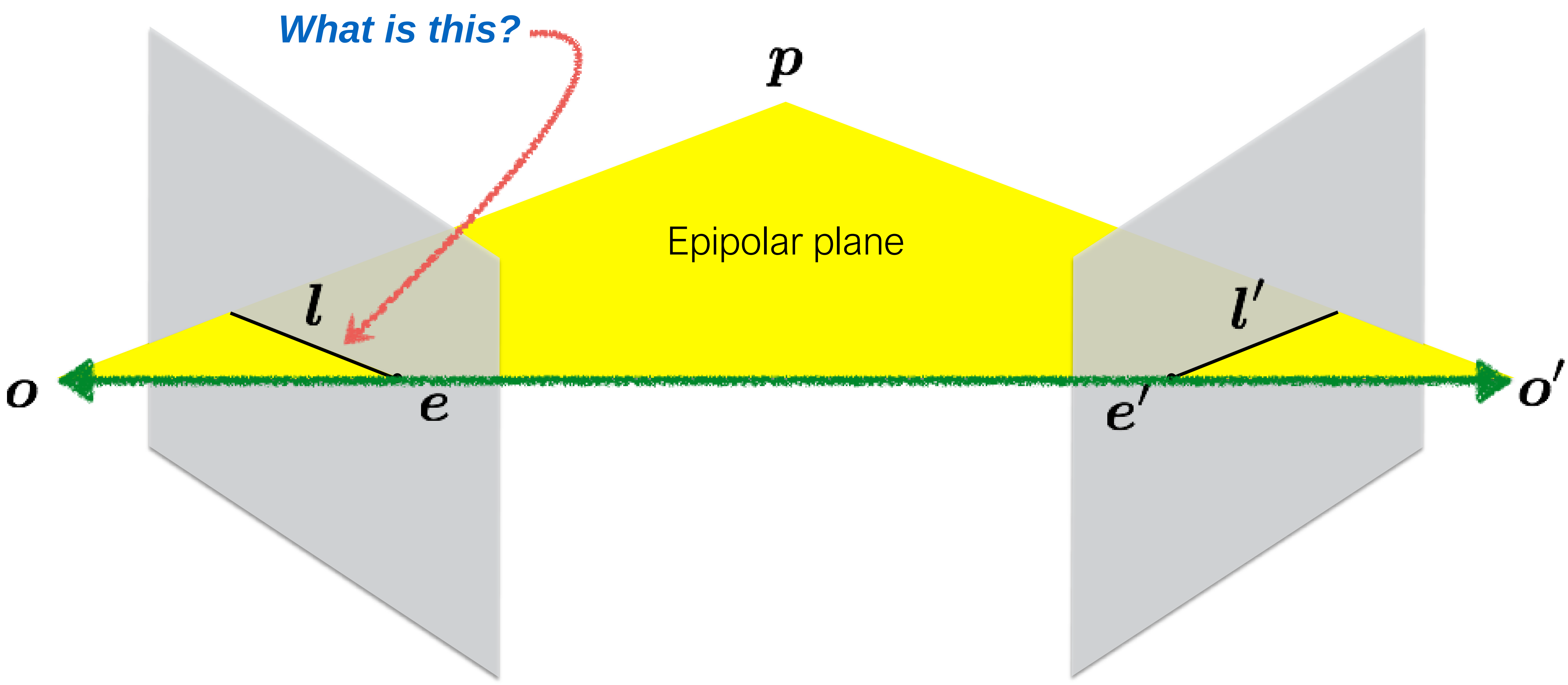
Quiz



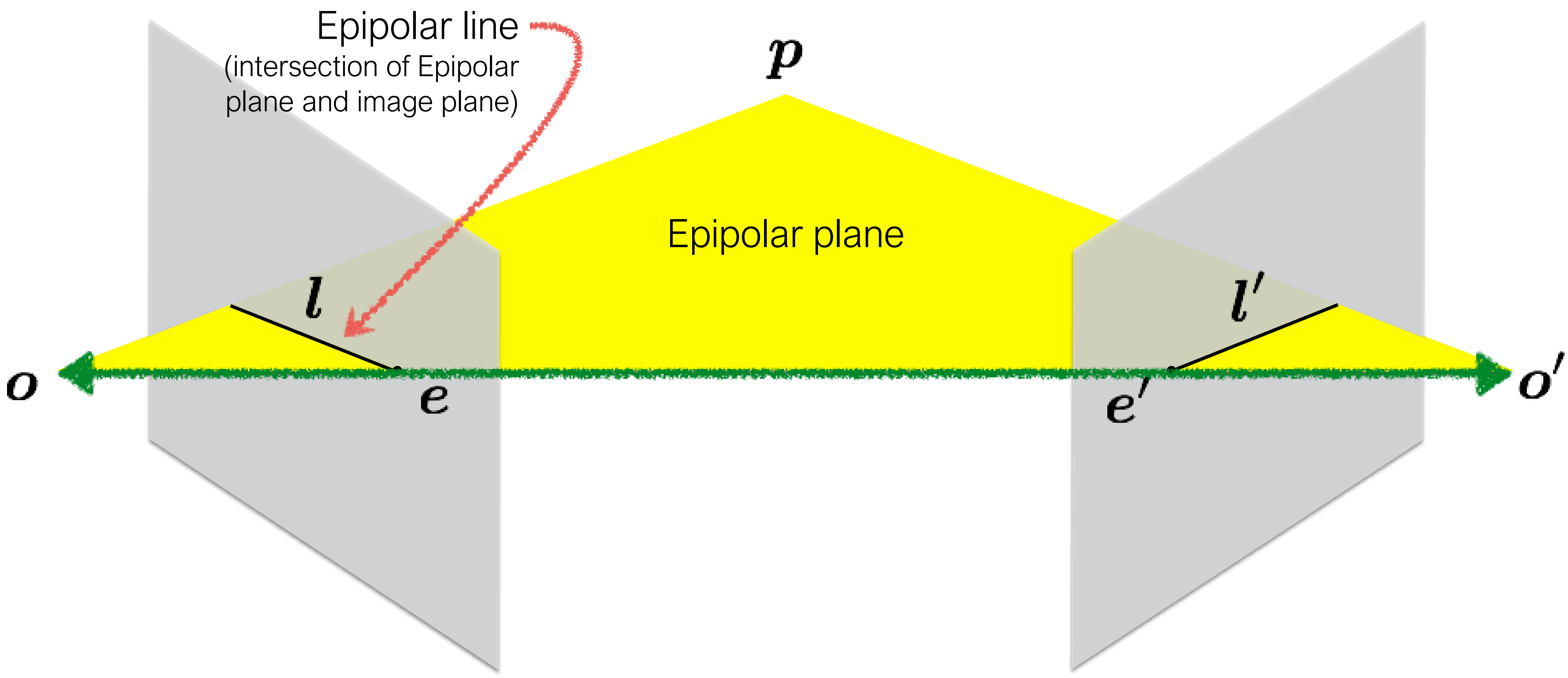
Quiz



Quiz

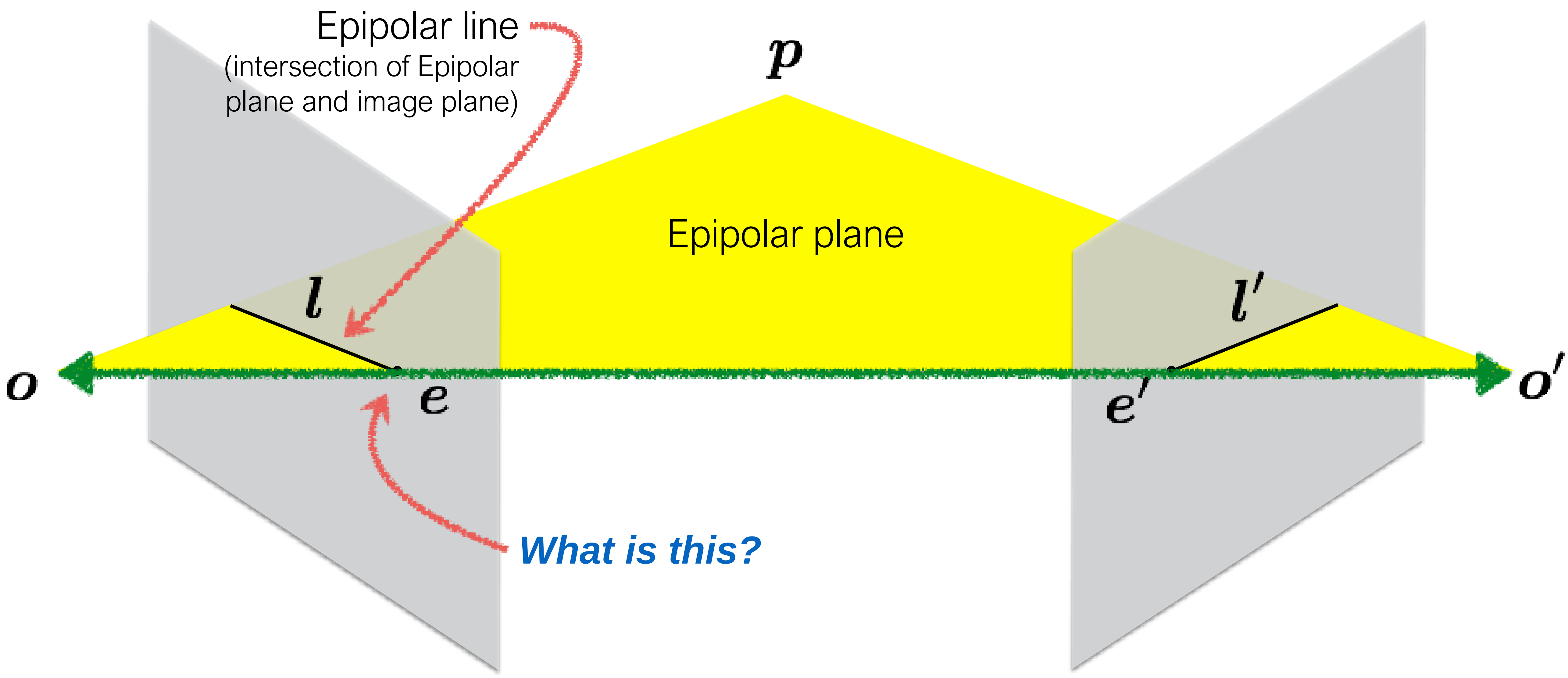


Quiz



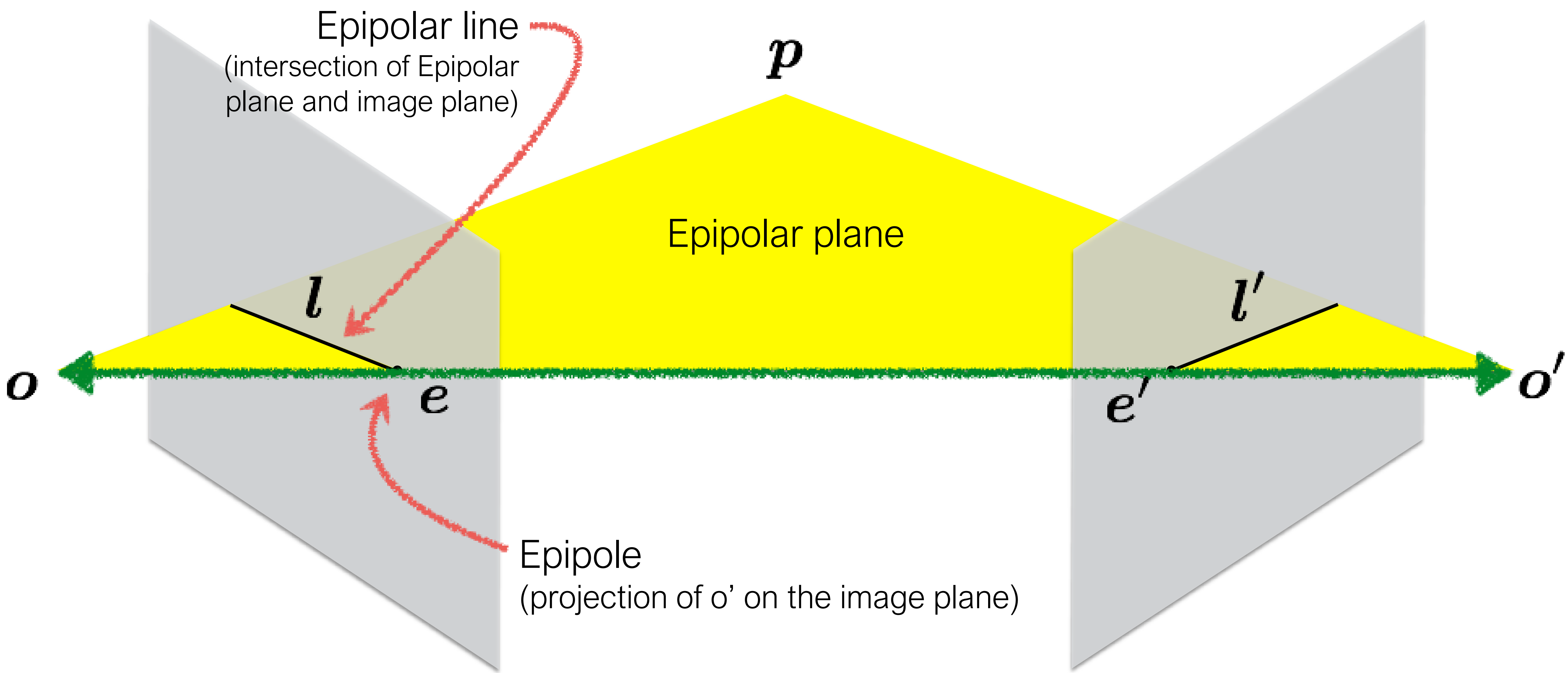
Epipolar line
(intersection of Epipolar plane and image plane)
p
Epipolar plane
l
l'
o
e
e'
o'

Quiz



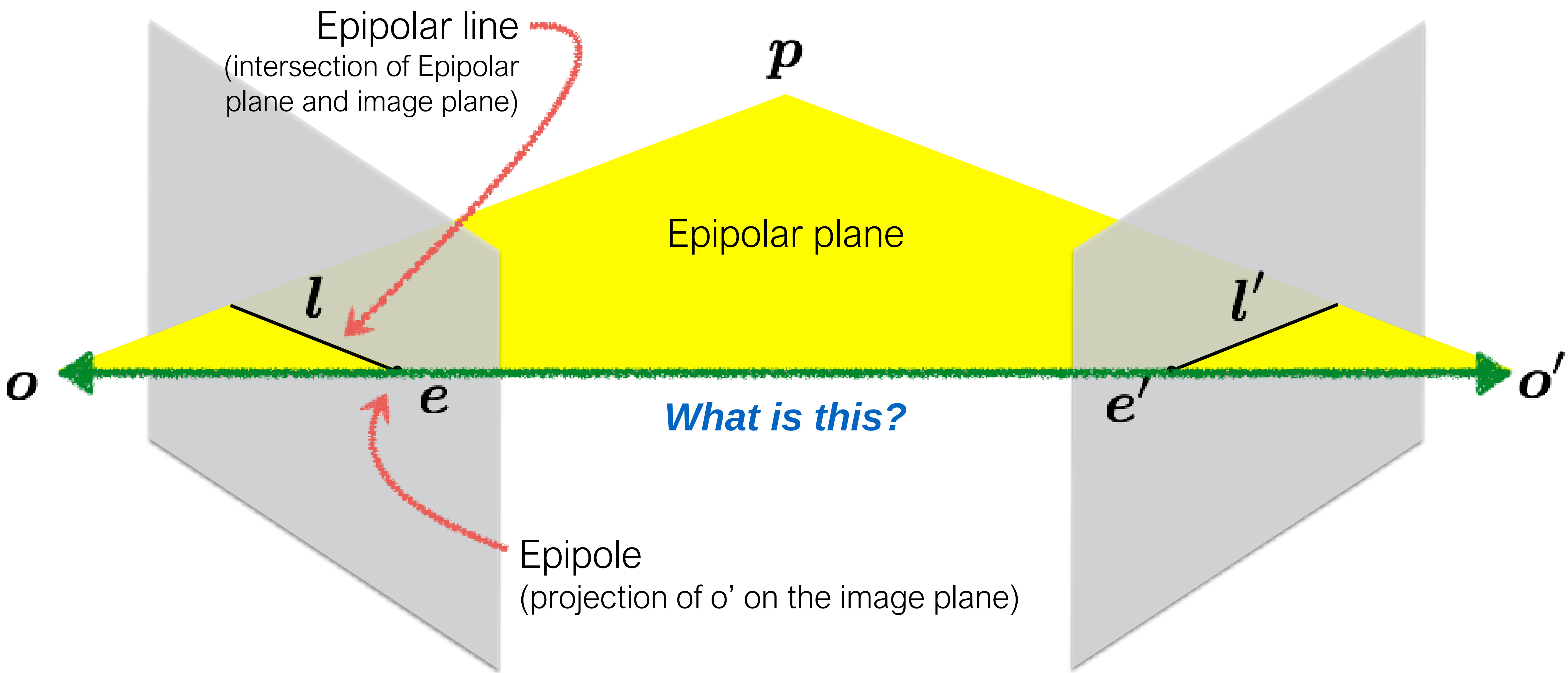
Epipolar line
(intersection of Epipolar plane and image plane)
$\boldsymbol{p}$
Epipolar plane
$\boldsymbol{l}$
$\boldsymbol{l}'$
$\boldsymbol{o}$
$\boldsymbol{e}$
$\boldsymbol{e}'$
$\boldsymbol{o}'$
What is this?

Quiz

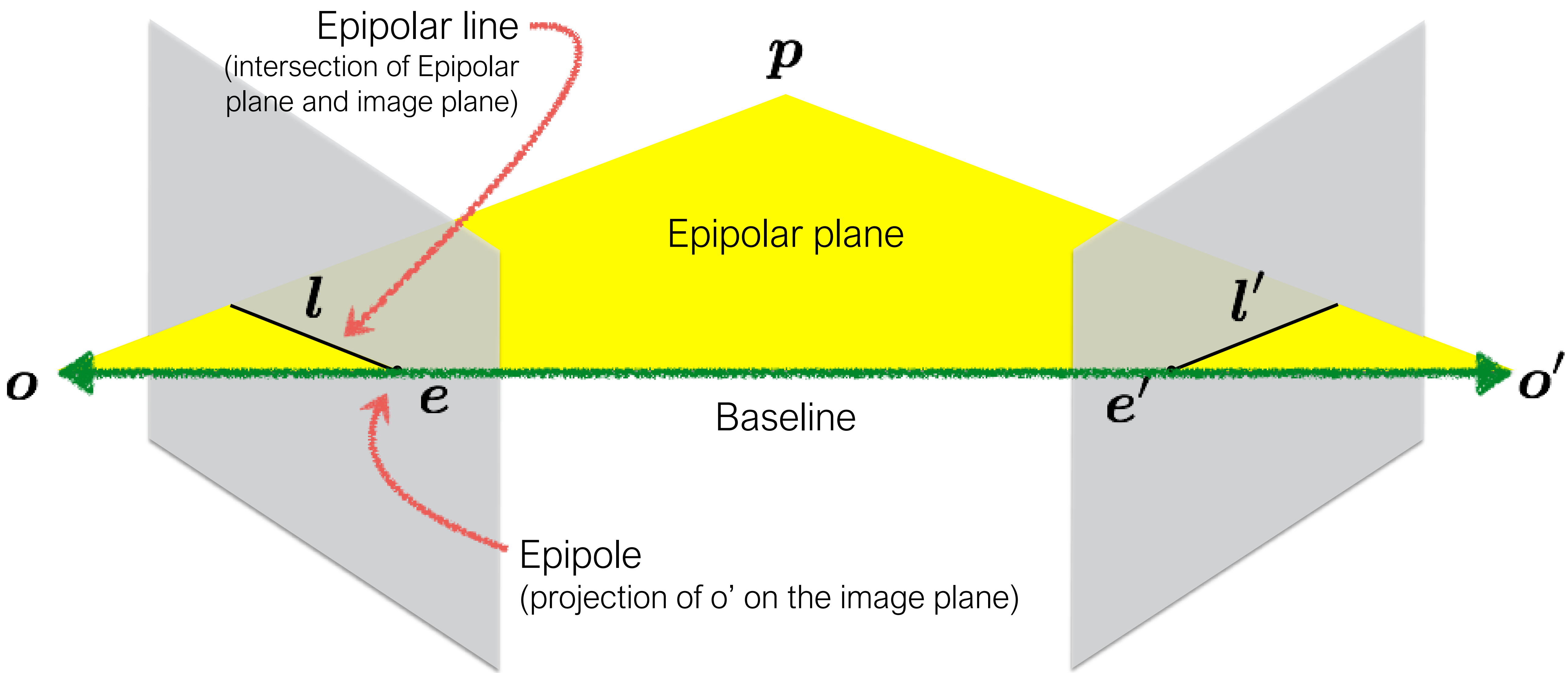


Epipolar line
(intersection of Epipolar plane and image plane)
p
Epipolar plane
l
l'
o
e
e'
o'
Epipole
(projection of o' on the image plane)

Quiz

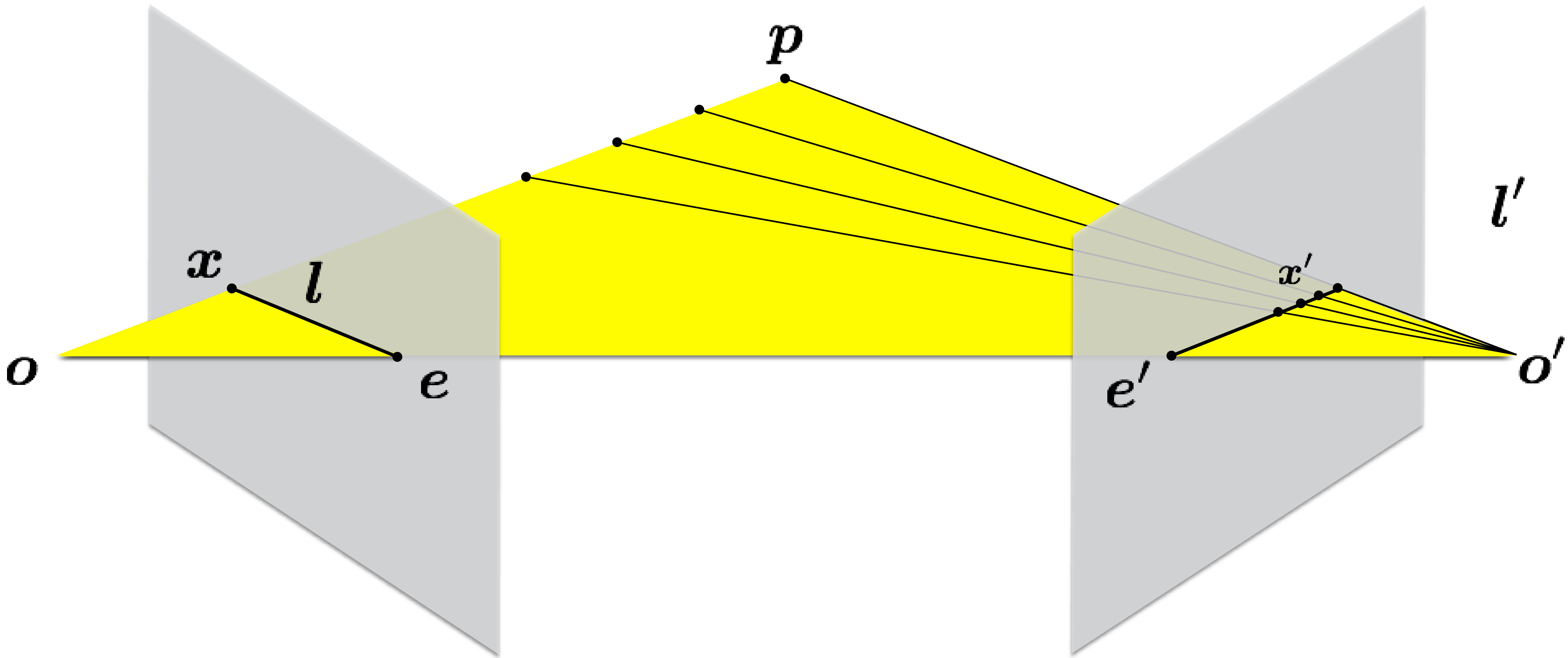

Epipolar line
(intersection of Epipolar plane and image plane)
p
Epipolar plane
l
l'
o
e
e'
o'
What is this?
Epipole
(projection of o' on the image plane)

Quiz



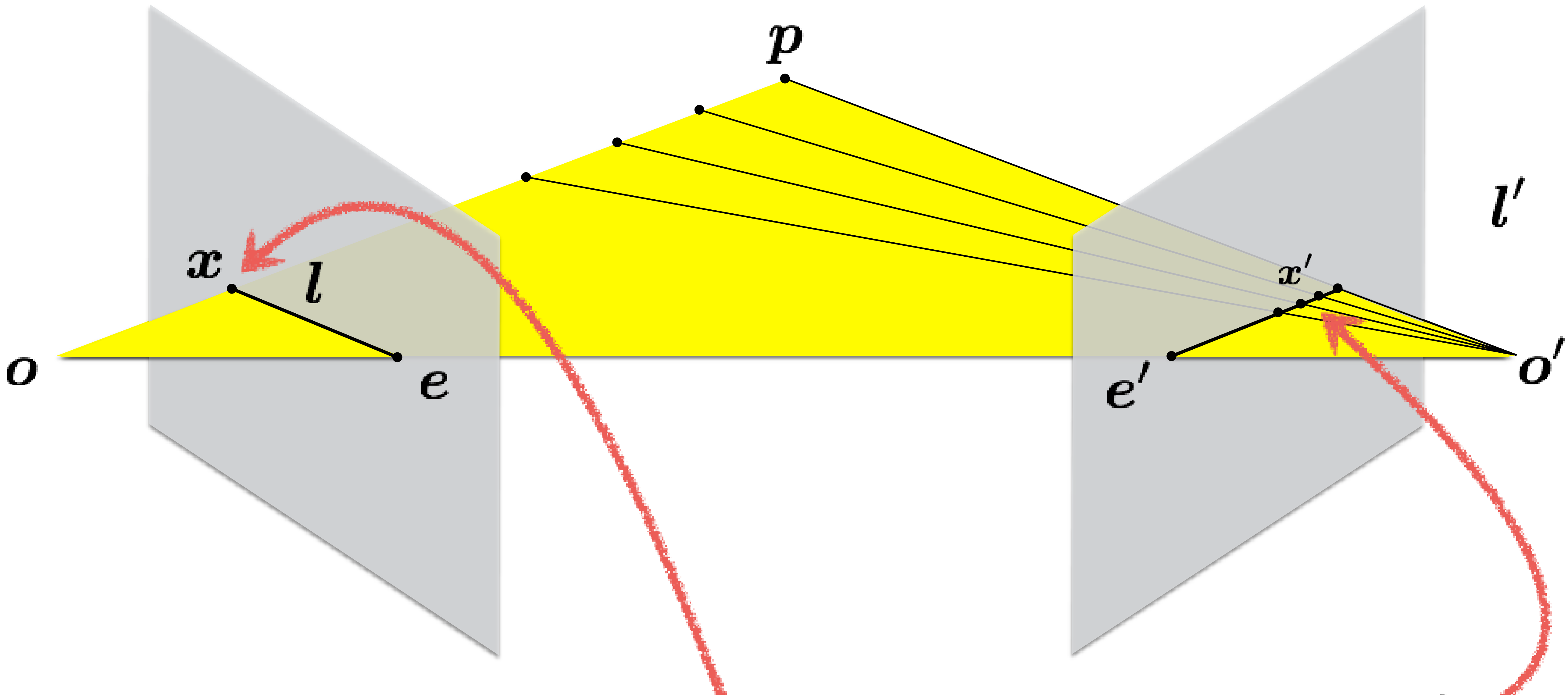
Epipolar line
(intersection of Epipolar plane and image plane)
p
Epipolar plane
l
l'
o
e
Baseline
e'
o'
Epipole
(projection of o’ on the image plane)

Epipolar constraint

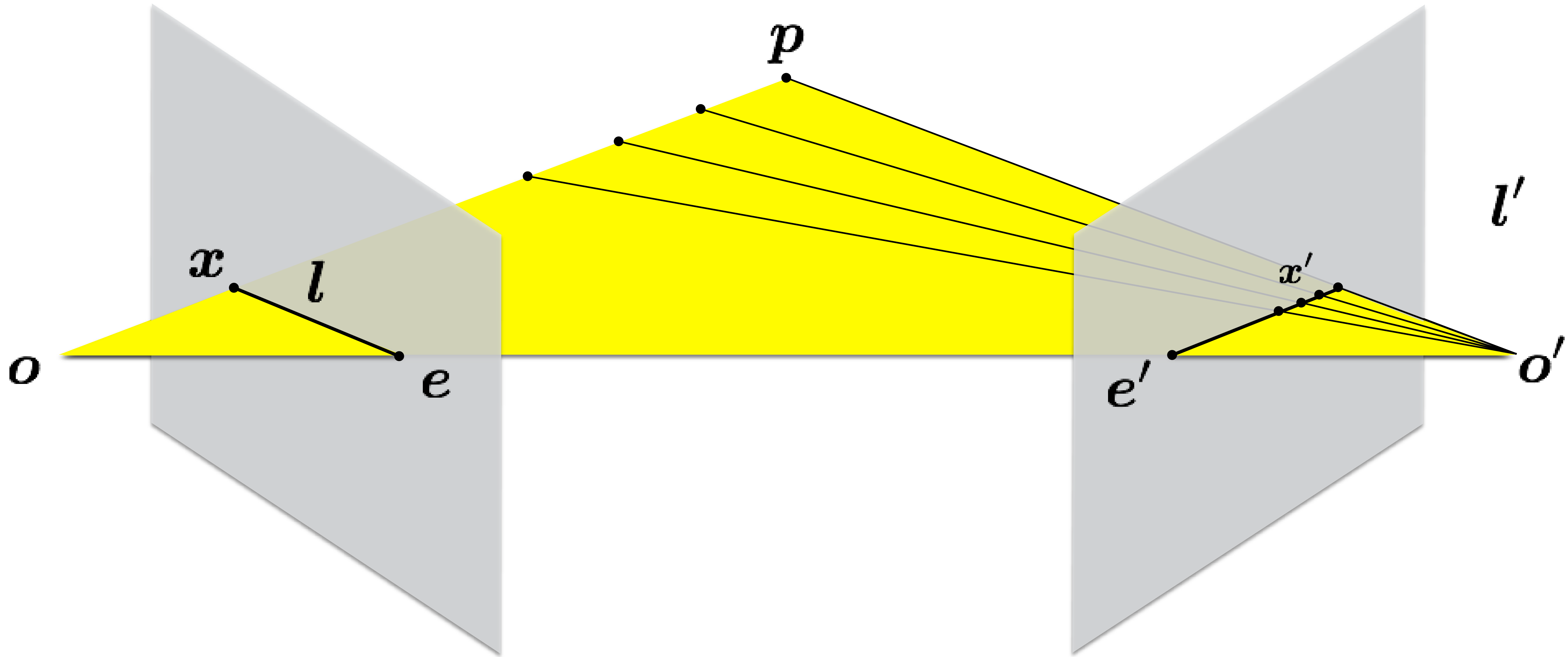


Potential matches for x lie on the epipolar line l'

Epipolar constraint



Potential matches for x lie on the epipolar line l'



The point **x** (left image) maps to a ____________ in the right image

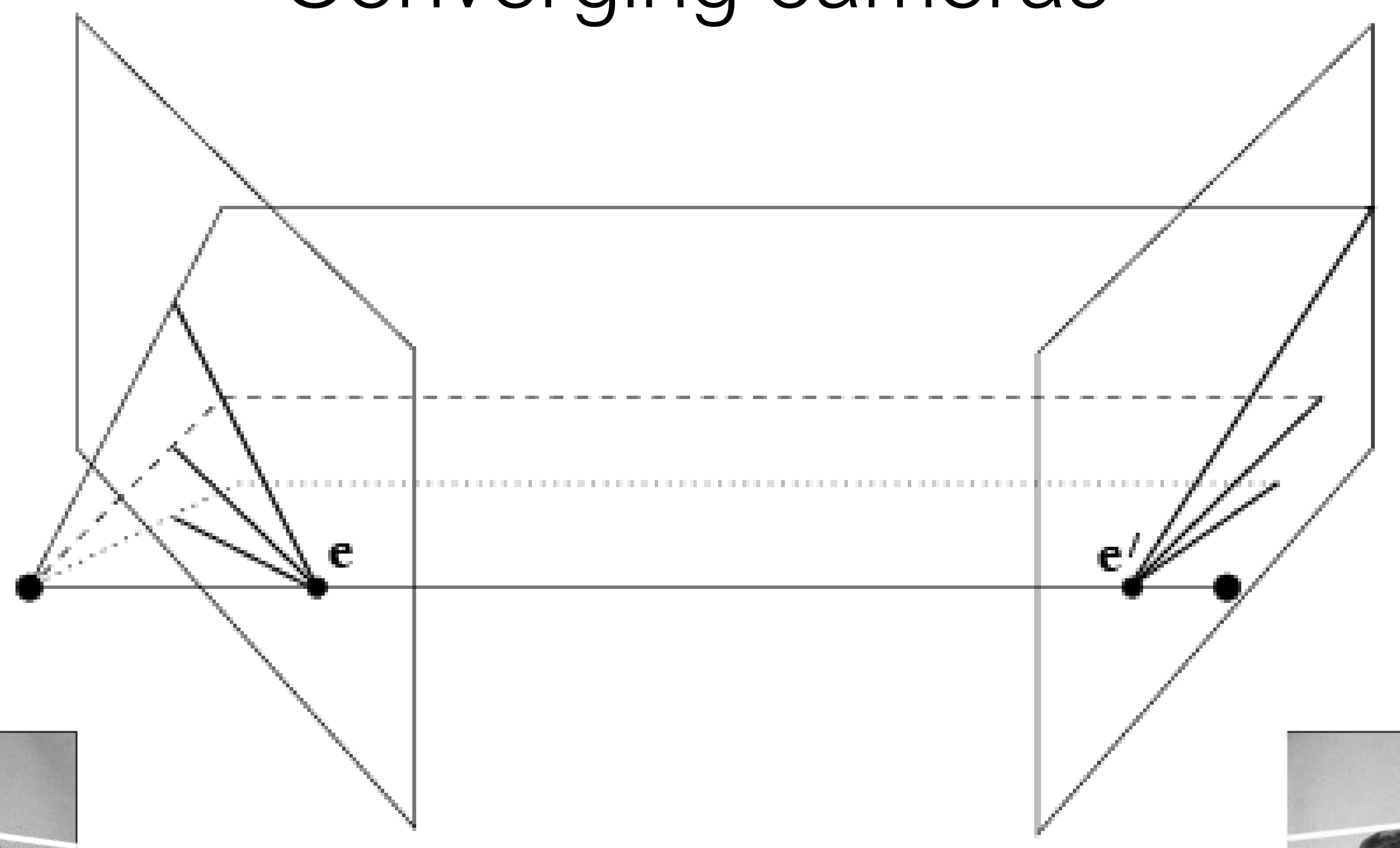
The baseline connects the ____________ and _____________

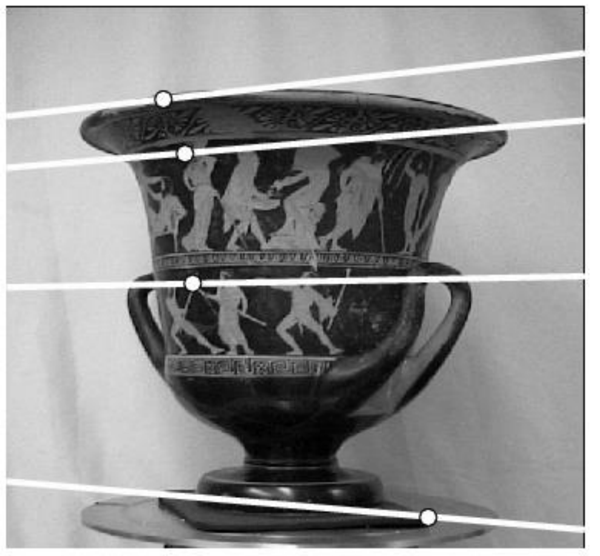
An epipolar line (left image) maps to a ___________ in the right image

An epipole **e** is a projection of the _______________ on the image plane

All epipolar lines in an image intersect at the _______________

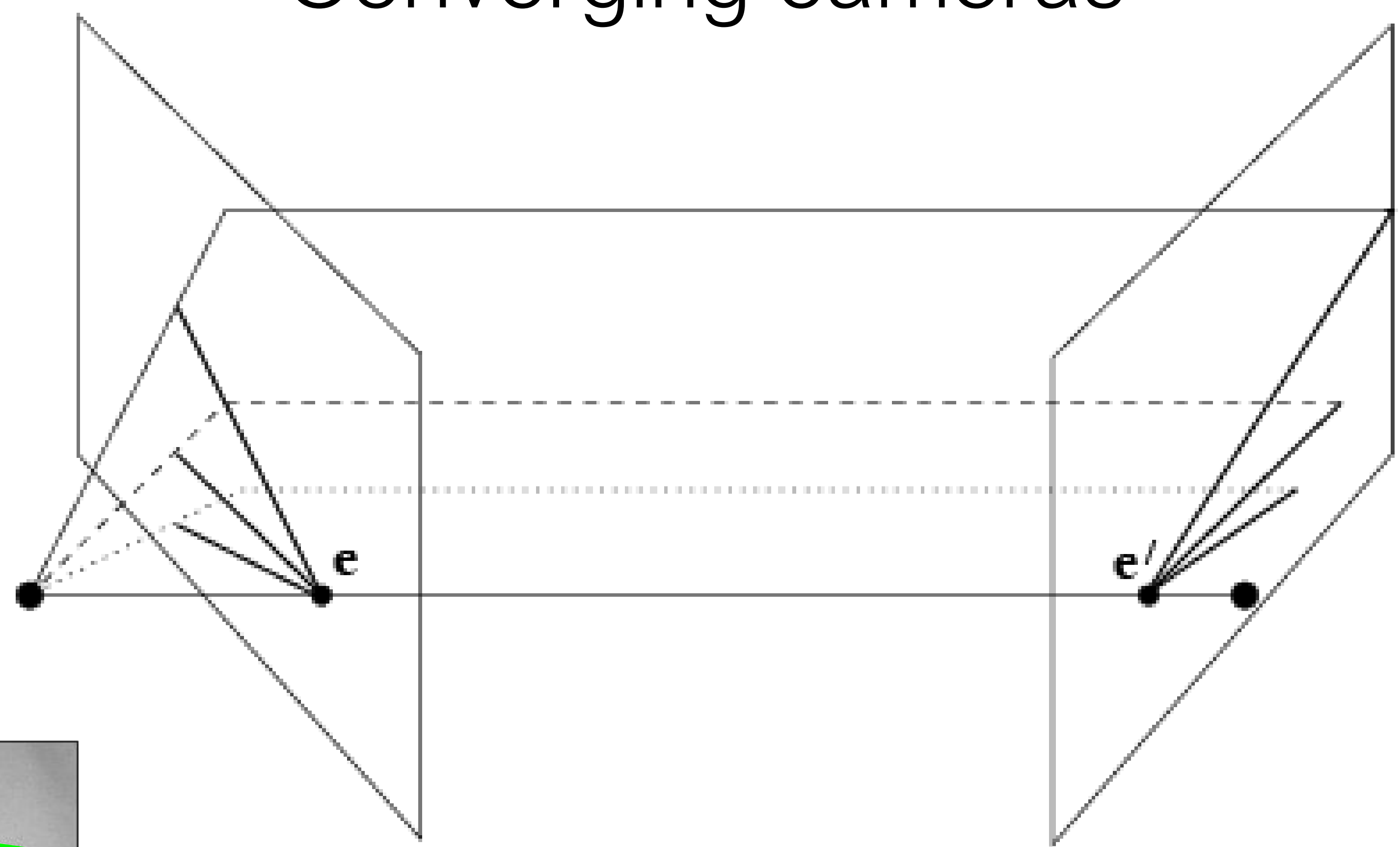
Converging cameras

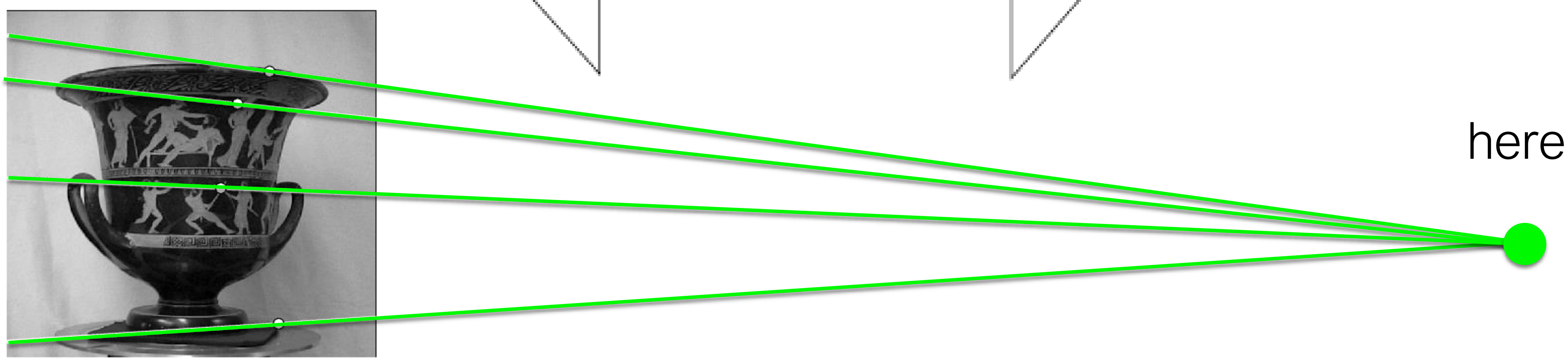




Where is the epipole in this image?

Converging cameras

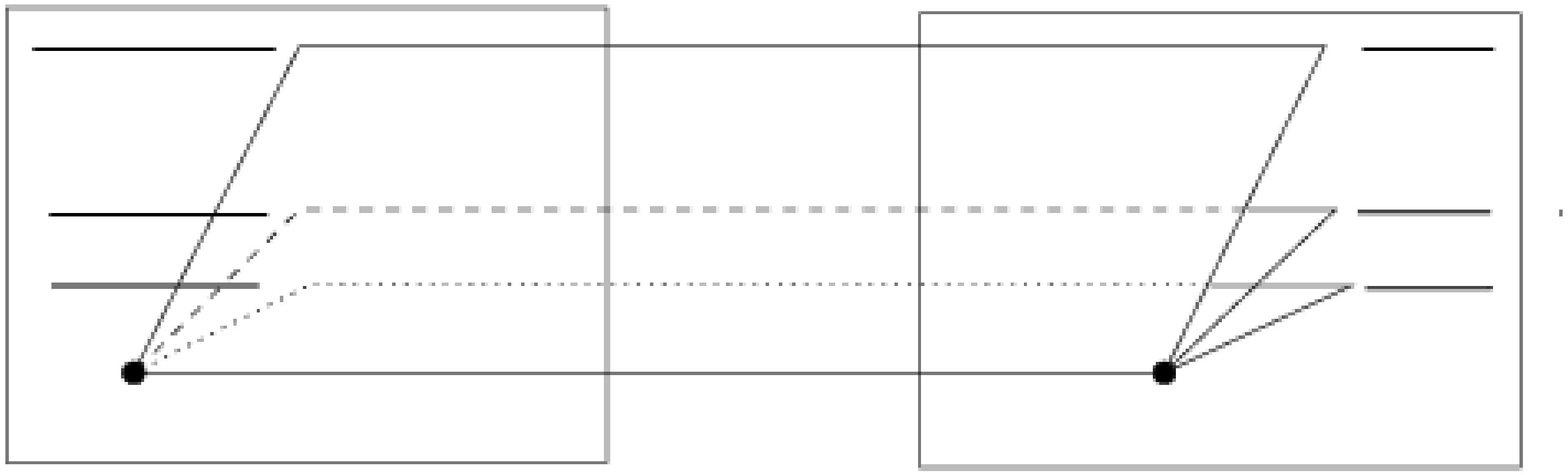


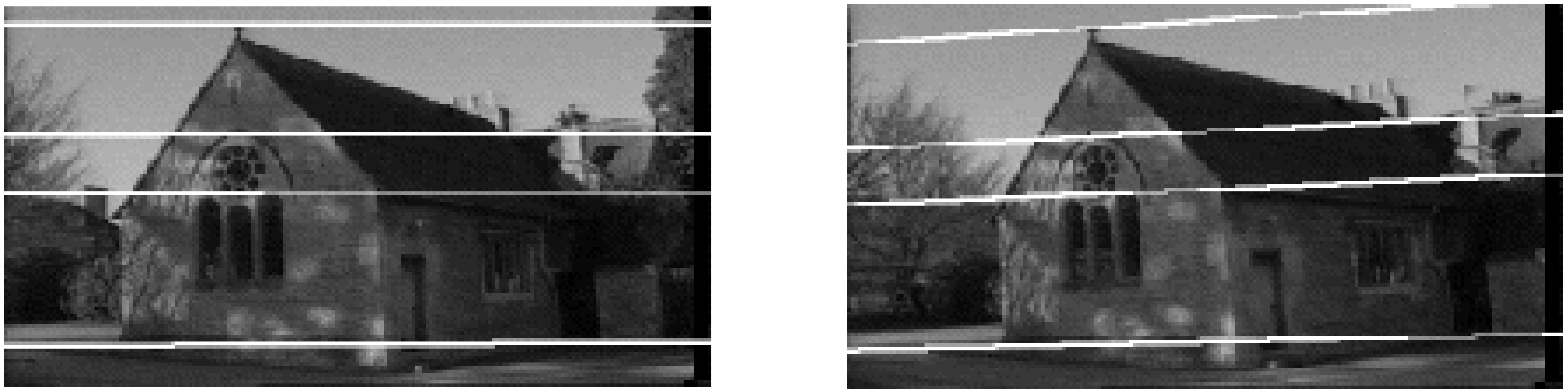


Where is the epipole in this image?

It's not always in the image

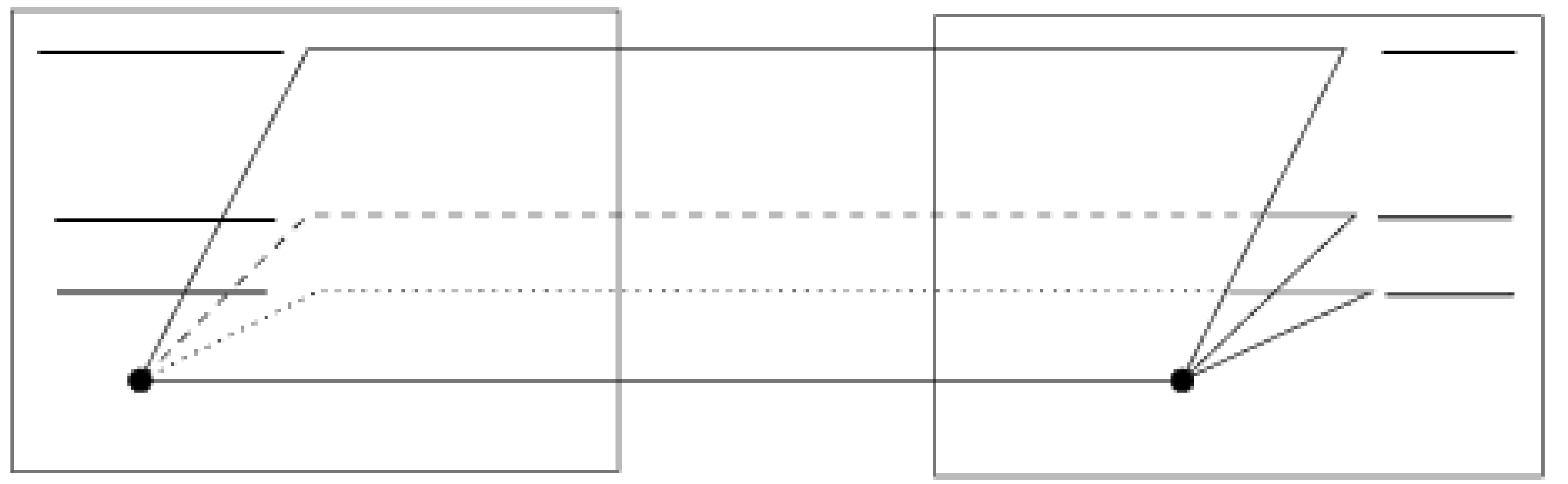
Parallel cameras





Where is the epipole?

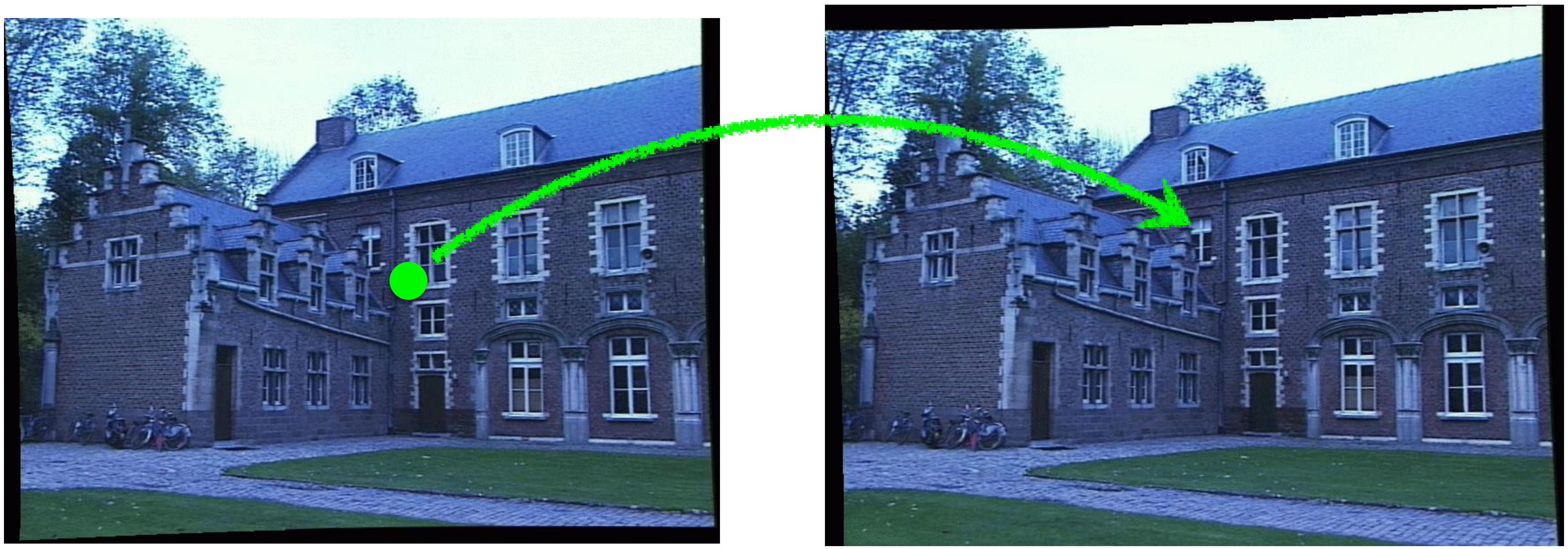
Parallel cameras



epipole at infinity

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image

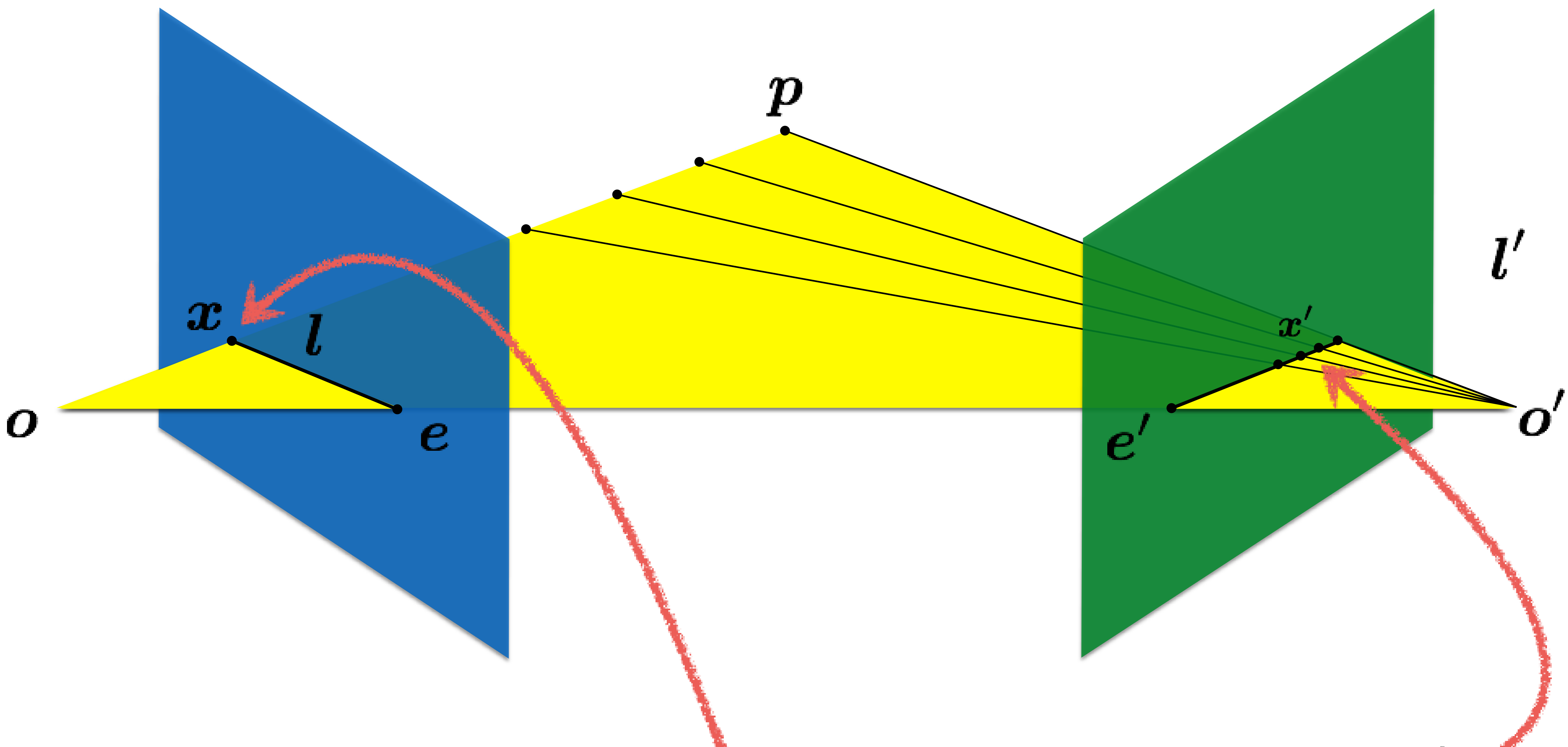


Left image

Right image

How would you do it?

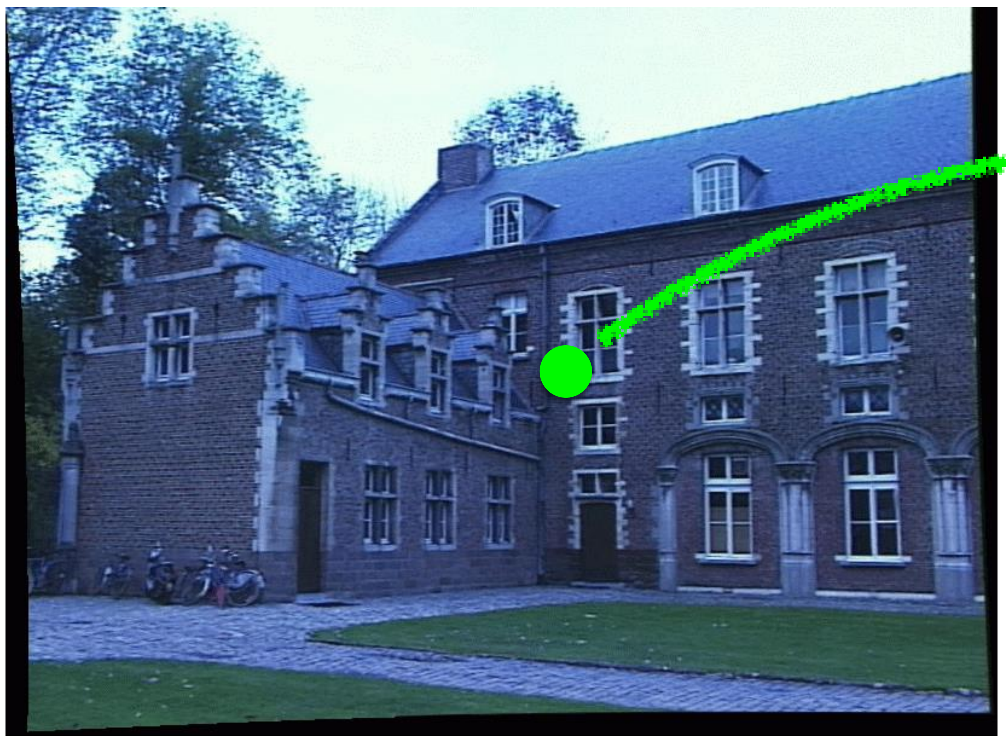
Recall:Epipolar constraint



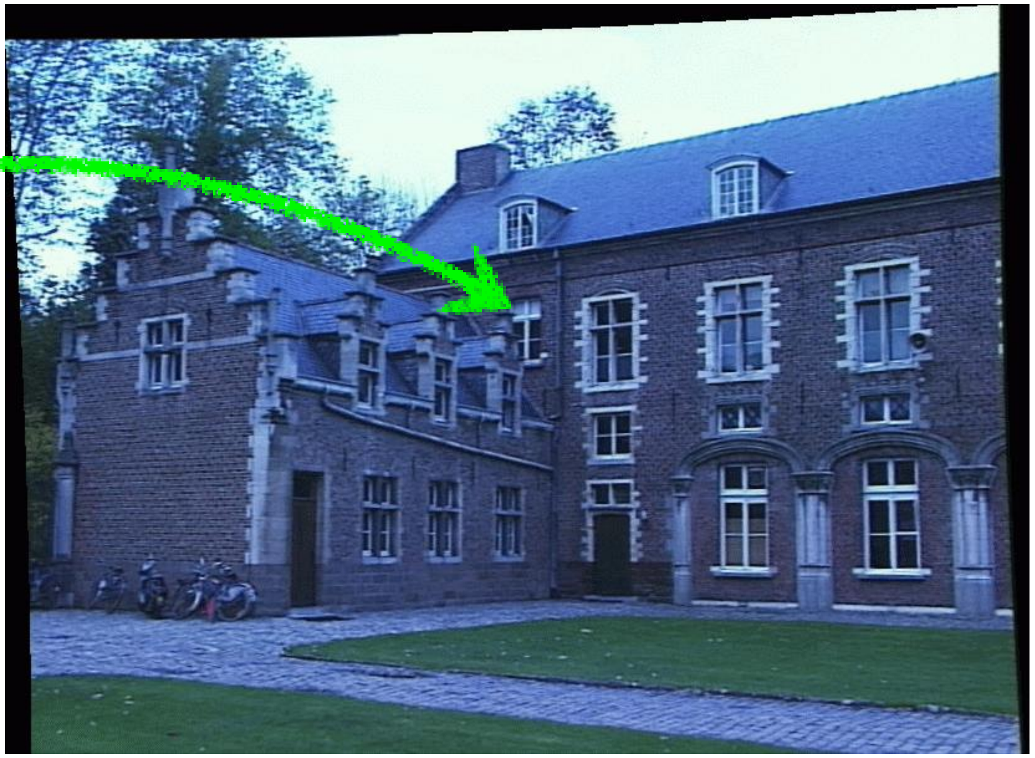
Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l}'$

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image



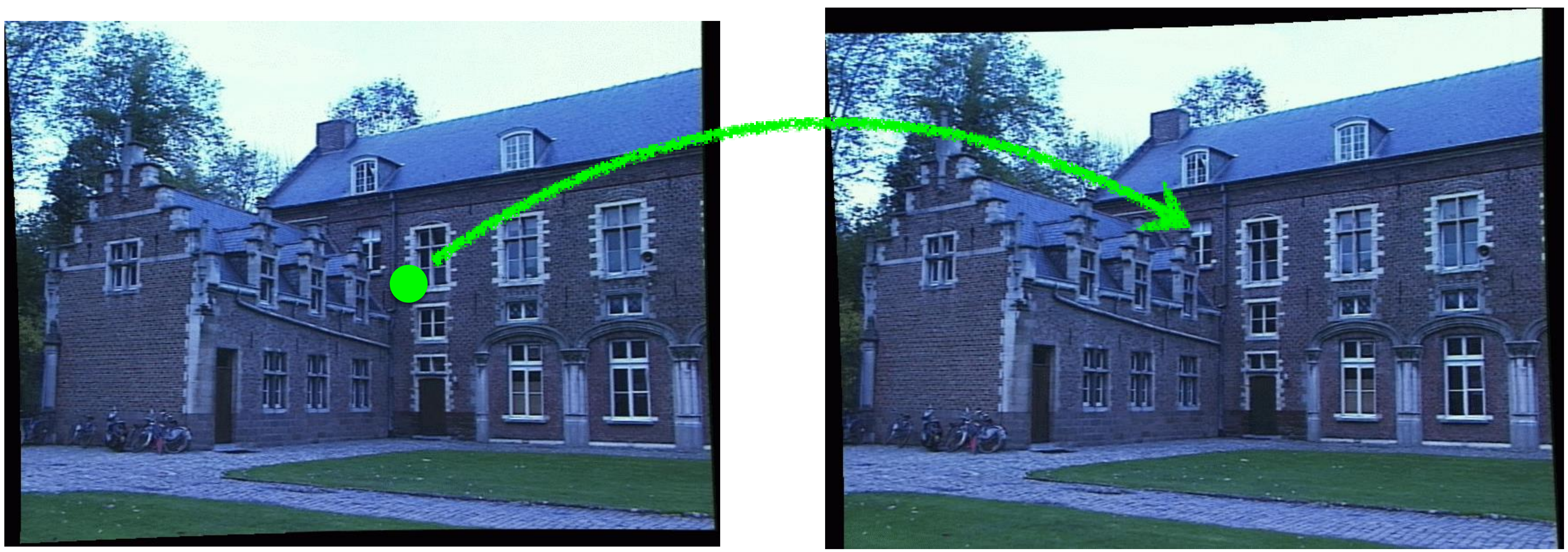
Right image

Want to avoid search over entire image

Epipolar constraint reduces search to a single line

The epipolar constraint is an important concept for stereo vision

Task: Match point in left image to point in right image



Left image

Right image

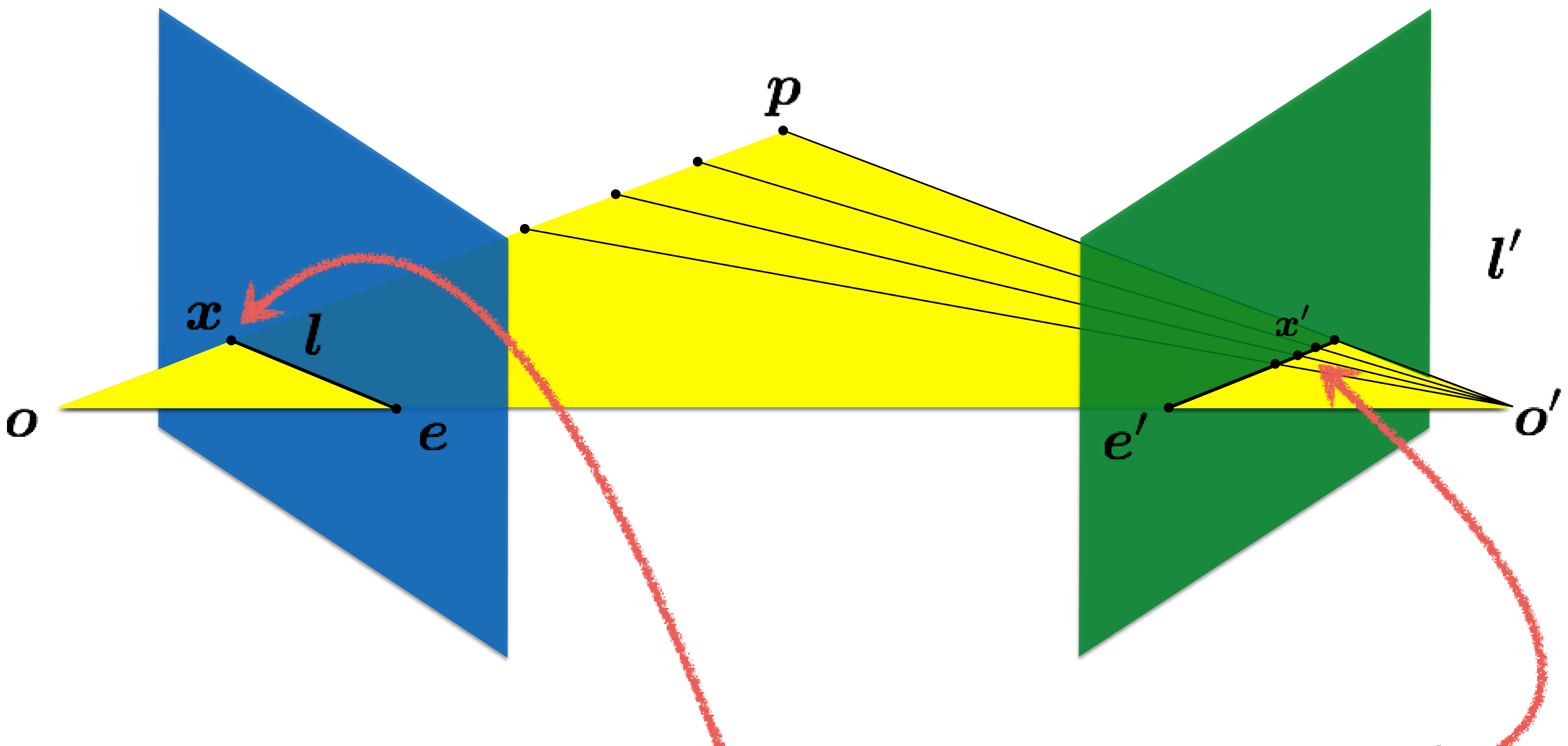
Want to avoid search over entire image

Epipolar constraint reduces search to a single line

How do you compute the epipolar line?

The essential matrix

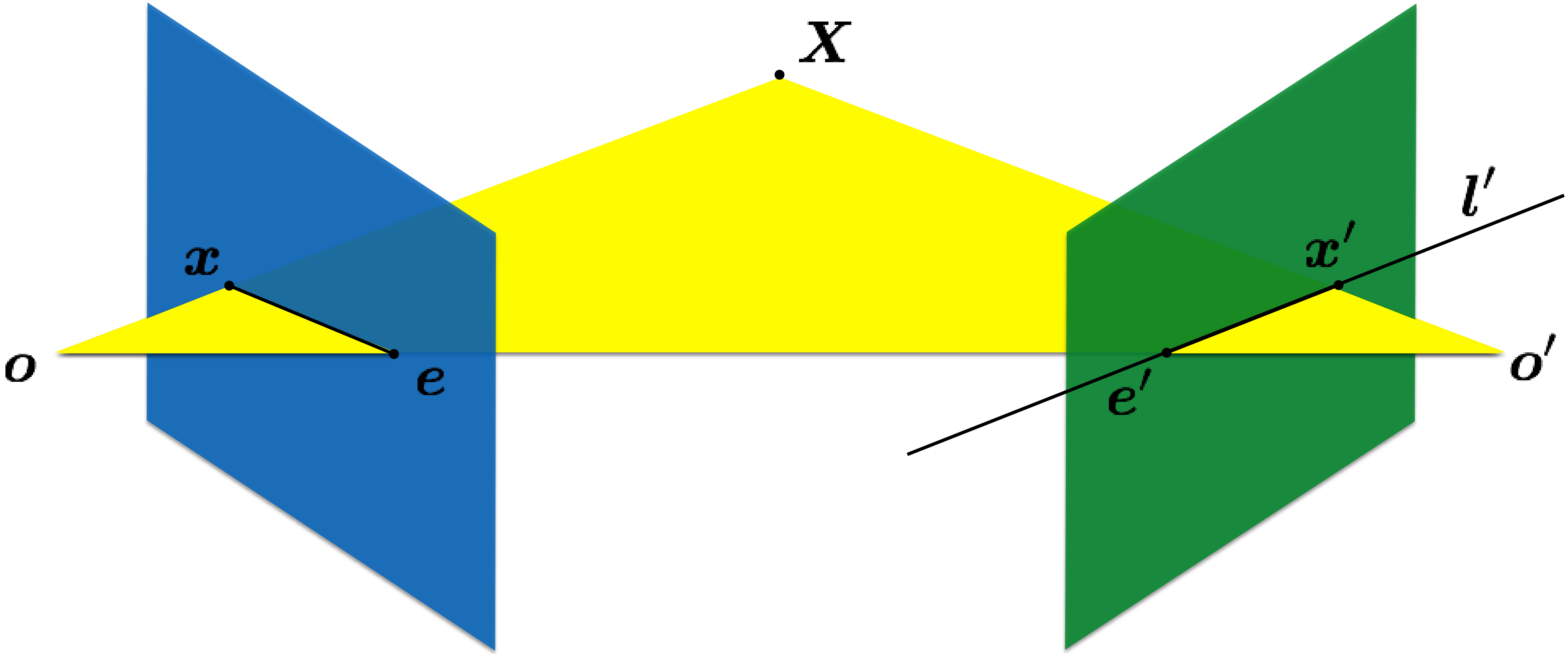
Recall:Epipolar constraint



Potential matches for x lie on the epipolar line l'

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second view.

$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$

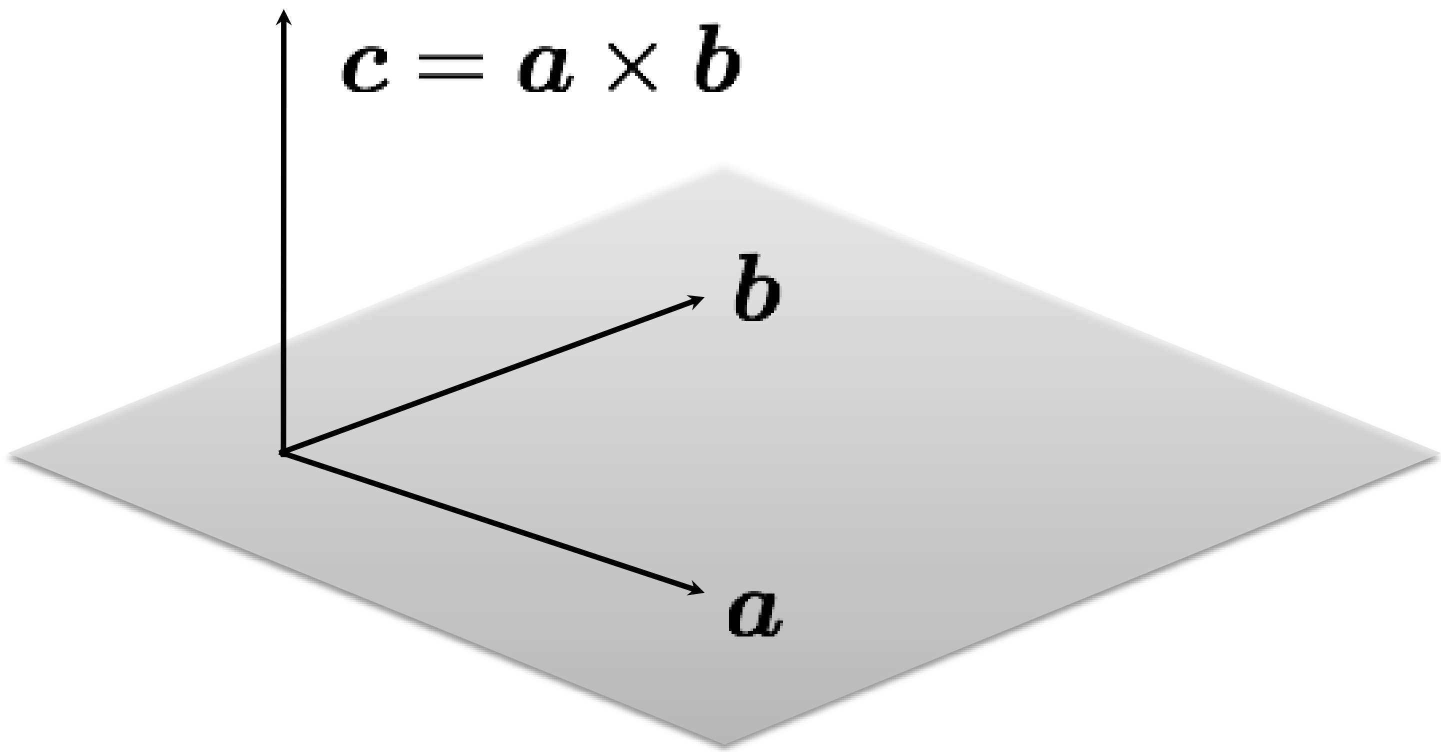


Motivation

The Essential Matrix is a 3 x 3 matrix that encodes **epipolar geometry**

Given a point in one image, multiplying by the **essential matrix** will tell us the **epipolar line** in the second view.

Recall: Dot Product



$\boldsymbol{c} \cdot \boldsymbol{a} = 0$ $\boldsymbol{c} \cdot \boldsymbol{b} = 0$

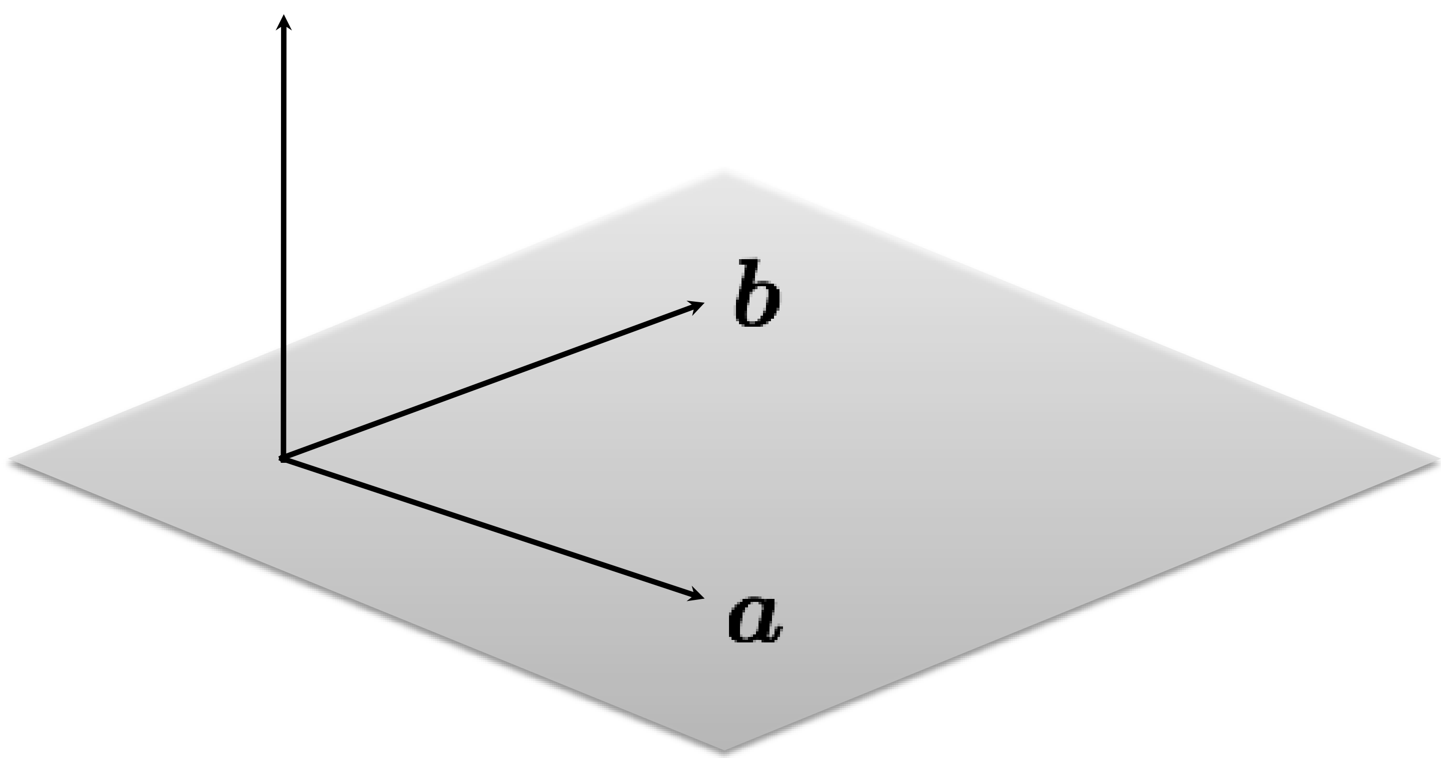
dot product of two orthogonal vectors is zero

Recall: Cross Product

Vector (cross) product

takes two vectors and returns a vector perpendicular to both

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$$



$$\boldsymbol{c} \cdot \boldsymbol{a} = 0 \qquad \boldsymbol{c} \cdot \boldsymbol{b} = 0$$

Cross product

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Can also be written as a matrix multiplication

$$\boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}]_\times \boldsymbol{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Skew symmetric

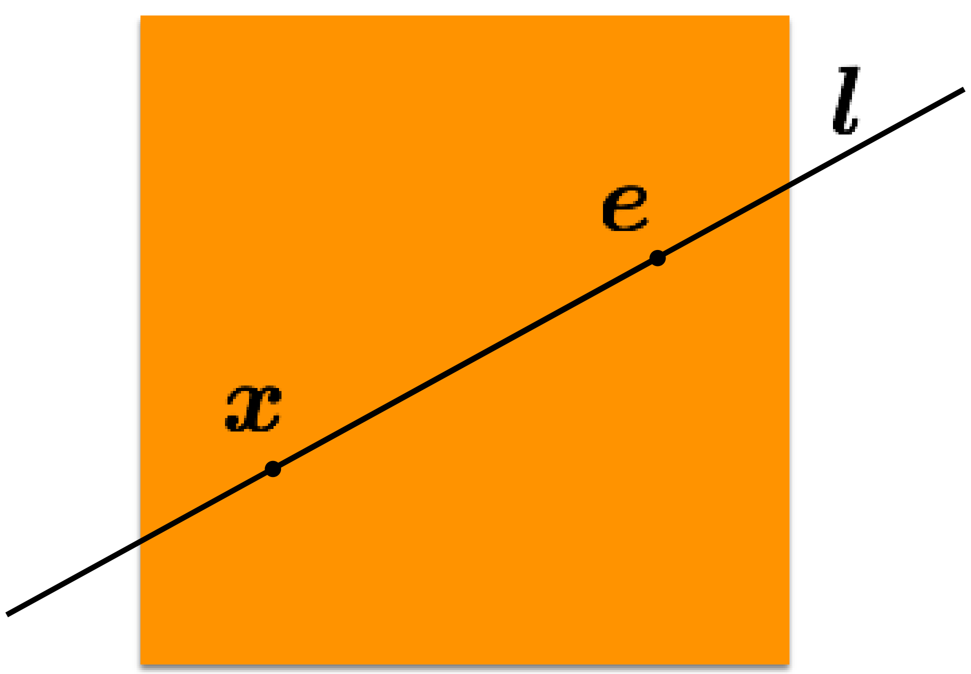
Representing the …

Epipolar Line

$$ax + by + c = 0$$

in vector form

$$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

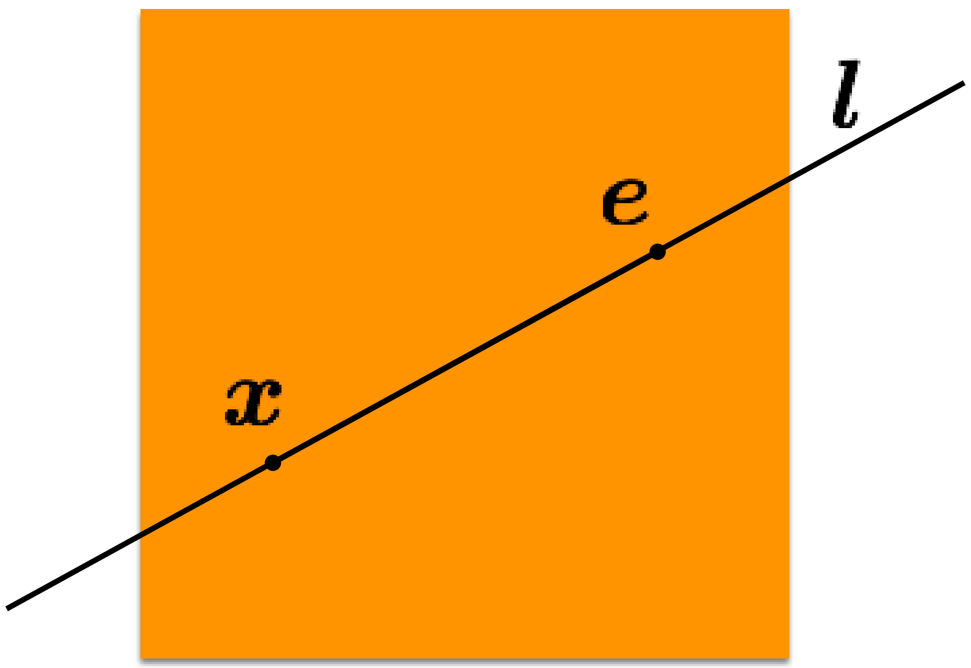
$$\boldsymbol{x}^\top \boldsymbol{l} = \ ?$$

Epipolar Line

$$ax + by + c = 0$$

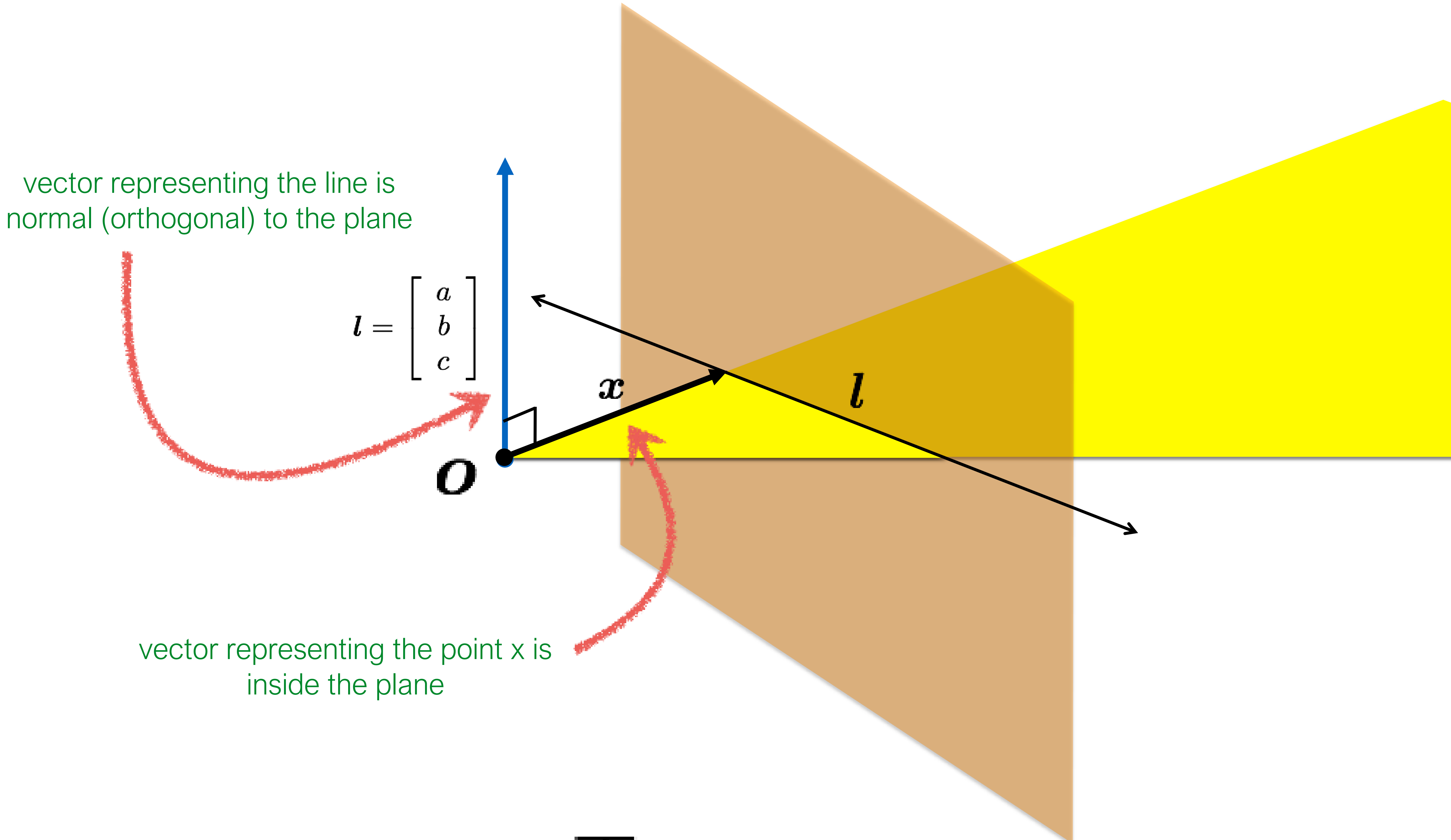
in vector form

$$\boldsymbol{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



If the point $\boldsymbol{x}$ is on the epipolar line $\boldsymbol{l}$ then

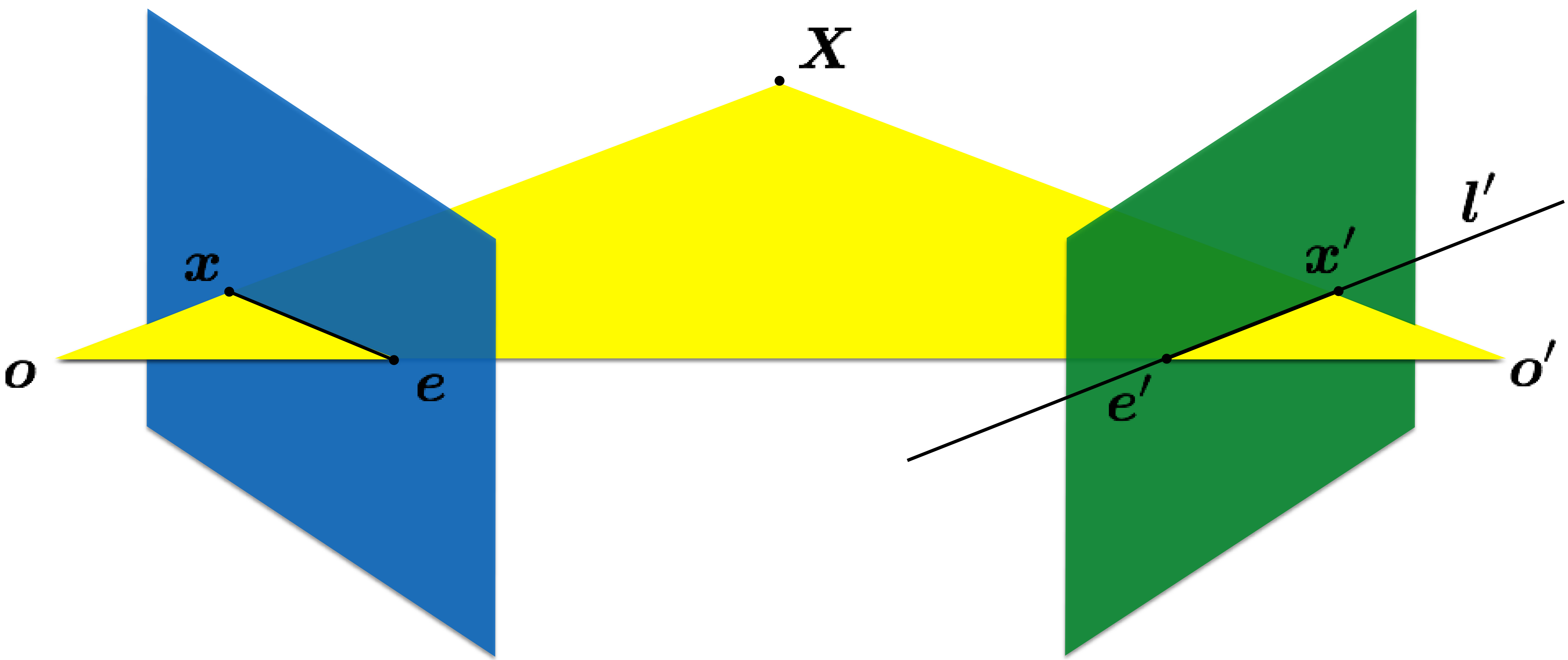
$$\boldsymbol{x}^\top \boldsymbol{l} = 0$$



Therefore: $$\boldsymbol{x}^\top \boldsymbol{l} = 0$$

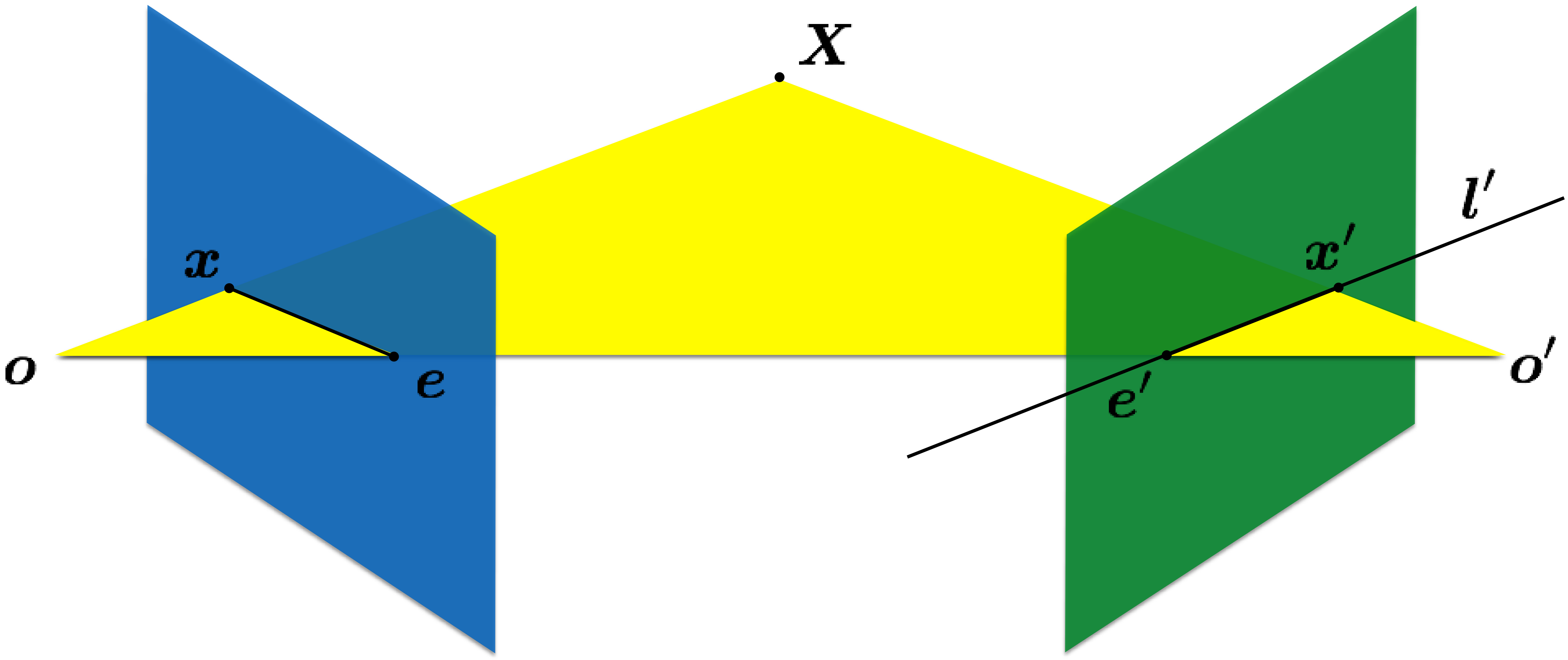
So if $\boldsymbol{x}^\top \boldsymbol{l} = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = \ ?$$



So if $\boldsymbol{x}^\top \boldsymbol{l} = 0$ and $\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'$ then

$$\boxed{\boldsymbol{x}'^\top \mathbf{E} \boldsymbol{x} = 0}$$



Essential Matrix vs Homography

What's the difference between the essential matrix and a homography?

Essential Matrix vs Homography

What's the difference between the essential matrix and a homography?

They are both 3 x 3 matrices but …

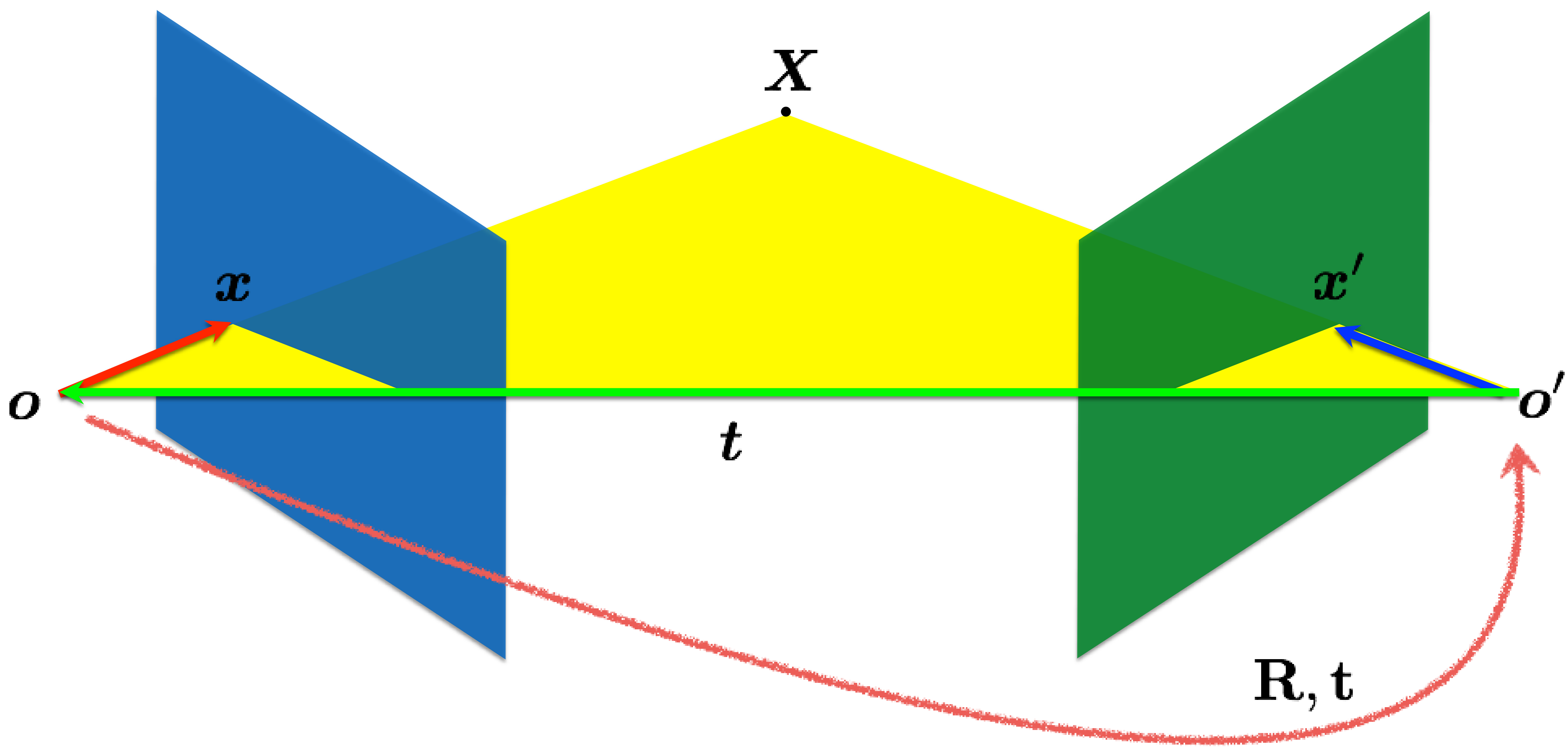
$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$$

Essential matrix maps a
point to a **line**

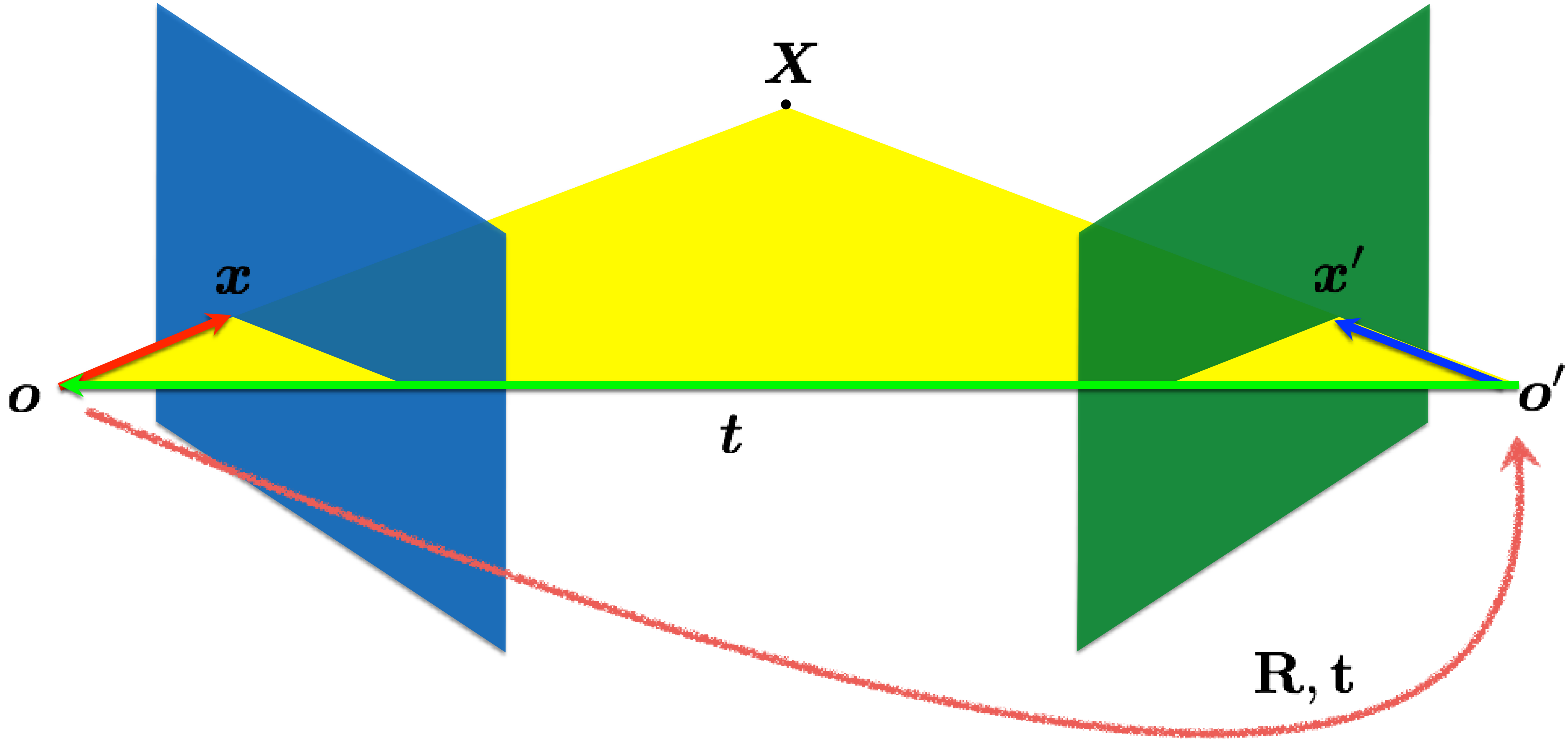
$$\boldsymbol{x}' = \mathbf{H}\boldsymbol{x}$$

Homography maps a
point to a **point**

Where does the Essential matrix come from?

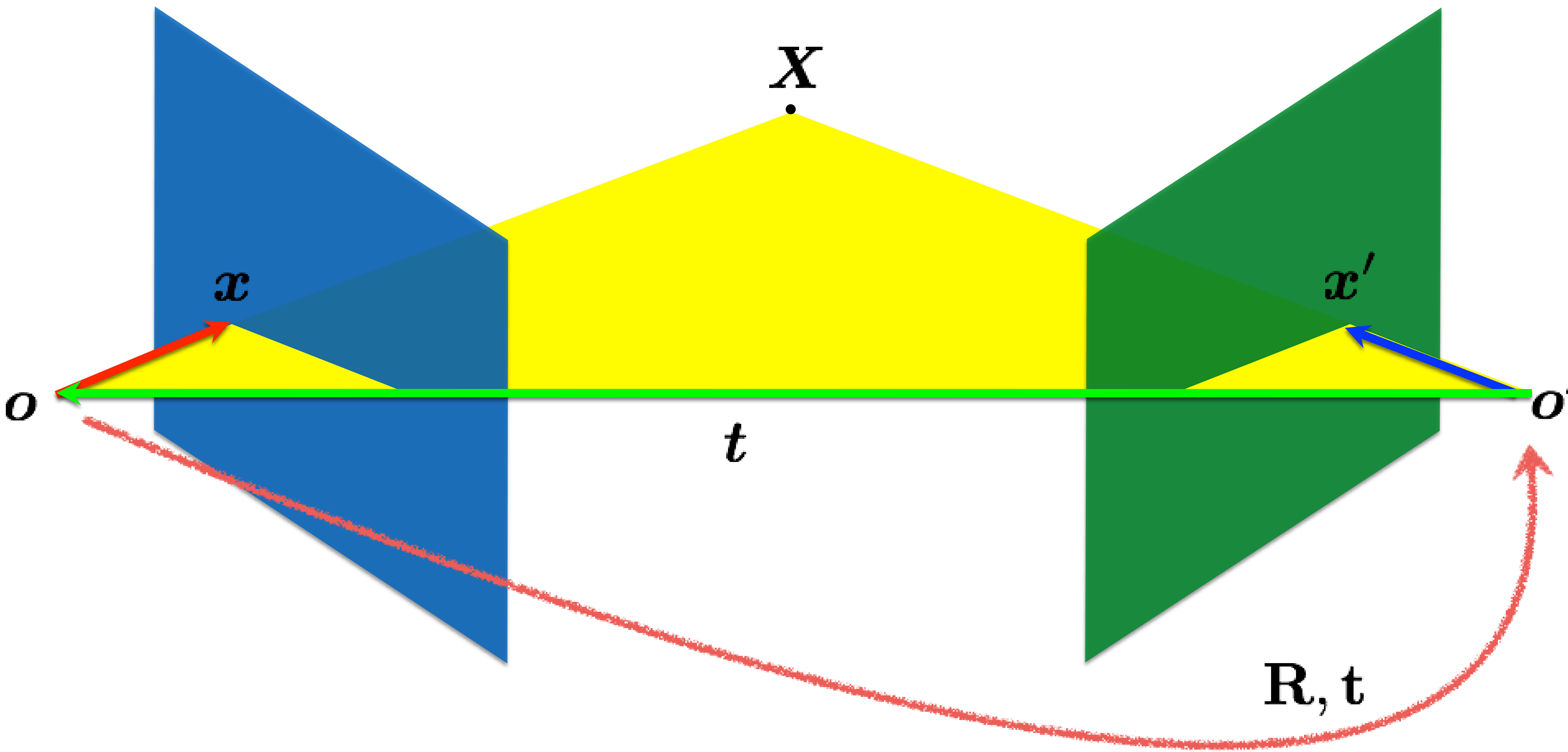


$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$



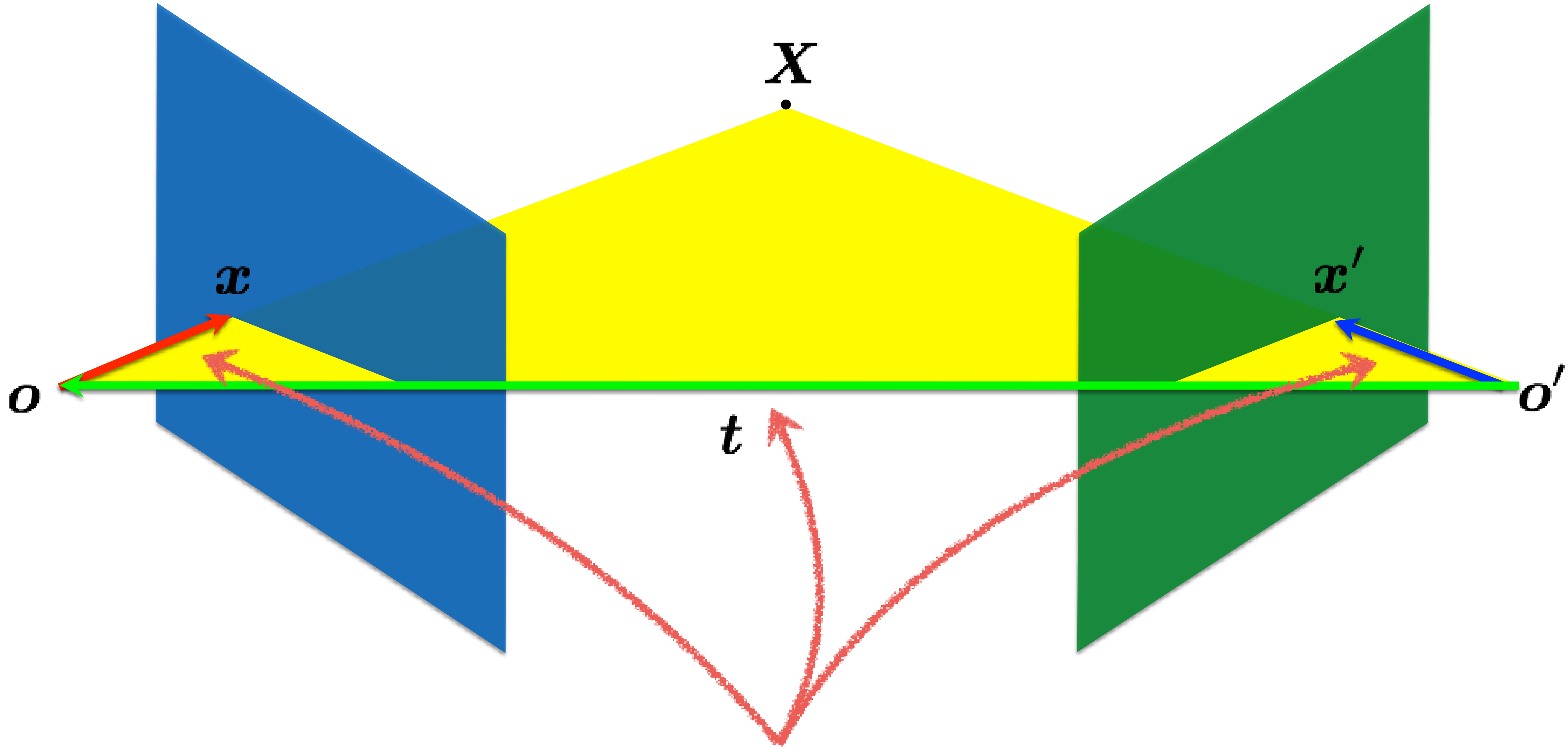
$$x' = \mathbf{R}(x - t)$$

Does this look familiar?



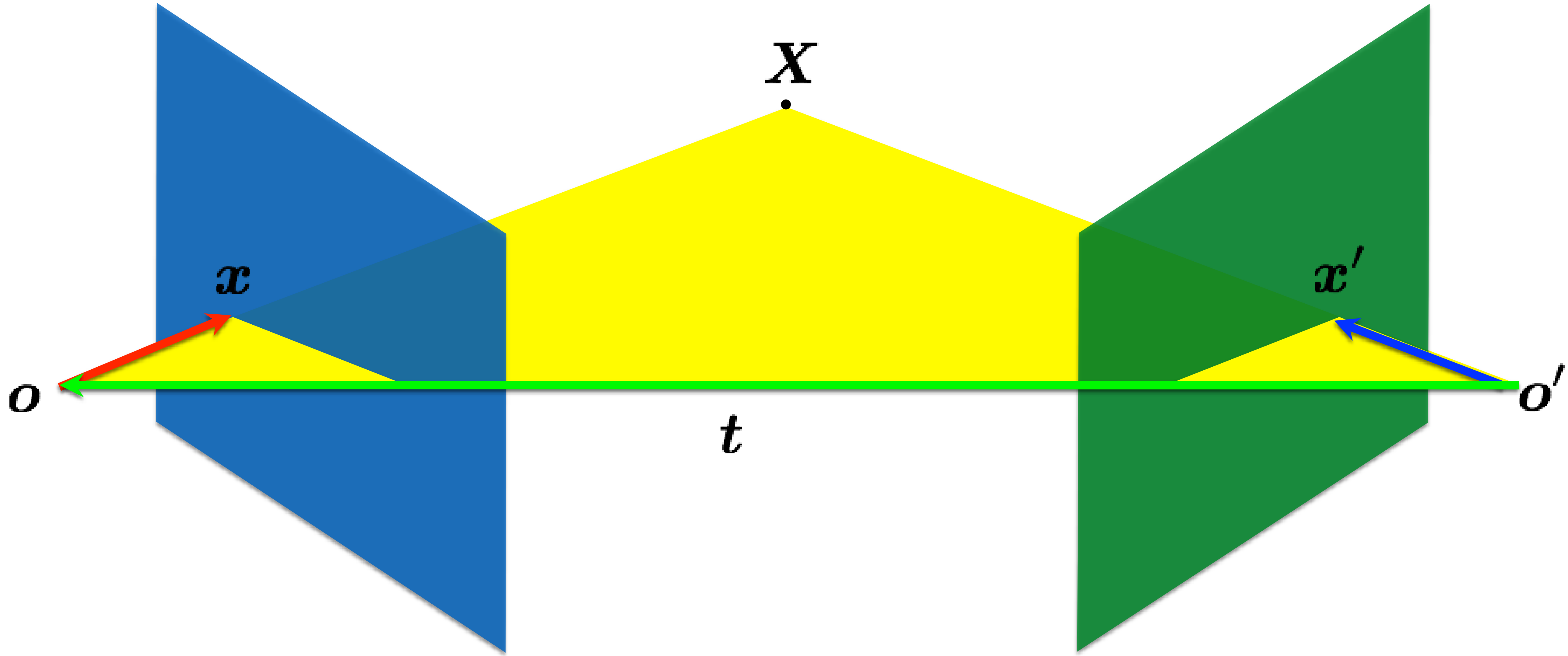
$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

Camera-camera transform just like **world-camera** transform



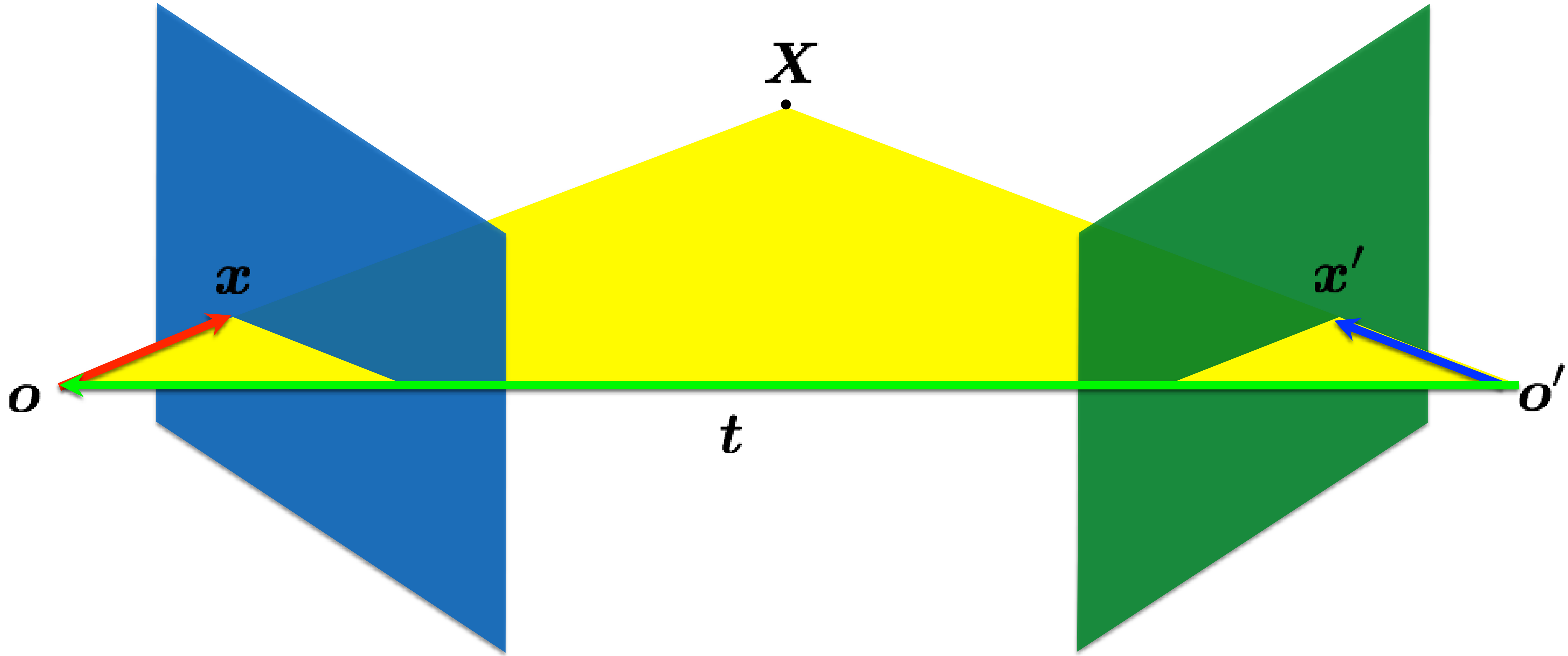
These three vectors are coplanar

$$\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$$



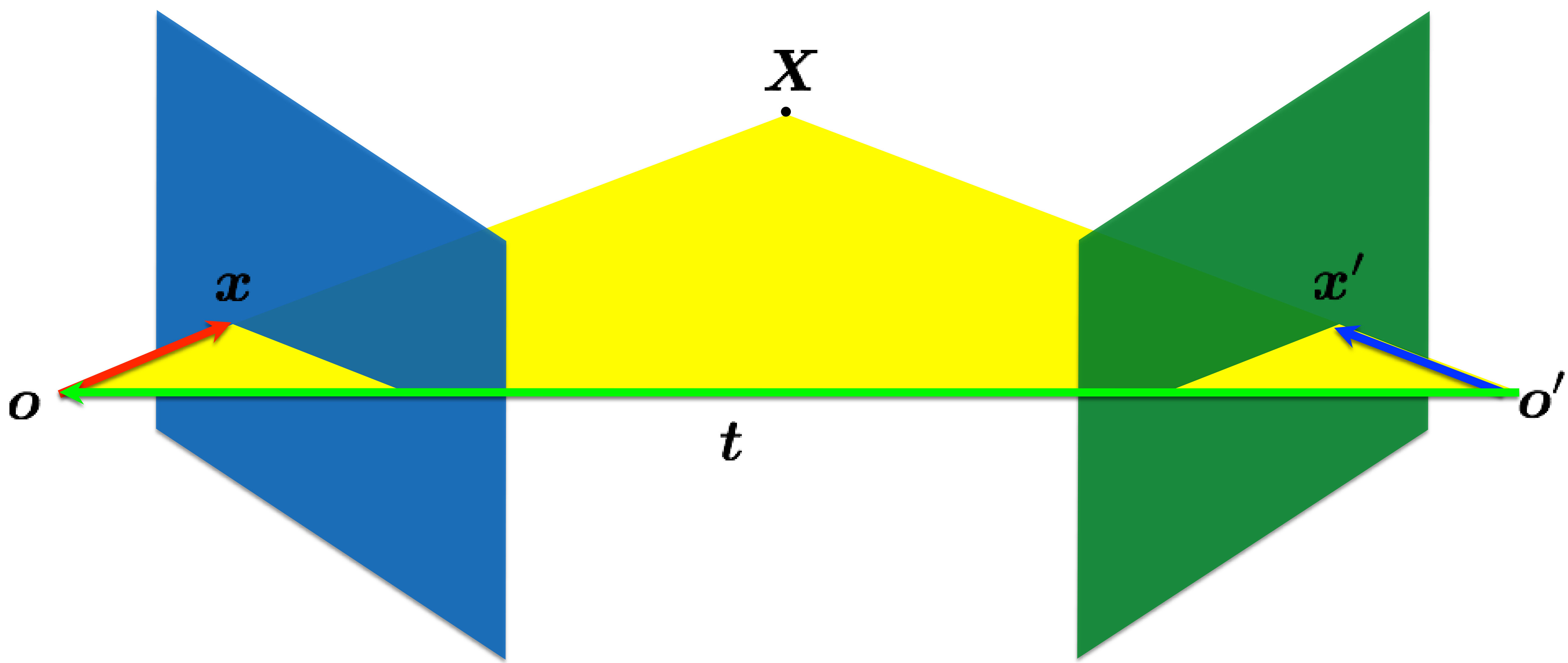
If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$\boldsymbol{x}^\top (\boldsymbol{t} \times \boldsymbol{x}) = \ ?$$



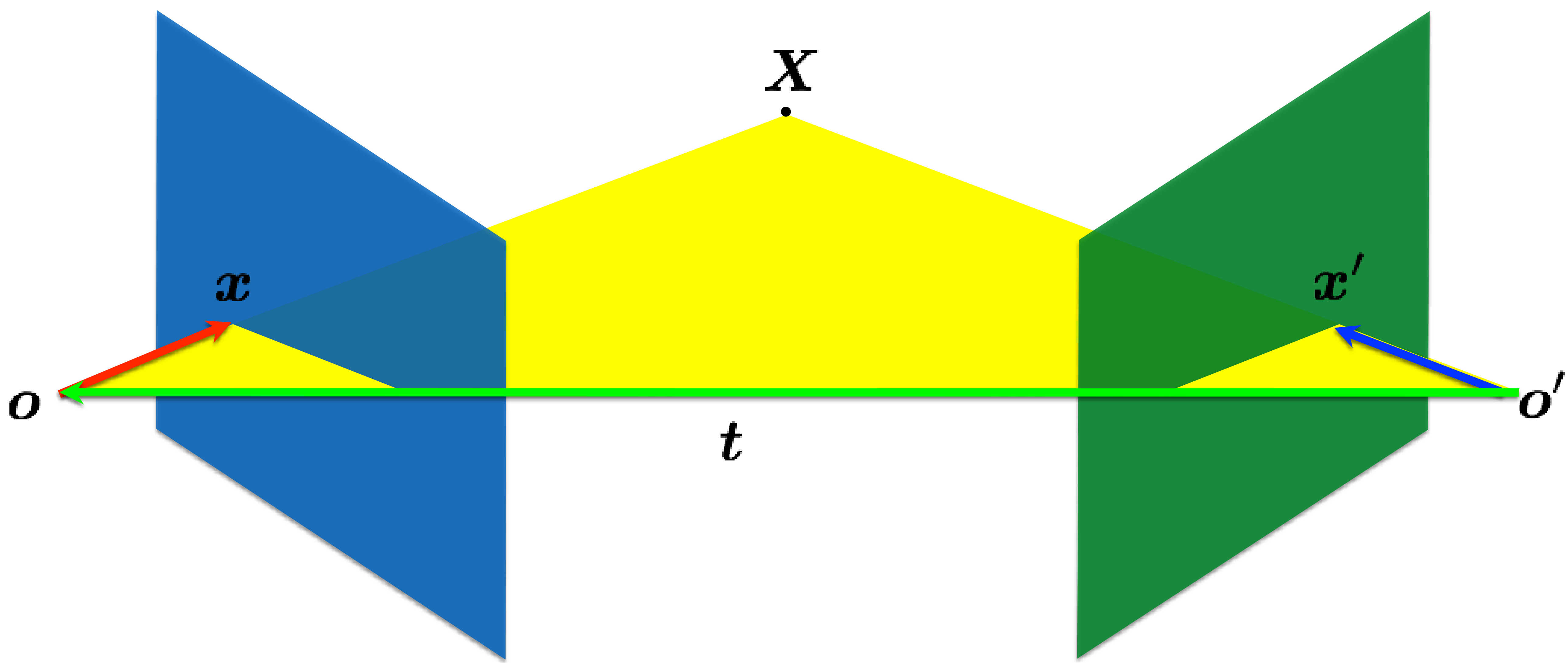
If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$\boldsymbol{x}^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$



If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = \; ?$$



If these three vectors are coplanar $\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{x}'$ then

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0$$

putting it together

rigid motion

$$\boldsymbol{x}' = \mathbf{R}(\boldsymbol{x} - \boldsymbol{t})$$

coplanarity

$$(\boldsymbol{x} - \boldsymbol{t})^\top (\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})(\boldsymbol{t} \times \boldsymbol{x}) = 0$$

$$(\boldsymbol{x}'^\top \mathbf{R})([\mathbf{t}_\times]\boldsymbol{x}) = 0$$

$$\boldsymbol{x}'^\top (\mathbf{R}[\mathbf{t}_\times])\boldsymbol{x} = 0$$

$$\boxed{\boldsymbol{x}'^\top \mathbf{E}\boldsymbol{x} = 0}$$

Essential Matrix
[Longuet-Higgins 1981]

properties of the E matrix

Longuet-Higgins equation $\quad \boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$

(points in normalized coordinates)

properties of the E matrix

Longuet-Higgins equation $$\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$$

Epipolar lines

$$\boldsymbol{x}^{\top}\boldsymbol{l} = 0 \qquad \boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$$

$$\boldsymbol{l}' = \mathbf{E}\boldsymbol{x} \qquad \boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$$

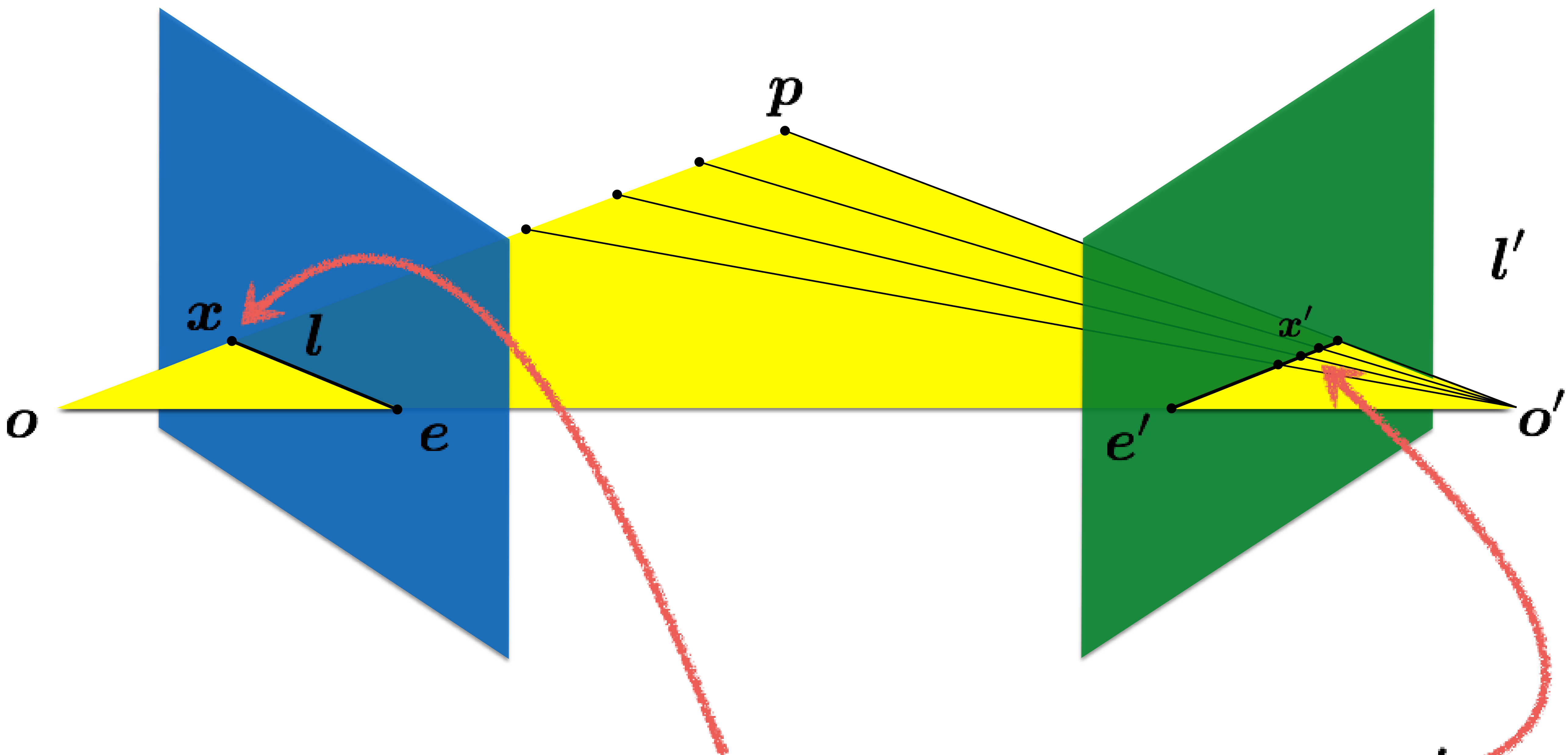
(points in normalized coordinates)

properties of the E matrix

Longuet-Higgins equation	$\boldsymbol{x}'^{\top}\mathbf{E}\boldsymbol{x} = 0$	
Epipolar lines	$\boldsymbol{x}^{\top}\boldsymbol{l} = 0$ $\boldsymbol{l}' = \mathbf{E}\boldsymbol{x}$	$\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$ $\boldsymbol{l} = \mathbf{E}^{T}\boldsymbol{x}'$
Epipoles	$\boldsymbol{e}'^{\top}\mathbf{E} = \mathbf{0}$	$\mathbf{E}\boldsymbol{e} = \mathbf{0}$

(points in normalized <u>camera</u> coordinates)

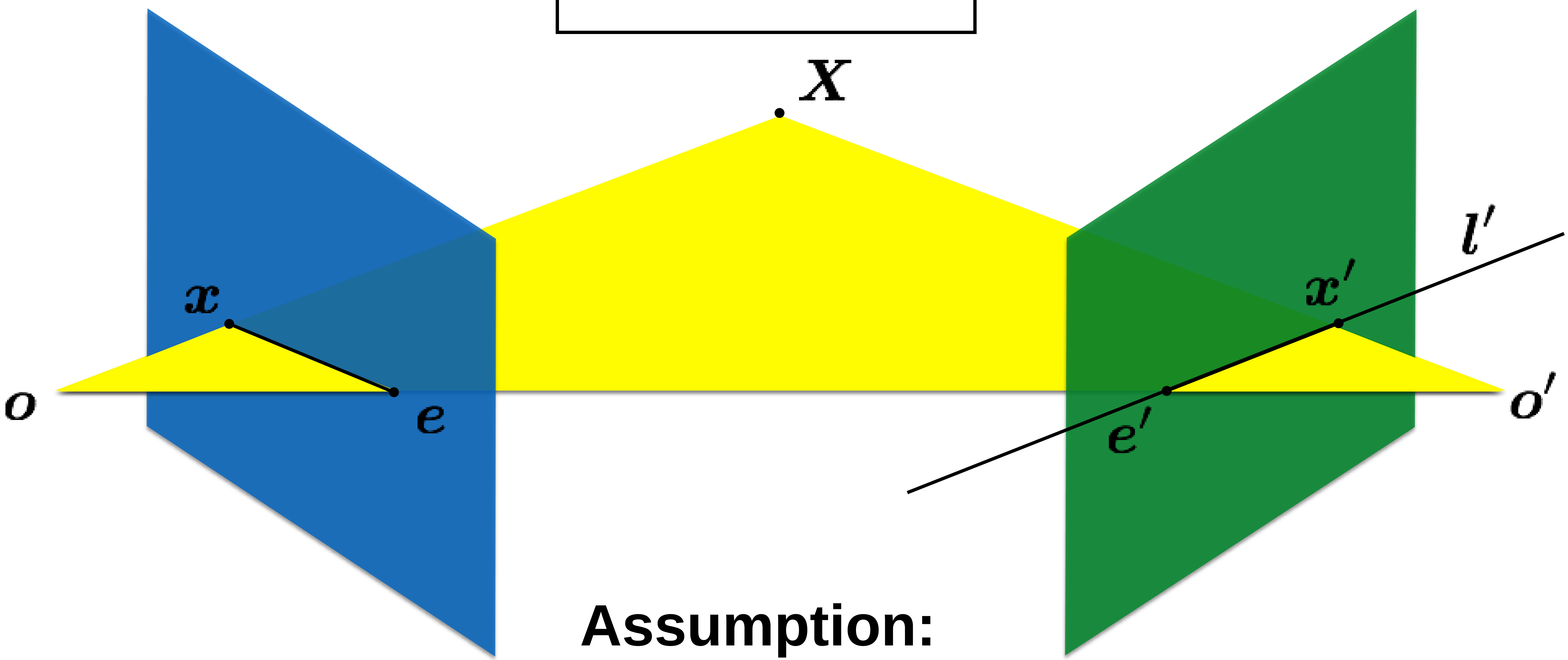
Recall:Epipolar constraint



Potential matches for $\boldsymbol{x}$ lie on the epipolar line $\boldsymbol{l'}$

Given a point in one image,
multiplying by the **essential matrix** will tell us
the **epipolar line** in the second view.

$$\boxed{\mathbf{E}\boldsymbol{x} = \boldsymbol{l}'}$$



Assumption:
points aligned to camera coordinate axis (calibrated camera)

How do you generalize to uncalibrated cameras?

The fundamental matrix

The

Fundamental matrix

is a

generalization

of the

Essential matrix,

where the assumption of

calibrated cameras

is removed

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

The Essential matrix operates on image points expressed in
normalized coordinates
(points have been aligned (normalized) to camera coordinates)

$$\hat{\boldsymbol{x}}' = \mathbf{K}^{-1}\boldsymbol{x}'$$

$$\underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1}\underbrace{\boldsymbol{x}}_{\text{image point}}$$

$$\hat{\boldsymbol{x}}'^{\top}\mathbf{E}\hat{\boldsymbol{x}} = 0$$

The Essential matrix operates on image points expressed in
normalized coordinates
(points have been aligned (normalized) to camera coordinates)

$$\hat{\boldsymbol{x}}' = \mathbf{K}^{-1}\boldsymbol{x}'$$

$$\underbrace{\hat{\boldsymbol{x}}}_{\text{camera point}} = \mathbf{K}^{-1}\underbrace{\boldsymbol{x}}_{\text{image point}}$$

Writing out the epipolar constraint in terms of image coordinates

$$\boldsymbol{x}'^{\top}\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}(\mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1})\boldsymbol{x} = 0$$

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

Same equation works in image coordinates!

$$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$$

it maps pixels to epipolar lines

properties of the ~~E~~ F matrix

Longuet-Higgins equation	$\boldsymbol{x}'^{\top}\mathbf{F}\boldsymbol{x} = 0$	
Epipolar lines	$\boldsymbol{x}^{\top}\boldsymbol{l} = 0$	$\boldsymbol{x}'^{\top}\boldsymbol{l}' = 0$
	$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$	$\boldsymbol{l} = \mathbf{F}^{T}\boldsymbol{x}'$
Epipoles	$\boldsymbol{e}'^{\top}\mathbf{F} = \mathbf{0}$	$\mathbf{F}\boldsymbol{e} = \mathbf{0}$

(points in **image** coordinates)

Breaking down the fundamental matrix

$$\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$$

$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

Depends on both intrinsic and extrinsic parameters

Breaking down the fundamental matrix

$$\mathbf{F} = \mathbf{K}'^{-\top}\mathbf{E}\mathbf{K}^{-1}$$

$$\mathbf{F} = \mathbf{K}'^{-\top}[\mathbf{t}_{\times}]\mathbf{R}\mathbf{K}^{-1}$$

Depends on both intrinsic and extrinsic parameters

How would you solve for F?

$$\boldsymbol{x}_m'^{\top}\mathbf{F}\boldsymbol{x}_m = 0$$

The 8-point algorithm

Assume you have *M* matched *image* points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \qquad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

How would you solve for the 3 x 3 ***F*** *matrix?*

Assume you have *M* matched *image* points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \qquad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

How would you solve for the 3 x 3 ***F*** *matrix?*

S V D

Assume you have *M* matched *image* points

$$\{\boldsymbol{x}_m, \boldsymbol{x}'_m\} \qquad m = 1, \ldots, M$$

Each correspondence should satisfy

$$\boldsymbol{x}'^{\top}_m \mathbf{F} \boldsymbol{x}_m = 0$$

How would you solve for the 3 x 3 ***F*** *matrix?*

Set up a homogeneous linear system with 9 unknowns

$$\boldsymbol{x}_m'^{\top} \mathbf{F} \boldsymbol{x}_m = 0$$

$$\begin{bmatrix} x_m' & y_m' & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

How many equation do you get from one correspondence?

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

ONE correspondence gives you ONE equation

$$\begin{aligned} x_m x'_m f_1 + x_m y'_m f_2 + x_m f_3 + \\ y_m x'_m f_4 + y_m y'_m f_5 + y_m f_6 + \\ x'_m f_7 + y'_m f_8 + f_9 = 0 \end{aligned}$$

$$\begin{bmatrix} x'_m & y'_m & 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \\ 1 \end{bmatrix} = 0$$

Set up a homogeneous linear system with 9 unknowns

$$\begin{bmatrix} x_1 x'_1 & x_1 y'_1 & x_1 & y_1 x'_1 & y_1 y'_1 & y_1 & x'_1 & y'_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_M x'_M & x_M y'_M & x_M & y_M x'_M & y_M y'_M & y_M & x'_M & y'_M & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \end{bmatrix} = \mathbf{0}$$

How many equations do you need?

Each point pair (according to epipolar constraint) contributes only one scalar equation

$$\boldsymbol{x}_m'^{\top}\mathbf{F}\boldsymbol{x}_m = 0$$

Note: This is different from the Homography estimation where each point pair contributes 2 equations.

We need at least 8 points

Hence, the 8 point algorithm!

How do you solve a homogeneous linear system?

$$\mathbf{A}\mathbf{X} = \mathbf{0}$$

How do you solve a homogeneous linear system?

$$\mathbf{AX} = \mathbf{0}$$

Total Least Squares

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

How do you solve a homogeneous linear system?

$$\mathbf{AX} = \mathbf{0}$$

Total Least Squares

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

SVD!

Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

See Hartley-Zisserman for why we do this

Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

How do we do this?

Eight-Point Algorithm

0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

How do we do this?

S V D !

Enforcing rank constraints

Problem: Given a matrix F, find the matrix F' of rank k that is closest to F,

$$\min_{\substack{F' \\ \mathrm{rank}(F')=k}} \quad ||F - F'||^2$$

Solution: Compute the singular value decomposition of F,

$$F = U\Sigma V^T$$

Form a matrix Σ' by replacing all but the k largest singular values in Σ with 0.

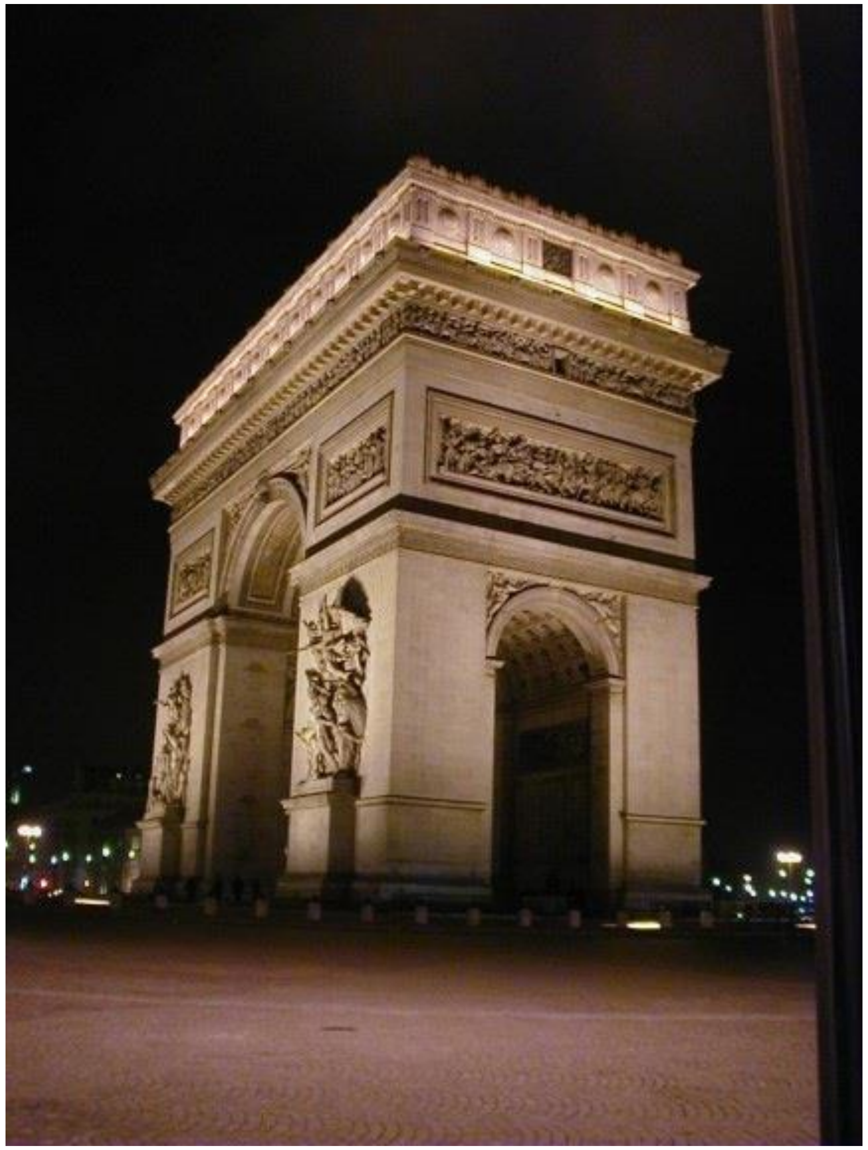
Then the problem solution is the matrix F' formed as,

$$F' = U\Sigma' V^T$$

Eight-Point Algorithm

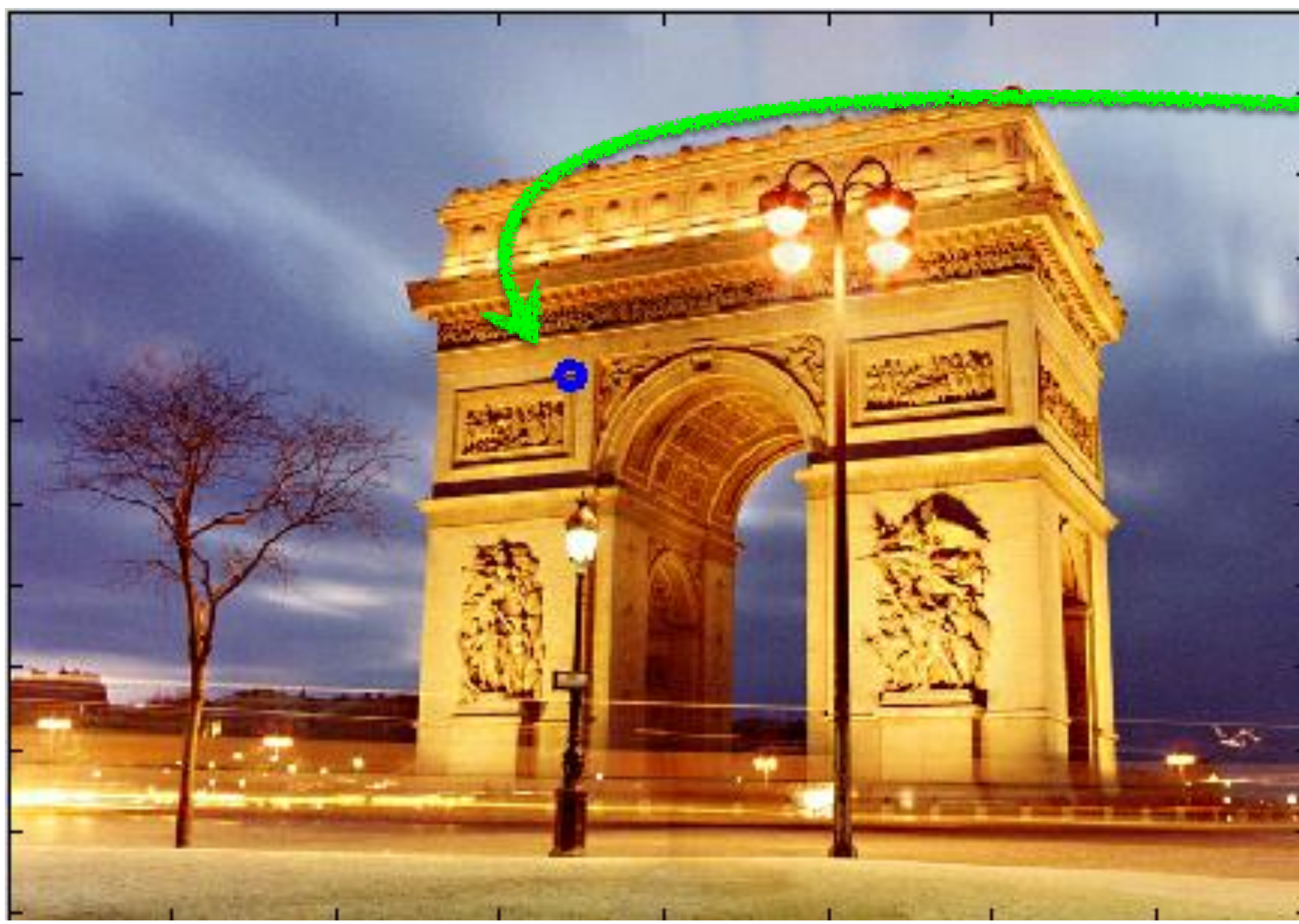
0. (Normalize points)
1. Construct the M x 9 matrix **A**
2. Find the SVD of **A**
3. Entries of **F** are the elements of column of **V** corresponding to the least singular value
4. (Enforce rank 2 constraint on F)
5. (Un-normalize F)

Example



epipolar lines

$$\mathbf{F} = \begin{bmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{bmatrix}$$



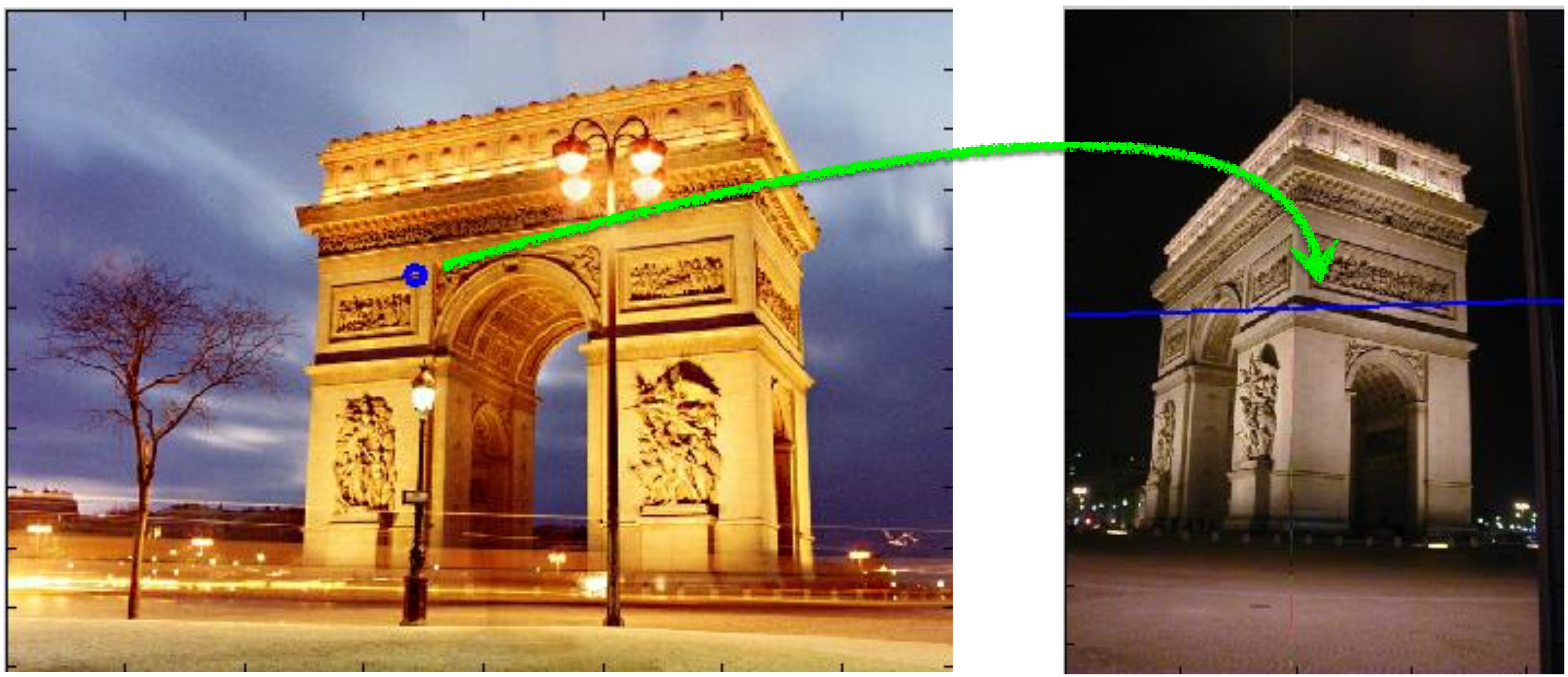
$$\boldsymbol{x} = \begin{bmatrix} 343.53 \\ 221.70 \\ 1.0 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$

$$\boldsymbol{l}' = \mathbf{F}\boldsymbol{x}$$

$$= \begin{bmatrix} 0.0295 \\ 0.9996 \\ -265.1531 \end{bmatrix}$$



Where is the epipole?

How would you compute it?

$$\mathbf{F}e = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

(hint: this is a homogeneous linear system)

$$\mathbf{F}\boldsymbol{e} = \mathbf{0}$$

The epipole is in the right null space of **F**

How would you solve for the epipole?

(hint: this is a homogeneous linear system)

S V D !

```
>> [u,d] = eigs(F' * F)
```

eigenvectors

```
u =
     -0.0013    0.2586   -0.9660
      0.0029   -0.9660   -0.2586
      1.0000    0.0032   -0.0005
```

eigenvalue

```
d = 1.0e8*
   -1.0000         0         0
         0   -0.0000         0
         0         0   -0.0000
```

```
>> [u,d] = eigs(F' * F)
```

eigenvectors

```
u =
     -0.0013     0.2586    -0.9660
      0.0029    -0.9660    -0.2586
      1.0000     0.0032    -0.0005
```

eigenvalue

```
d = 1.0e8*
    -1.0000          0          0
          0    -0.0000          0
          0          0    -0.0000
```

```
>> [u,d] = eigs(F' * F)
```

eigenvectors

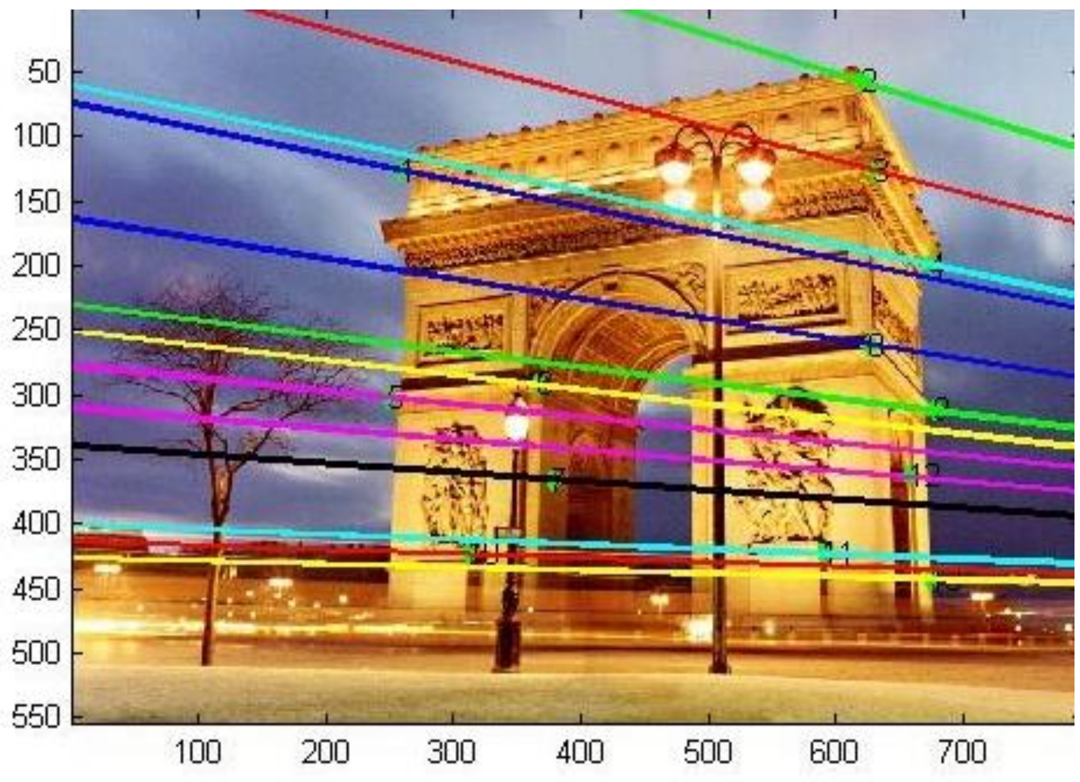
```
u =
     -0.0013     0.2586    -0.9660
      0.0029    -0.9660    -0.2586
      1.0000     0.0032    -0.0005
```

eigenvalue

```
d = 1.0e8*
    -1.0000          0          0
          0    -0.0000          0
          0          0    -0.0000
```

Eigenvector associated with smallest eigenvalue

```
>> uu = u(:,3)
(   -0.9660     -0.2586     -0.0005)
```




```
>> [u,d] = eigs(F' * F)
```

eigenvectors

```
u =
     -0.0013     0.2586   -0.9660
      0.0029    -0.9660   -0.2586
      1.0000     0.0032   -0.0005
```

eigenvalue

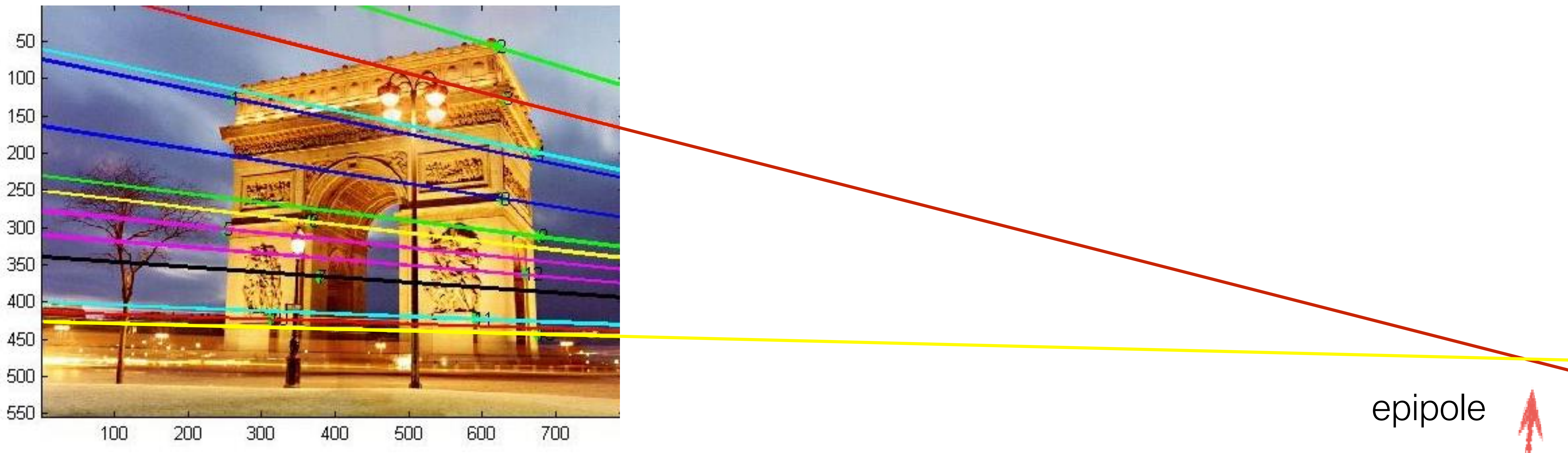
```
d = 1.0e8*
   -1.0000          0          0
         0    -0.0000          0
         0          0    -0.0000
```

Eigenvector associated with smallest eigenvalue

```
>> uu = u(:,3)
(  -0.9660    -0.2586    -0.0005)
```

Epipole projected to image coordinates

```
>> uu / uu(3)
(1861.02     498.21     1.0)
```



Epipole projected to image coordinates

```
>> uu / uu(3)
(1861.02     498.21     1.0)
```

References

Basic reading:

- Szeliski textbook, Sections 7.1, 7.2, 11.1.
- Hartley and Zisserman, Chapters 9, 11, 12.