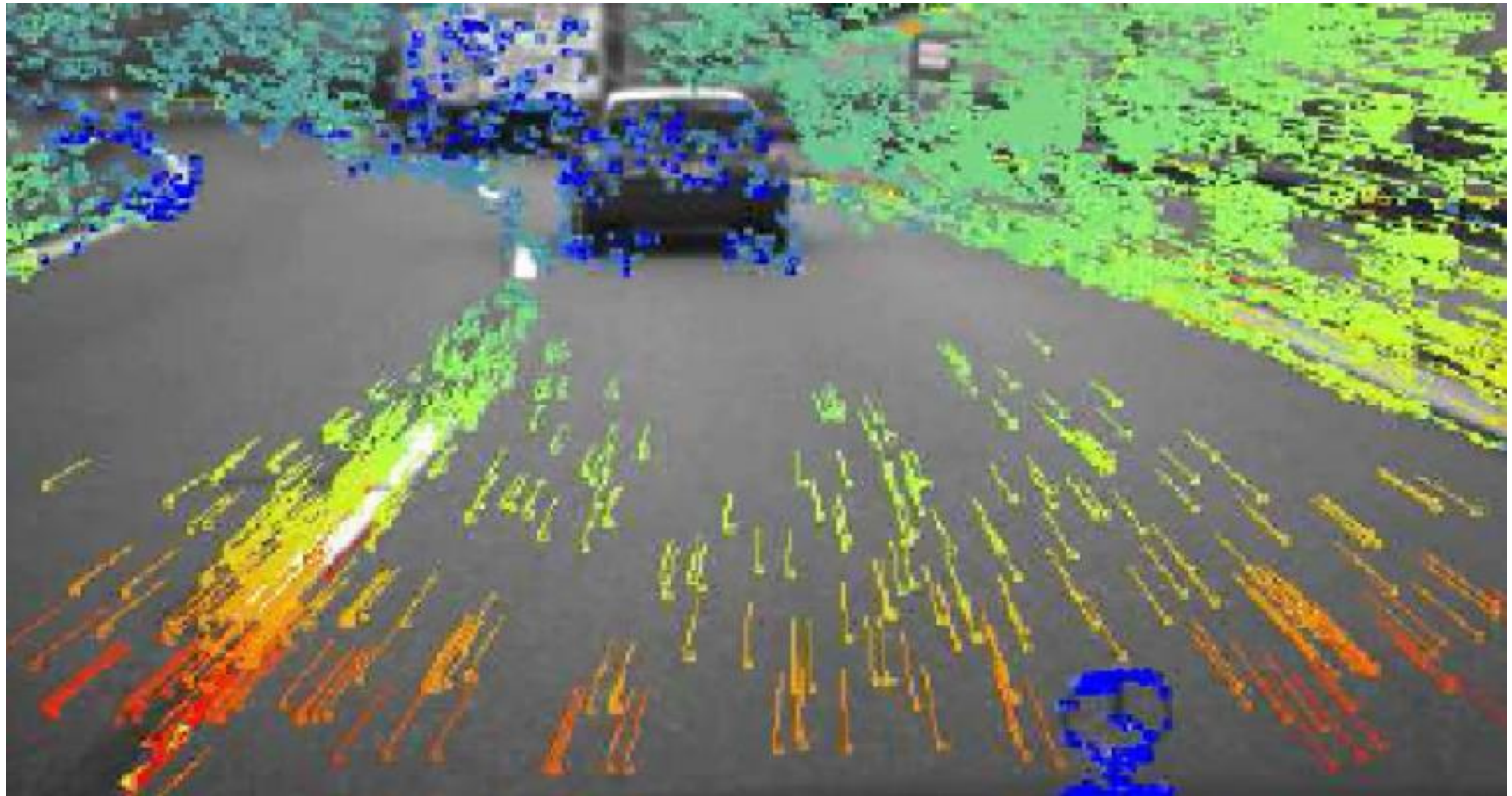# Optical flow

# Course announcements

- Homework 6 has been posted and is due on April 24$^{th}$.
    - Any questions about the homework?
    - How many of you have looked at/started/finished homework 6?

- Wednesday's office hours will be covered by Yannis at the graphics lounge.

# Overview of today's lecture

- Quick intro to vision for video.

- Optical flow.

- Constant flow.

- Horn-Schunck flow.

# Slide credits

Most of these slides were adapted from:

- Kris Kitani (16-385, Spring 2017).

# Course overview

1. Image processing. ← Lectures 1 – 7
See also 18-793: Image and Video Processing

2. Geometry-based vision. ← Lectures 7 – 12
See also 16-822: Geometry-based Methods in Vision

3. Physics-based vision. ← Lectures 13 – 16
See also 16-823: Physics-based Methods in Vision
See also 15-463: Computational Photography

4. Semantic vision. ← Lectures 17 – 21
See also 16-824: Vision Learning and Recognition

5. Dealing with motion. ← We are starting this part now

# Computer vision for video

$$\begin{bmatrix} I_x(\boldsymbol{p}_1) & I_y(\boldsymbol{p}_1) \\ I_x(\boldsymbol{p}_2) & I_y(\boldsymbol{p}_2) \\ \vdots & \vdots \\ I_x(\boldsymbol{p}_{25}) & I_y(\boldsymbol{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\boldsymbol{p}_1) \\ I_t(\boldsymbol{p}_2) \\ \vdots \\ I_t(\boldsymbol{p}_{25}) \end{bmatrix}$$

$$\min_{\boldsymbol{u}, \boldsymbol{v}} \sum_{ij} \left\{ E_d(i,j) + \lambda E_s(i,j) \right\}$$

Constant Flow                    Horn-Schunck

**Optical Flow**

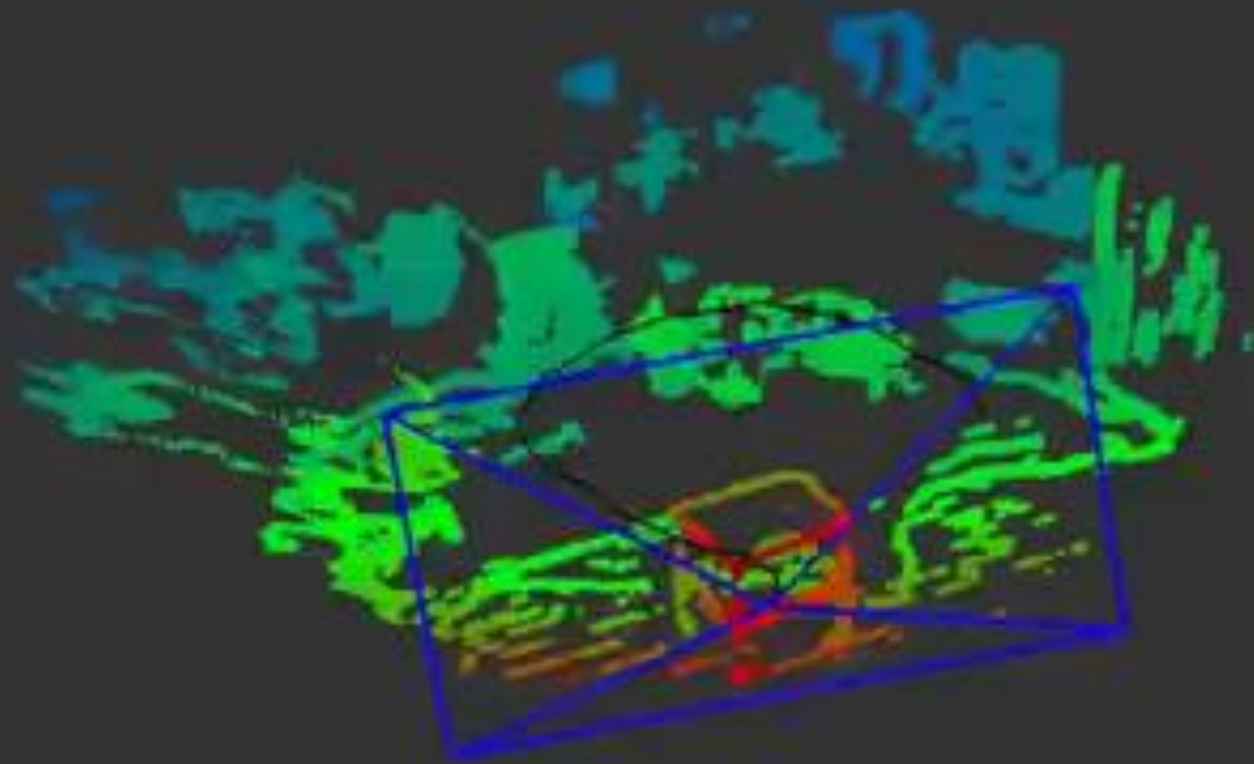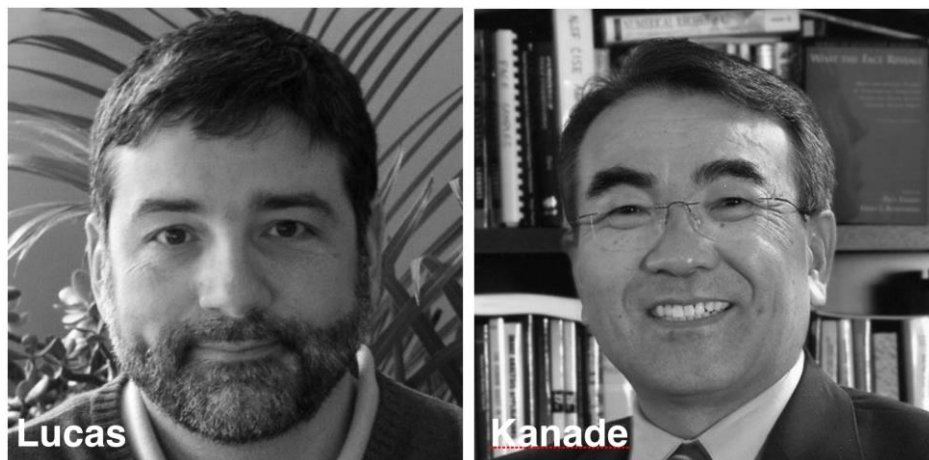# Optical flow used for feature tracking on a drone
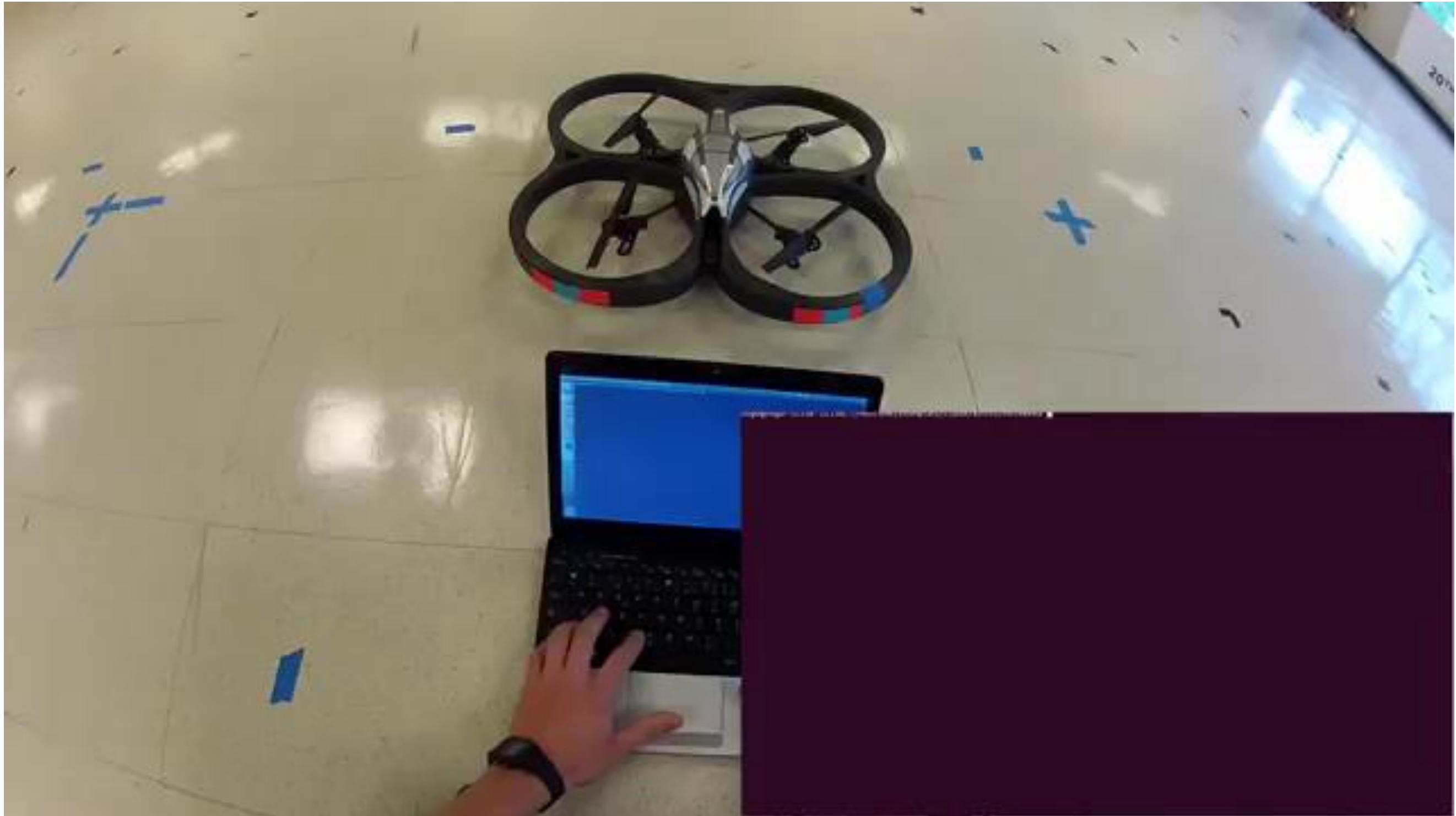
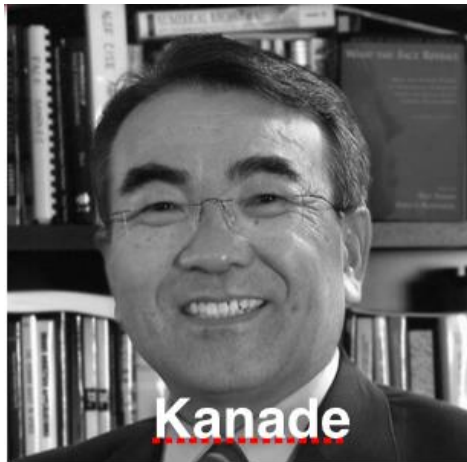# optical flow used for motion estimation in visual odometry
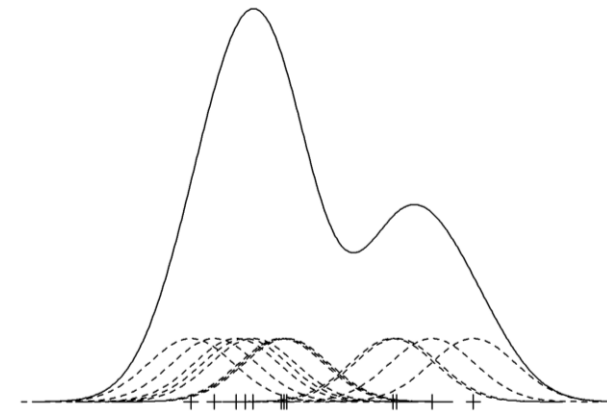
Lucas-Kanade
(Forward additive)

Baker-Matthews
(Inverse Compositional)

**Image Alignment**

KLT

Mean shift

Kalman Filtering
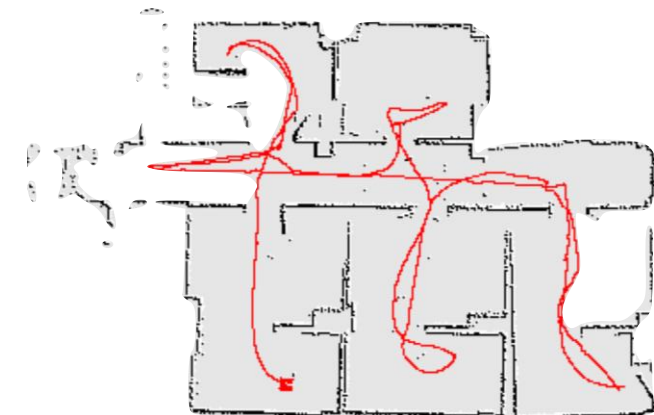
SLAM

**Tracking in Video**

# Optical flow

# Optical Flow

## Problem Definition

Given two consecutive image frames,
estimate the motion of each pixel

## Assumptions

Brightness constancy

Small motion

# Optical Flow

(Problem definition)
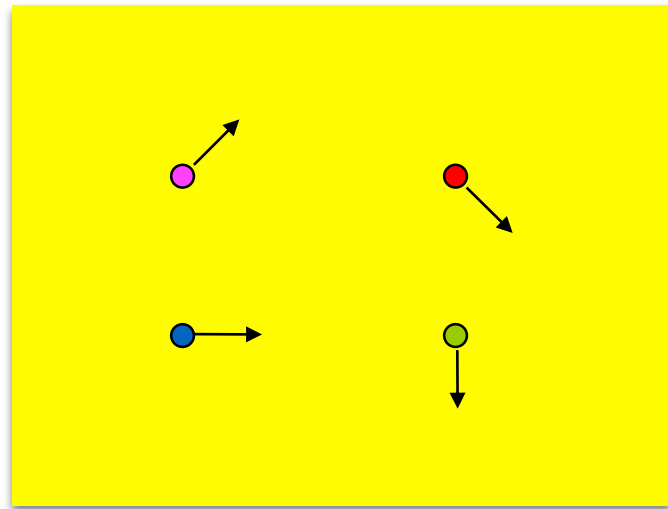


$$I(x, y, t)$$
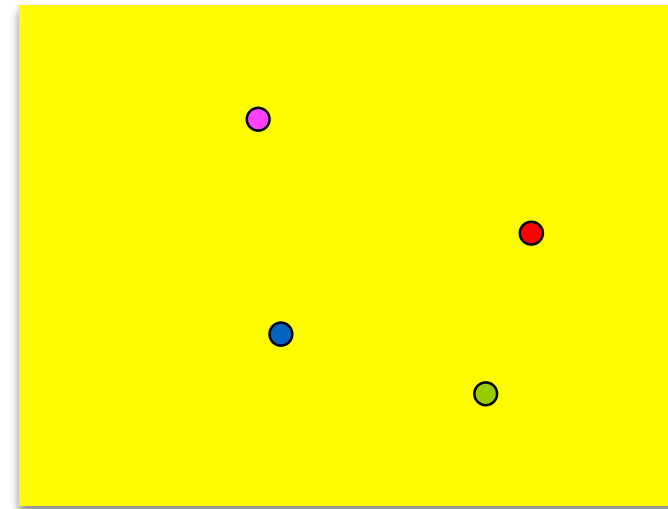
$$I(x, y, t')$$

Estimate the motion (flow) between these two consecutive images

*How is this different from estimating a 2D transform?*

# Key Assumptions

(unique to optical flow)

## **Color Constancy**

(Brightness constancy for intensity images)

Implication: allows for pixel to pixel comparison
(not image features)

## **Small Motion**

(pixels only move a little bit)

Implication: linearization of the brightness
constancy constraint

# Approach
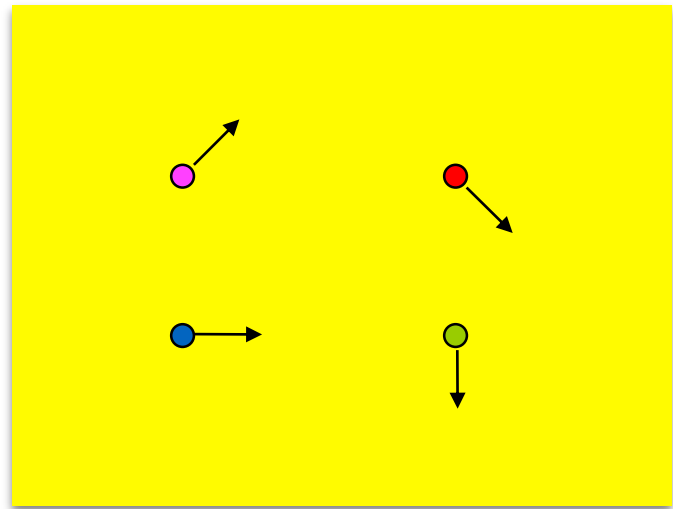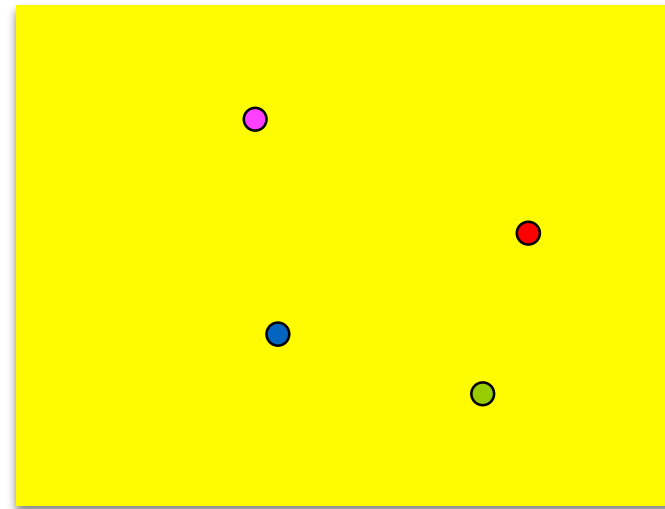


$$I(x, y, t) \qquad I(x, y, t')$$

Look for nearby pixels with the same color

(small motion)          (color constancy)

# Assumption 1
# Brightness constancy

Scene point moving through image sequence



$(x(1), y(1))$

$(x(2), y(2))$

...

$(x(k), y(k))$

# Brightness constancy

Scene point moving through image sequence



$I(x, y, 1)$     $I(x, y, 2)$     $I(x, y, k)$

# Brightness constancy

Scene point moving through image sequence



**Assumption:Brightness of the point will remain the same**

# Brightness constancy

Scene point moving through image sequence



$I(x, y, 1)$      $I(x, y, 2)$      $I(x, y, k)$
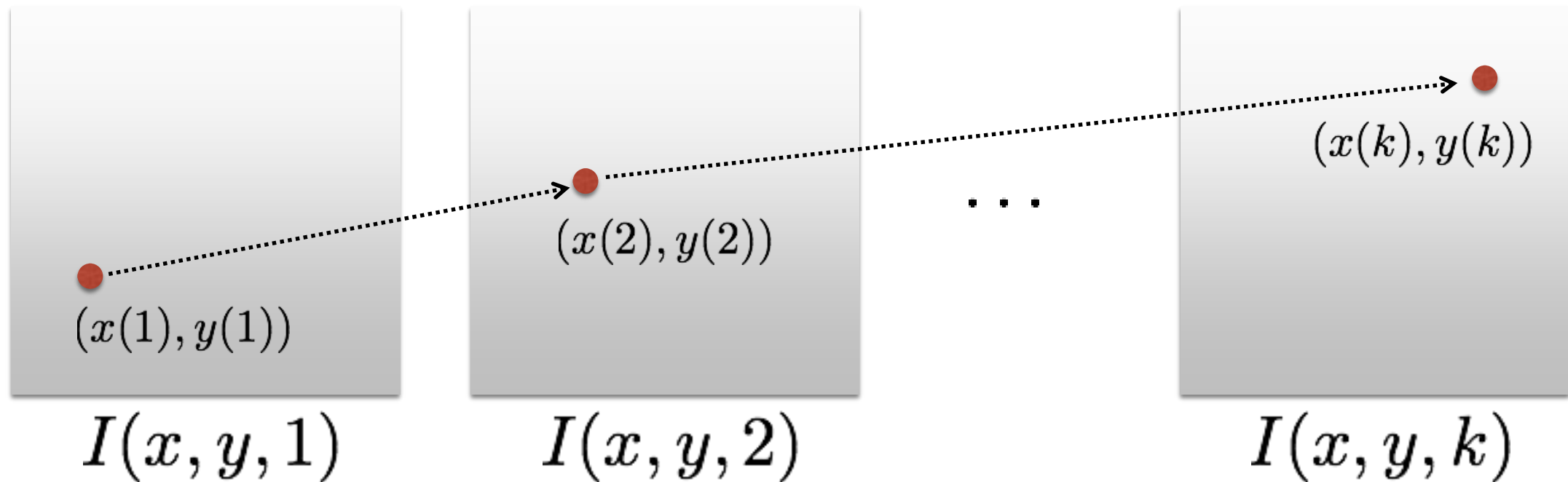
**Assumption:Brightness of the point will remain the same**

$$I(x(t), y(t), t) = C$$

constant

# Assumption 2
# Small motion



$I(x, y, t)$      $I(x, y, t + \delta t)$

# Small motion



$$(x + u\delta t, y + v\delta t)$$

$$(x, y)$$

$$(x, y)$$

$$I(x, y, t)$$    $$I(x, y, t + \delta t)$$

# Small motion



$$I(x, y, t) \qquad\qquad I(x, y, t + \delta t)$$

Optical flow (velocities): $(u, v)$     Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

# Small motion

$(x + u\delta t, y + v\delta t)$

$(x, y)$

$(x, y)$
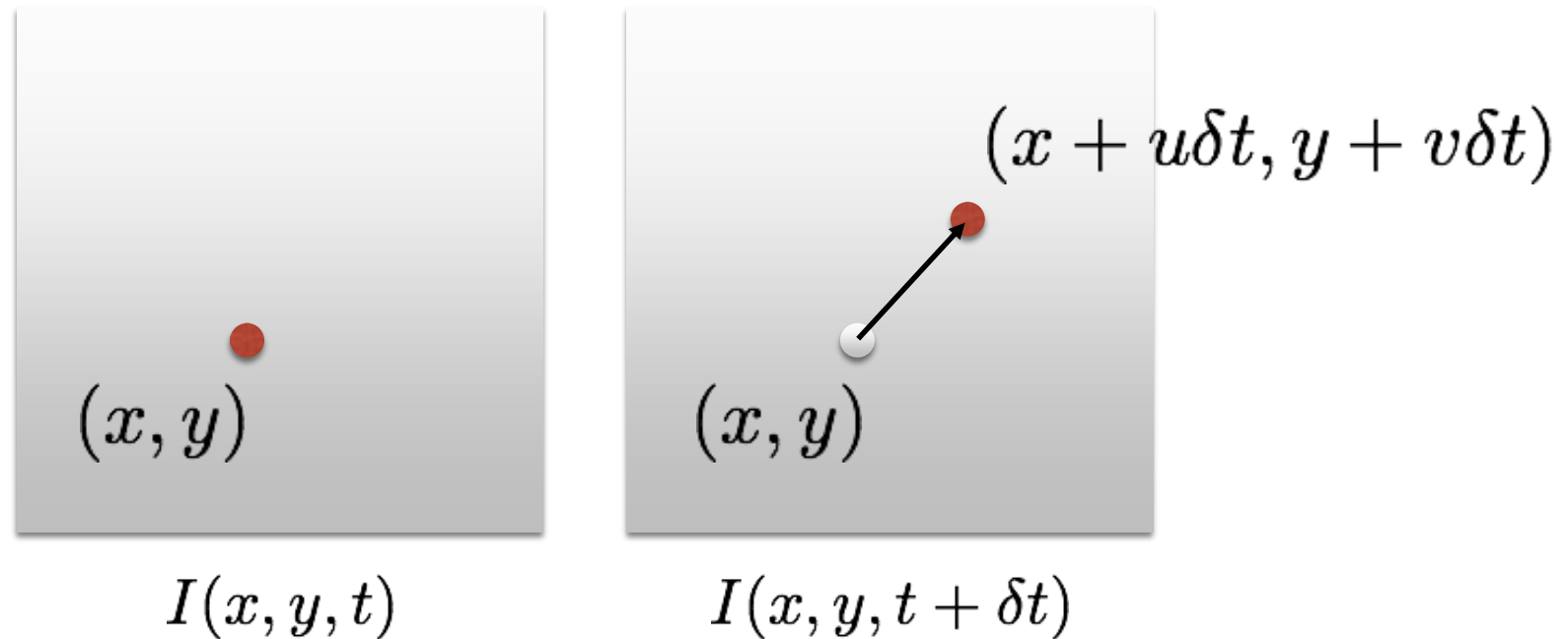
$I(x, y, t)$

$I(x, y, t + \delta t)$

Optical flow (velocities): $(u, v)$     Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

For a _really small space-time step_…

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

… the brightness between two consecutive image frames is the same

These assumptions yield the …

**Brightness Constancy Equation**

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

total derivative                    partial derivative

*Equation is not obvious. Where does this come from?*

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

**Insight:**
If the time step is really small,
we can *linearize* the intensity function

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion
(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

assuming small motion

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

partial derivative

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

fixed point

assuming small motion

cancel terms

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

**Multivariable Taylor Series Expansion**
(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

cancel terms

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

divide by $\delta t$

take limit $\delta t \rightarrow 0$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

**Multivariable Taylor Series Expansion**
(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

divide by $\delta t$

take limit $\delta t \to 0$

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion
(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

divide by $\delta t$

take limit $\delta t \to 0$

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

**Brightness Constancy Equation**

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

**Brightness Constancy Equation**

$$I_x u + I_y v + I_t = 0$$

(x-flow)   (y-flow)

shorthand notation

$$\nabla I^\top \boldsymbol{v} + I_t = 0$$

(1 x 2)   (2 x 1)

vector form

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

temporal gradient

*How do you compute these terms?*

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

Forward difference
Sobel filter
Scharr filter
…

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference
Sobel filter
Scharr filter

…

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference
Sobel filter
Scharr filter

…

frame differencing

# Frame differencing

$$t \qquad\qquad t+1 \qquad\qquad I_t = \frac{\partial I}{\partial t}$$

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 10 | 10 | 10 | 10 |
| 1 | 10 | 10 | 10 | 10 |
| 1 | 10 | 10 | 10 | 10 |
| 1 | 10 | 10 | 10 | 10 |

**-**

| 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 10 | 10 | 10 |
| 1 | 1 | 10 | 10 | 10 |
| 1 | 1 | 10 | 10 | 10 |

**=**

| 0 | 0 | 0 | 0 | 0 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 9 | 9 | 9 | 9 |
| 0 | 9 | 0 | 0 | 0 |
| 0 | 9 | 0 | 0 | 0 |
| 0 | 9 | 0 | 0 | 0 |

(example of a forward difference)

# Example:
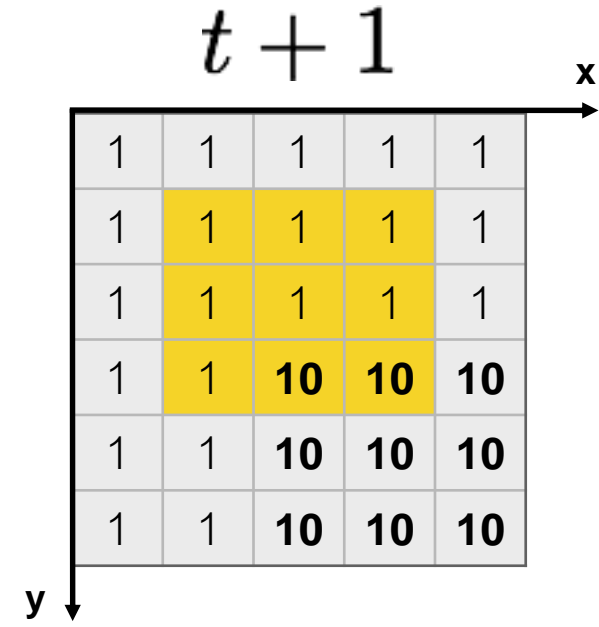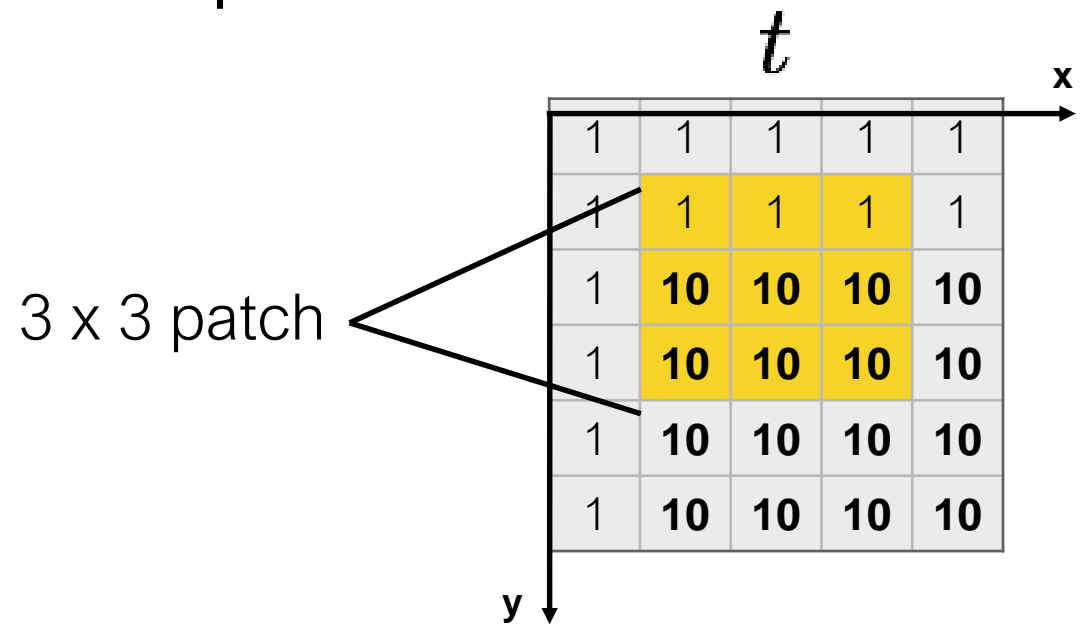


$$I_x = \frac{\partial I}{\partial x}$$
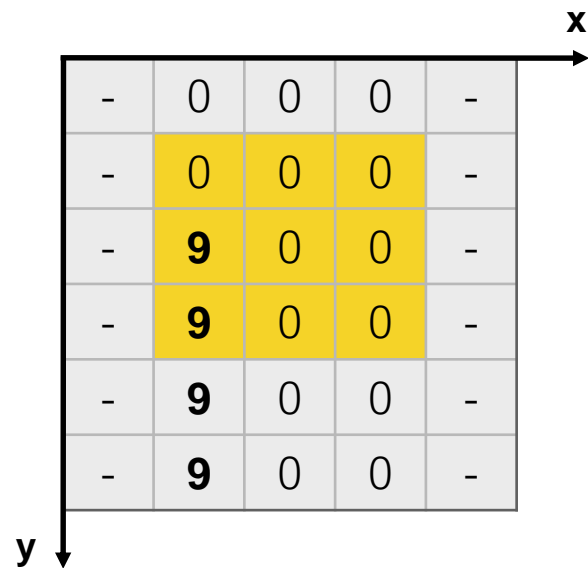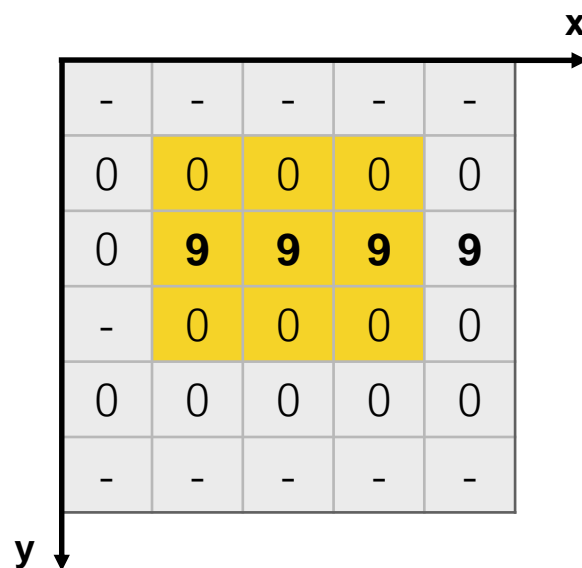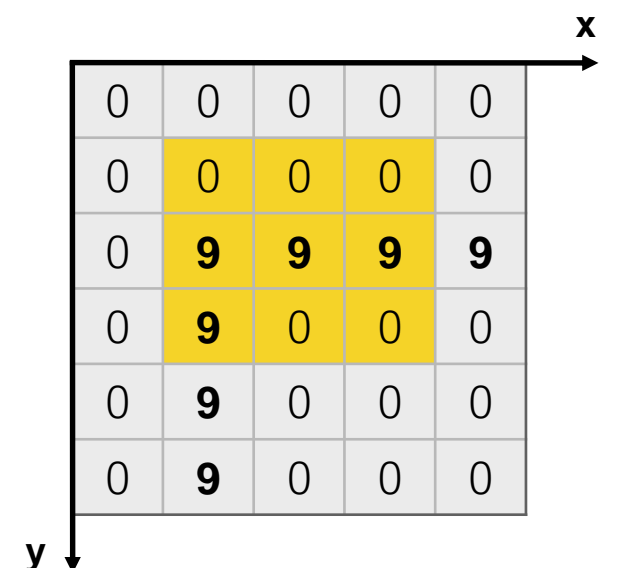


-1 0 1

$$I_y = \frac{\partial I}{\partial y}$$



-1
0
1

$$I_t = \frac{\partial I}{\partial t}$$

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference
Sobel filter
Scharr filter

…

How do you compute this?

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …



| spatial derivative | optical flow | temporal derivative |
|---|---|---|
| $I_x = \dfrac{\partial I}{\partial x} \quad I_y = \dfrac{\partial I}{\partial y}$ | $u = \dfrac{dx}{dt} \quad v = \dfrac{dy}{dt}$ | $I_t = \dfrac{\partial I}{\partial t}$ |

**spatial derivative**

Forward difference
Sobel filter
Scharr filter

…

**optical flow**

**We need to solve for this!**
(this is the unknown in the optical flow problem)

**temporal derivative**

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference
Sobel filter
Scharr filter

…

$$(u, v)$$

Solution lies on a line

Cannot be found uniquely
with a single constraint

frame differencing

Solution lies on a straight line

$$I_x u + I_y v + I_t = 0$$

many combinations of u and v will satisfy the equality

$u$

$v$
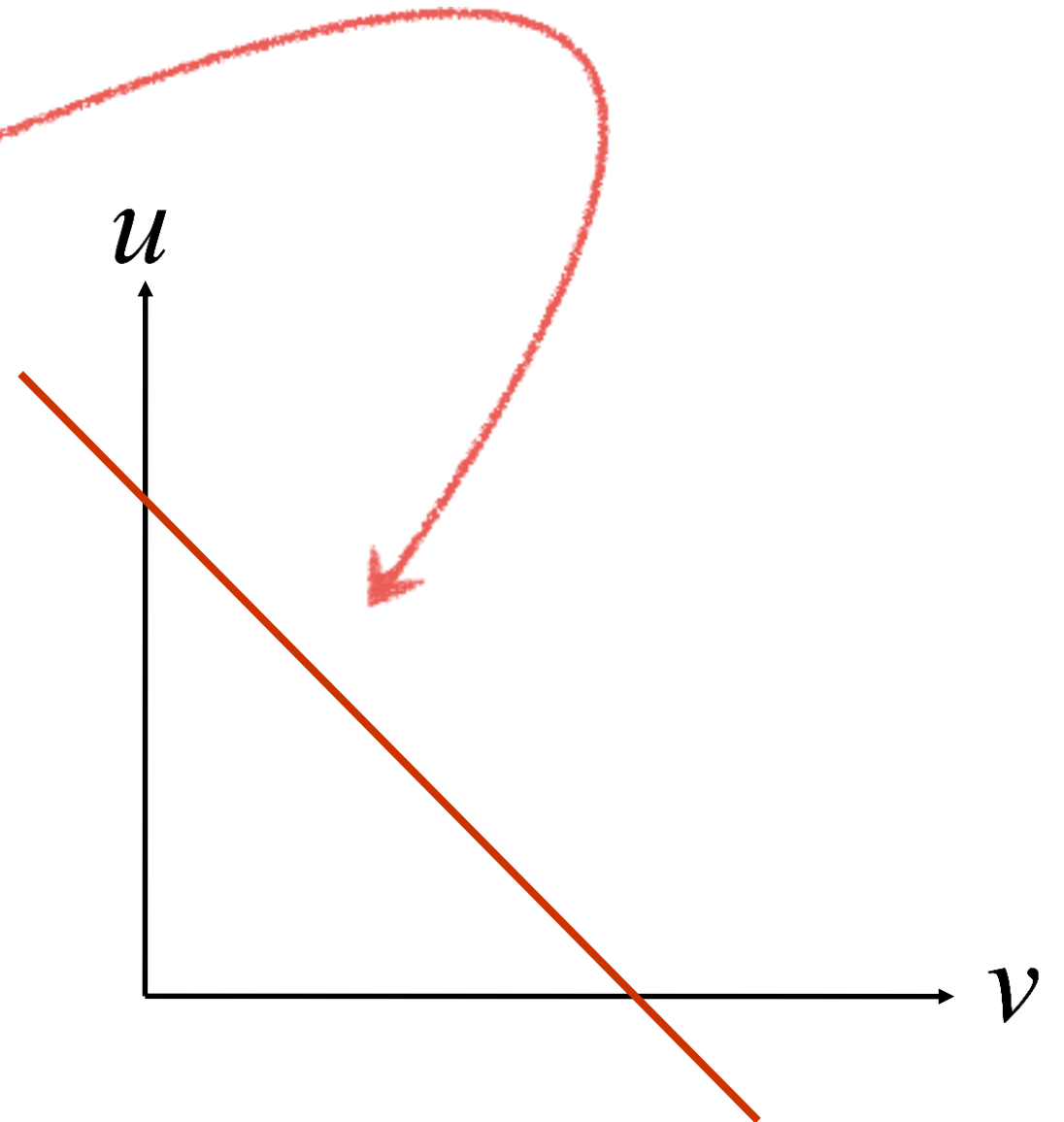
The solution cannot be determined uniquely with a
single constraint (a single pixel)

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

*How can we use the brightness constancy equation to estimate the optical flow?*

**unknown**

$$I_x u + I_y v + I_t = 0$$

**known**

*We need at least _____ equations to solve for 2 unknowns.*

**unknown**

$$I_x u + I_y v + I_t = 0$$

**known**

*Where do we get more equations (constraints)?*

# Horn-Schunck Optical Flow (1981)

brightness constancy

small motion

**'smooth' flow**

(flow can vary from pixel to pixel)

global method
(dense)

# Lucas-Kanade Optical Flow (1981)

method of differences

**'constant' flow**

(flow is constant for all pixels)

local method
(sparse)

# Constant flow

*Where do we get more equations (constraints)?*

$$I_x u + I_y v + I_t = 0$$

Assume that the surrounding patch (say 5x5) has
'**constant flow**'

## Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us ⬛ equations

## Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$I_x(\boldsymbol{p_1})u + I_y(\boldsymbol{p_1})v = -I_t(\boldsymbol{p_1})$$

$$I_x(\boldsymbol{p_2})u + I_y(\boldsymbol{p_2})v = -I_t(\boldsymbol{p_2})$$

$$\vdots$$

$$I_x(\boldsymbol{p_{25}})u + I_y(\boldsymbol{p_{25}})v = -I_t(\boldsymbol{p_{25}})$$

## Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

Using a 5 x 5 image patch, gives us 25 equations

$$\begin{bmatrix} I_x(\boldsymbol{p}_1) & I_y(\boldsymbol{p}_1) \\ I_x(\boldsymbol{p}_2) & I_y(\boldsymbol{p}_2) \\ \vdots & \vdots \\ I_x(\boldsymbol{p}_{25}) & I_y(\boldsymbol{p}_{25}) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} I_t(\boldsymbol{p}_1) \\ I_t(\boldsymbol{p}_2) \\ \vdots \\ I_t(\boldsymbol{p}_{25}) \end{bmatrix}$$

Matrix form

# Assumptions:

Flow is locally smooth

Neighboring pixels have same displacement

## Using a 5 x 5 image patch, gives us 25 equations

$$\underbrace{\begin{bmatrix} I_x(\boldsymbol{p}_1) & I_y(\boldsymbol{p}_1) \\ I_x(\boldsymbol{p}_2) & I_y(\boldsymbol{p}_2) \\ \vdots & \vdots \\ I_x(\boldsymbol{p}_{25}) & I_y(\boldsymbol{p}_{25}) \end{bmatrix}}_{\substack{A \\ 25 \times 2}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\substack{x \\ 2 \times 1}} = - \underbrace{\begin{bmatrix} I_t(\boldsymbol{p}_1) \\ I_t(\boldsymbol{p}_2) \\ \vdots \\ I_t(\boldsymbol{p}_{25}) \end{bmatrix}}_{\substack{b \\ 25 \times 1}}$$

*How many equations? How many unknowns? How do we solve this?*

## Least squares approximation

$$\hat{x} = \arg\min_{x} ||Ax - b||^2$$ is equivalent to solving $$A^\top A\hat{x} = A^\top b$$

$$\hat{x} = \arg\min_{x} ||Ax - b||^2 \quad \text{is equivalent to solving} \quad A^\top A\hat{x} = A^\top b$$

To obtain the least squares solution solve:

$$
\underset{A^\top A}{\left[\begin{array}{cc} \sum\limits_{p \in P} I_x I_x & \sum\limits_{p \in P} I_x I_y \\ \sum\limits_{p \in P} I_y I_x & \sum\limits_{p \in P} I_y I_y \end{array}\right]} \underset{\hat{x}}{\left[\begin{array}{c} u \\ v \end{array}\right]} = - \underset{A^\top b}{\left[\begin{array}{c} \sum\limits_{p \in P} I_x I_t \\ \sum\limits_{p \in P} I_y I_t \end{array}\right]}
$$

where the summation is over each pixel $p$ in patch $P$

$$x = (A^\top A)^{-1} A^\top b$$

$$\hat{x} = \arg\min_{x} ||Ax - b||^2 \quad \text{is equivalent to solving} \quad A^\top A \hat{x} = A^\top b$$

To obtain the least squares solution solve:

$$A^\top A \qquad\qquad \hat{x} \qquad\qquad A^\top b$$

$$\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = - \begin{bmatrix} \sum_{p \in P} I_x I_t \\ \sum_{p \in P} I_y I_t \end{bmatrix}$$

where the summation is over each pixel $p$ in patch $P$

Sometimes called 'Lucas-Kanade Optical Flow'
(can be interpreted to be a special case of the LK method with a translational warp model)

# When is this solvable?

$$A^\top A \hat{x} = A^\top b$$

$A^\mathsf{T}A$ should be invertible

$A^\mathsf{T}A$ should not be too small

$\lambda_1$ and $\lambda_2$ should not be too small

$A^\mathsf{T}A$ should be well conditioned

$\lambda_1/\lambda_2$ should not be too large ($\lambda_1$=larger eigenvalue)

*Where have you seen this before?*

$$A^\top A = \begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

*Where have you seen this before?*

$$\begin{bmatrix} \sum_{p \in P} I_x I_x & \sum_{p \in P} I_x I_y \\ \sum_{p \in P} I_y I_x & \sum_{p \in P} I_y I_y \end{bmatrix}$$

Harris Corner Detector!

# Implications

- Corners are when λ1, λ2 are big; this is also when Lucas-Kanade optical flow works best

- Corners are regions with two different directions of gradient (at least)
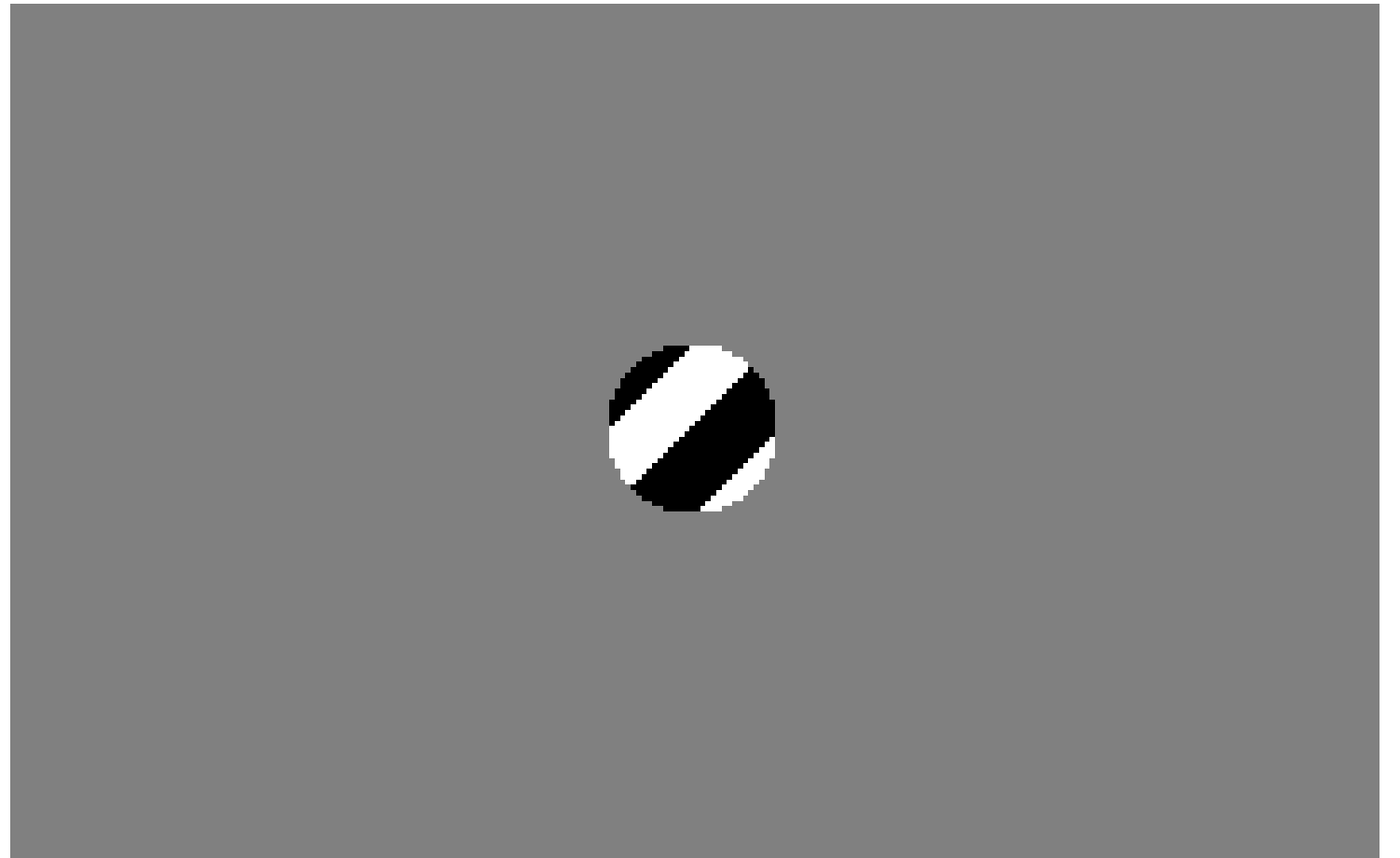
- Corners are good places to compute flow!

*What happens when you have no 'corners'?*

*You want to compute optical flow.*
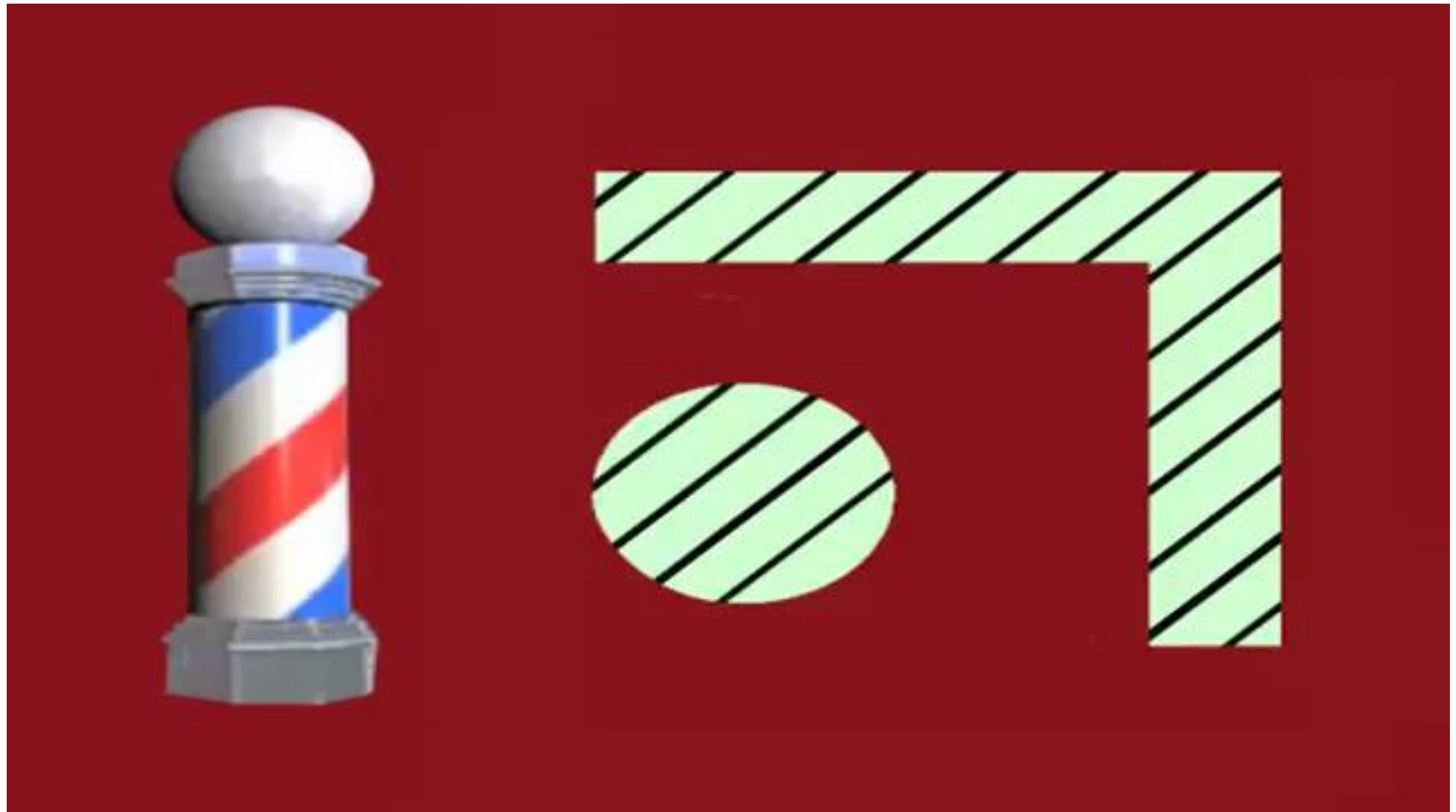*What happens if the image patch contains only a line?*
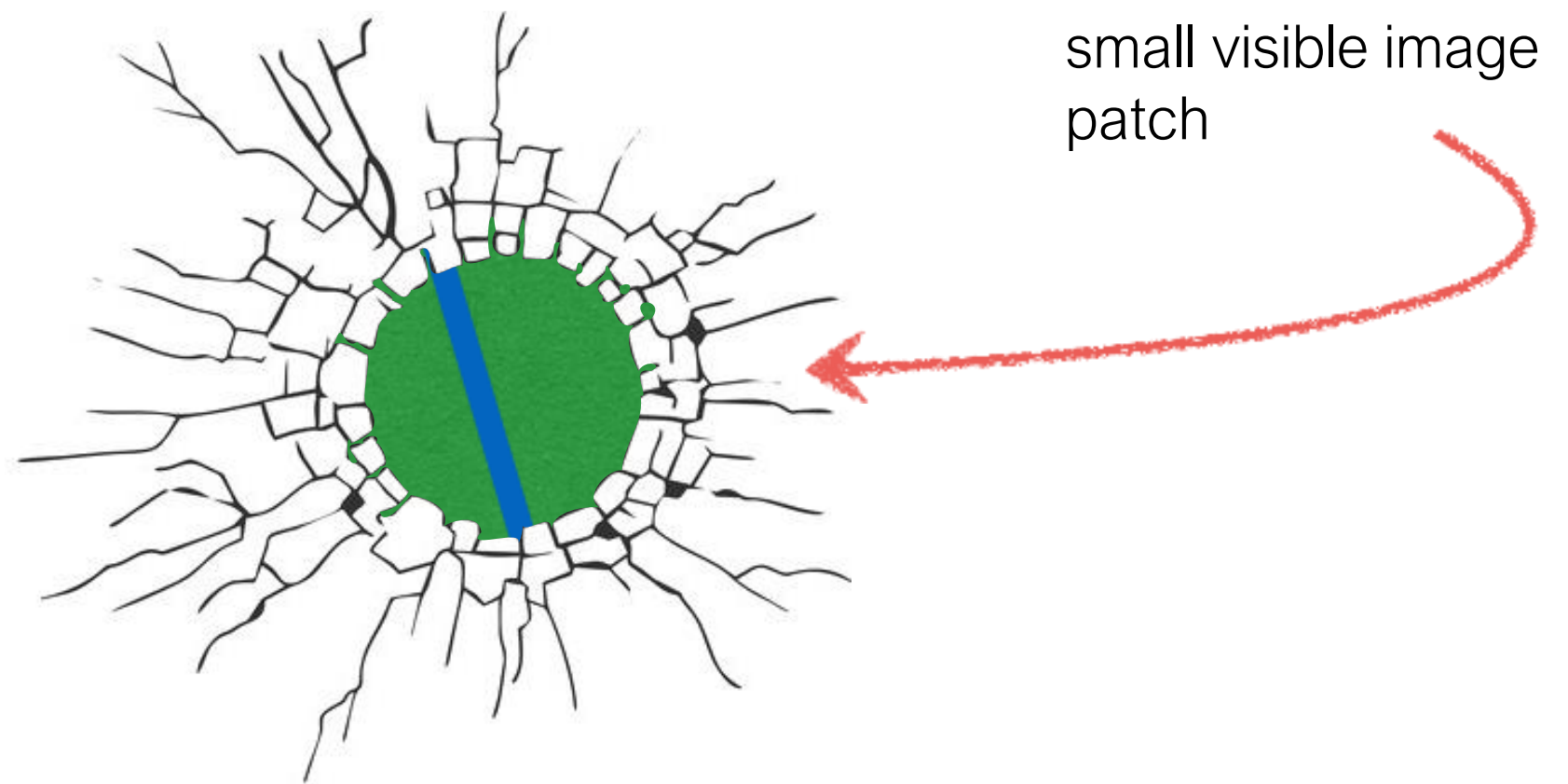
# Barber's pole illusion
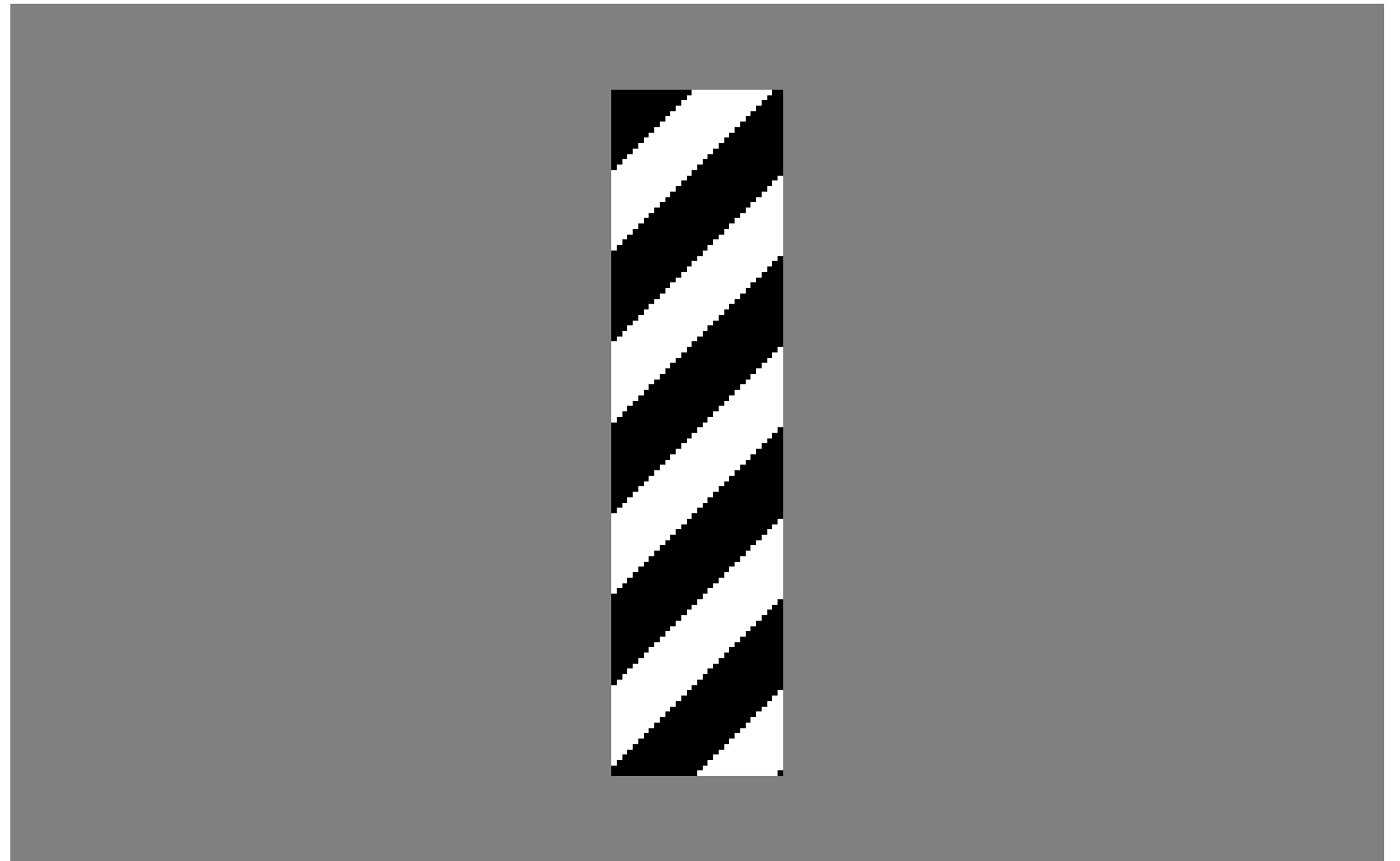
# Barber's pole illusion
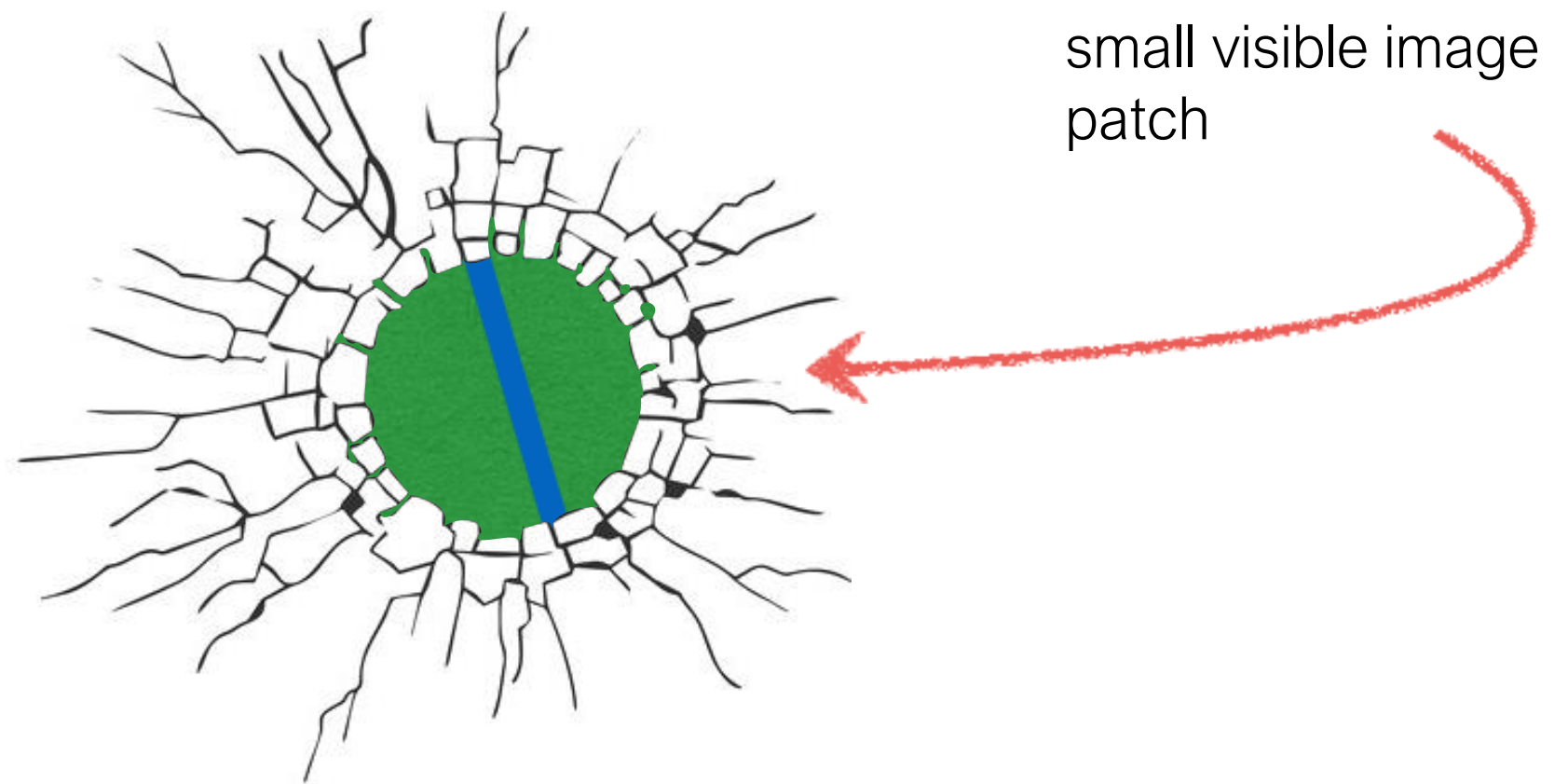
# Barber's pole illusion

# Aperture Problem



small visible image patch

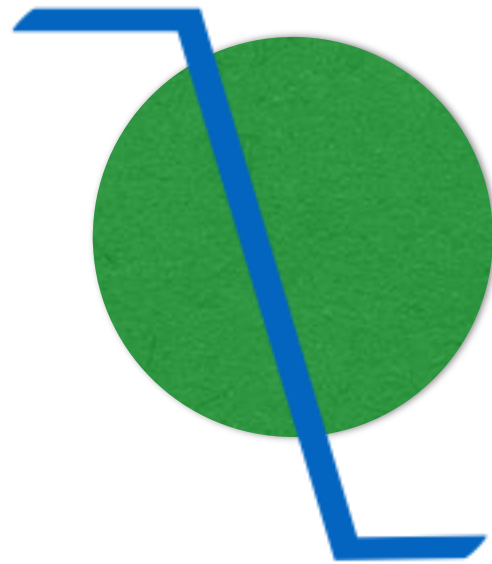*In which direction is the line moving?*

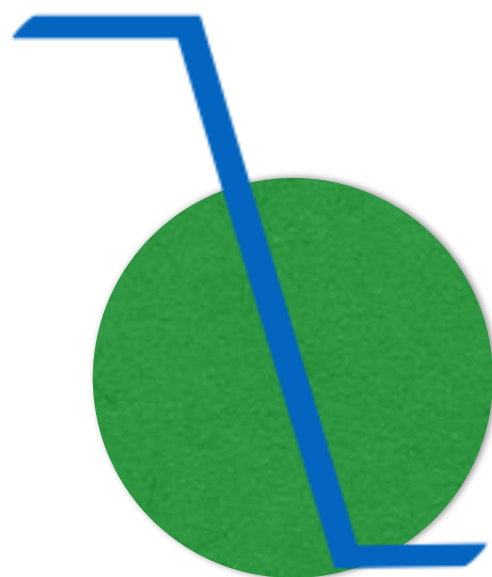# Aperture Problem



small visible image patch

*In which direction is the line moving?*
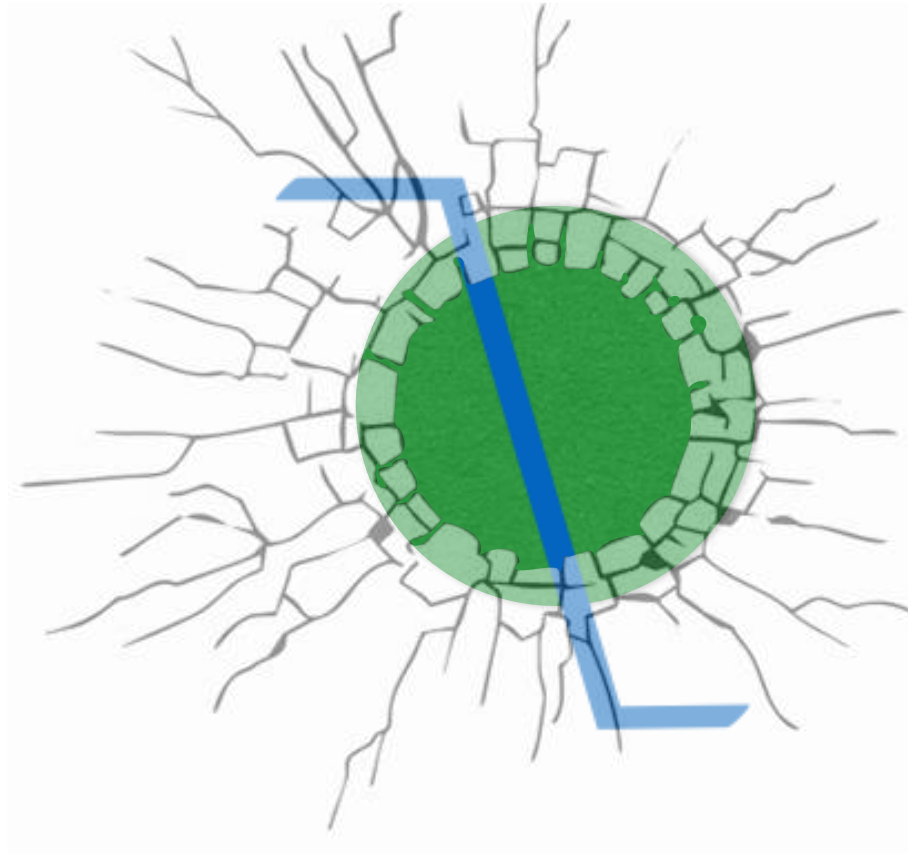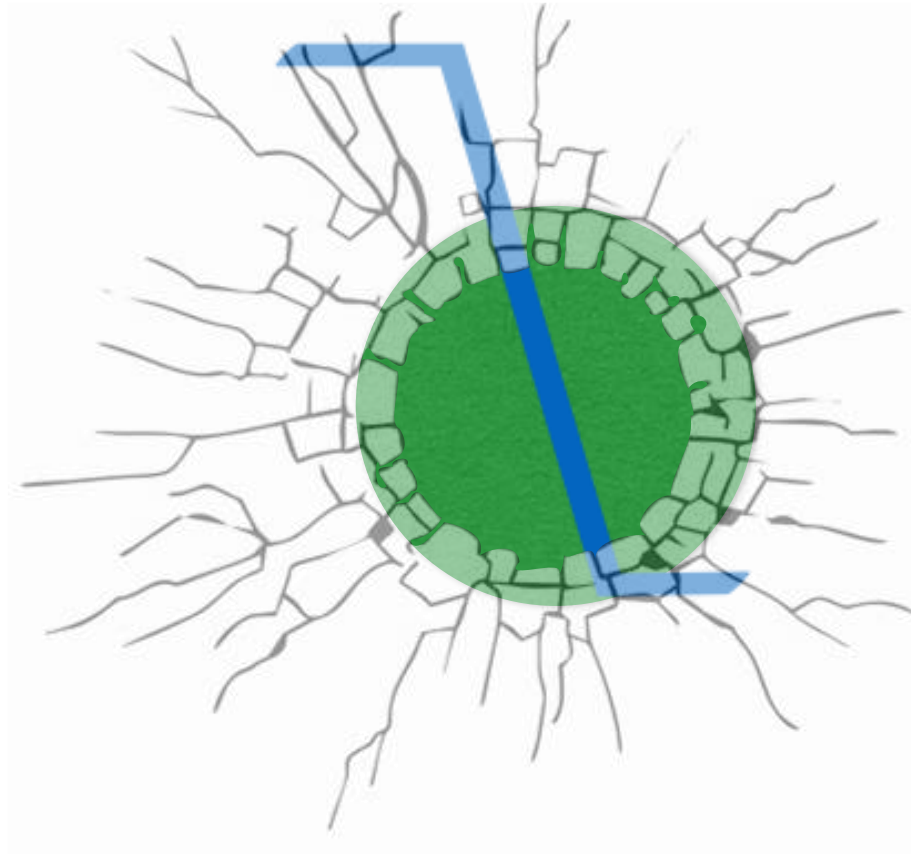
# Aperture Problem

# Aperture Problem

# Aperture Problem

# Aperture Problem

Want patches with different gradients to the avoid aperture problem

Want patches with different gradients to
the avoid aperture problem

H(x,y) = y

I(x,y)

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 |

x

y

optical flow: (1,1)

| - | - | - | - |
|---|---|---|---|
| - | 1 | 1 | 1 |
| - | 2 | 2 | 2 |
| - | 3 | 3 | 3 |
| - | 4 | 4 | 4 |

x

y
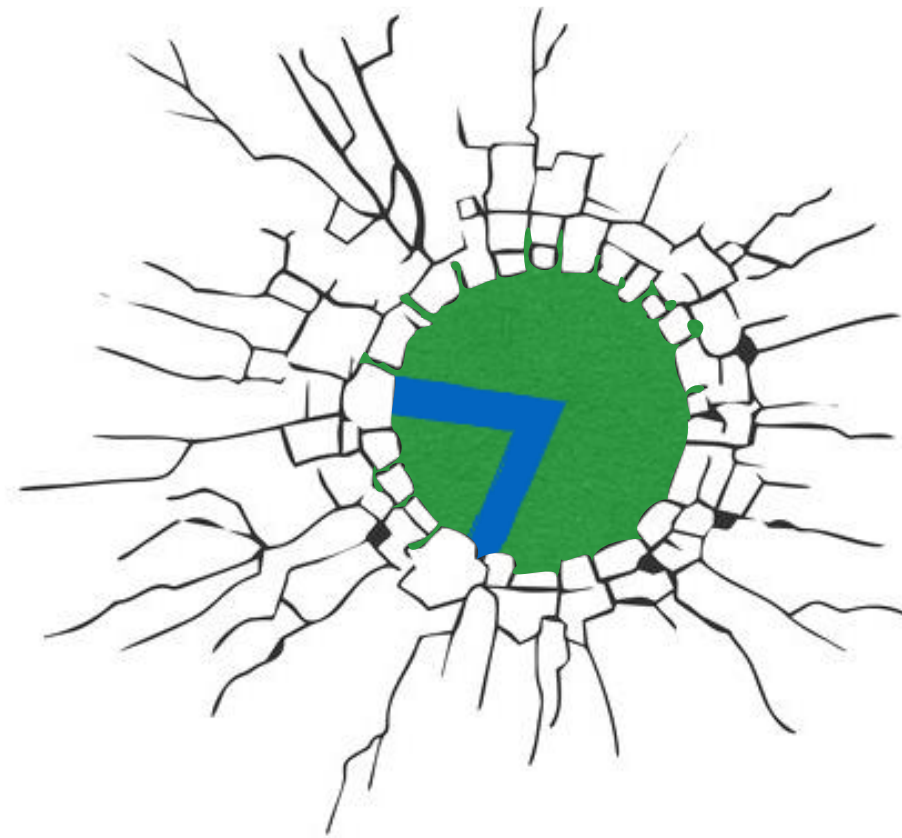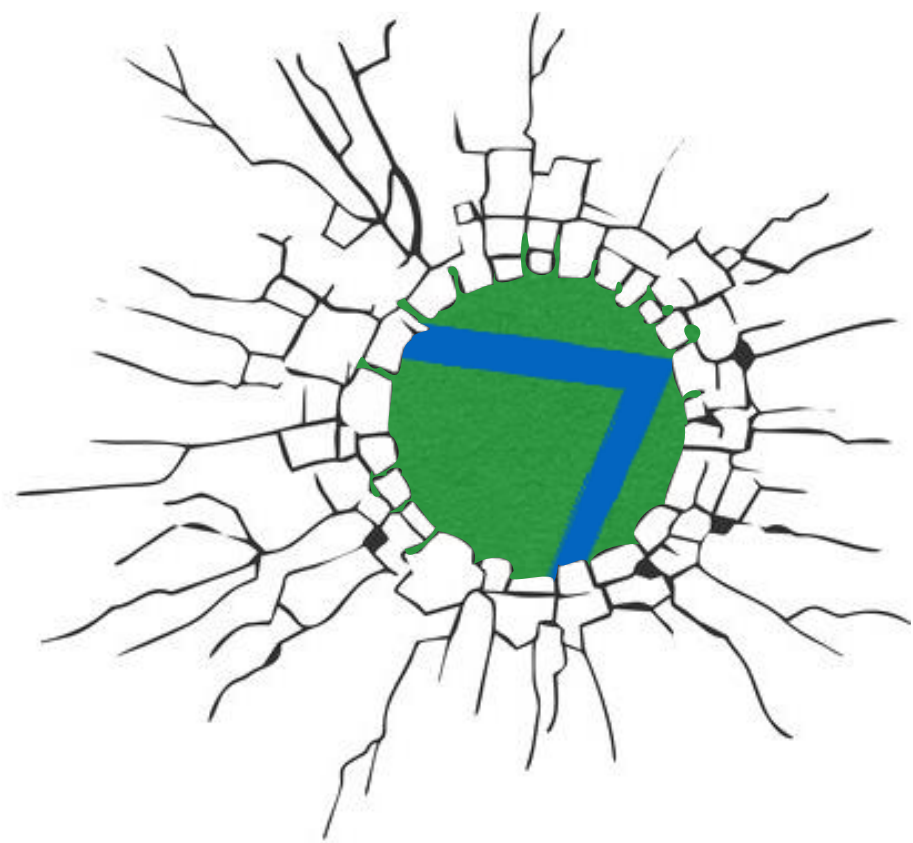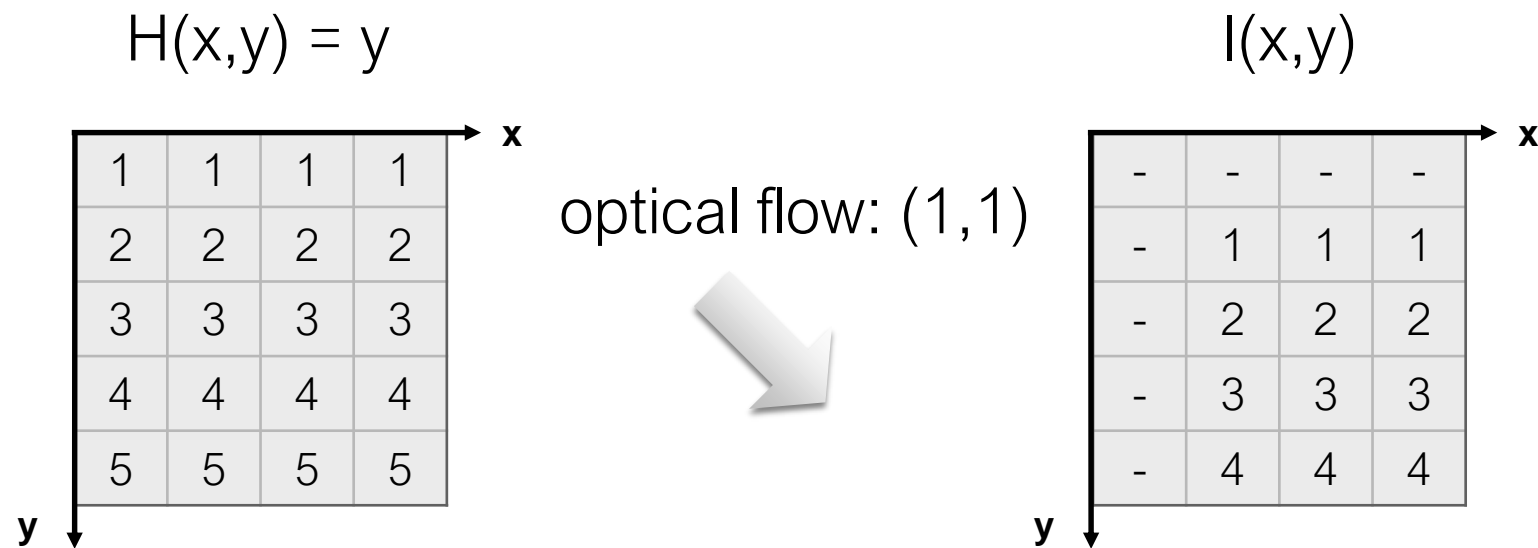
$$I_x u + I_y v + I_t = 0$$

**Compute gradients**

**Solution:**

$$I_x(3,3) = 0$$
$$I_y(3,3) = 1$$
$$I_t(3,3) = I(3,3) - H(3,3) = -1$$
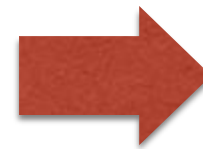
$$v = 1$$

We recover the v of the optical flow but not the u.
***This is the aperture problem.***

# Horn-Schunck optical flow

# Horn-Schunck Optical Flow (1981)

brightness constancy

small motion

## 'smooth' flow

(flow can vary from pixel to pixel)

global method
(dense)

# Lucas-Kanade Optical Flow (1981)

method of differences

## 'constant' flow

(flow is constant for all pixels)

local method
(sparse)

# Smoothness

**most objects in the world are rigid or deform elastically
moving together coherently**

**we expect optical flow fields to be smooth**

# Key idea

(of Horn-Schunck optical flow)

Enforce
**brightness constancy**

Enforce
**smooth flow field**

to compute optical flow

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# Enforce
# **brightness constancy**

$$I_x u + I_y v + I_t = 0$$

For every pixel,

$$\min_{u,v} \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

# Enforce
# **brightness constancy**

$$I_x u + I_y v + I_t = 0$$

For every pixel,

$$\min_{u,v} \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

lazy notation for $I_x(i, j)$

# Key idea

(of Horn-Schunck optical flow)

Enforce
**brightness constancy**

Enforce
**smooth flow field**

to compute optical flow

# Enforce **smooth flow field**
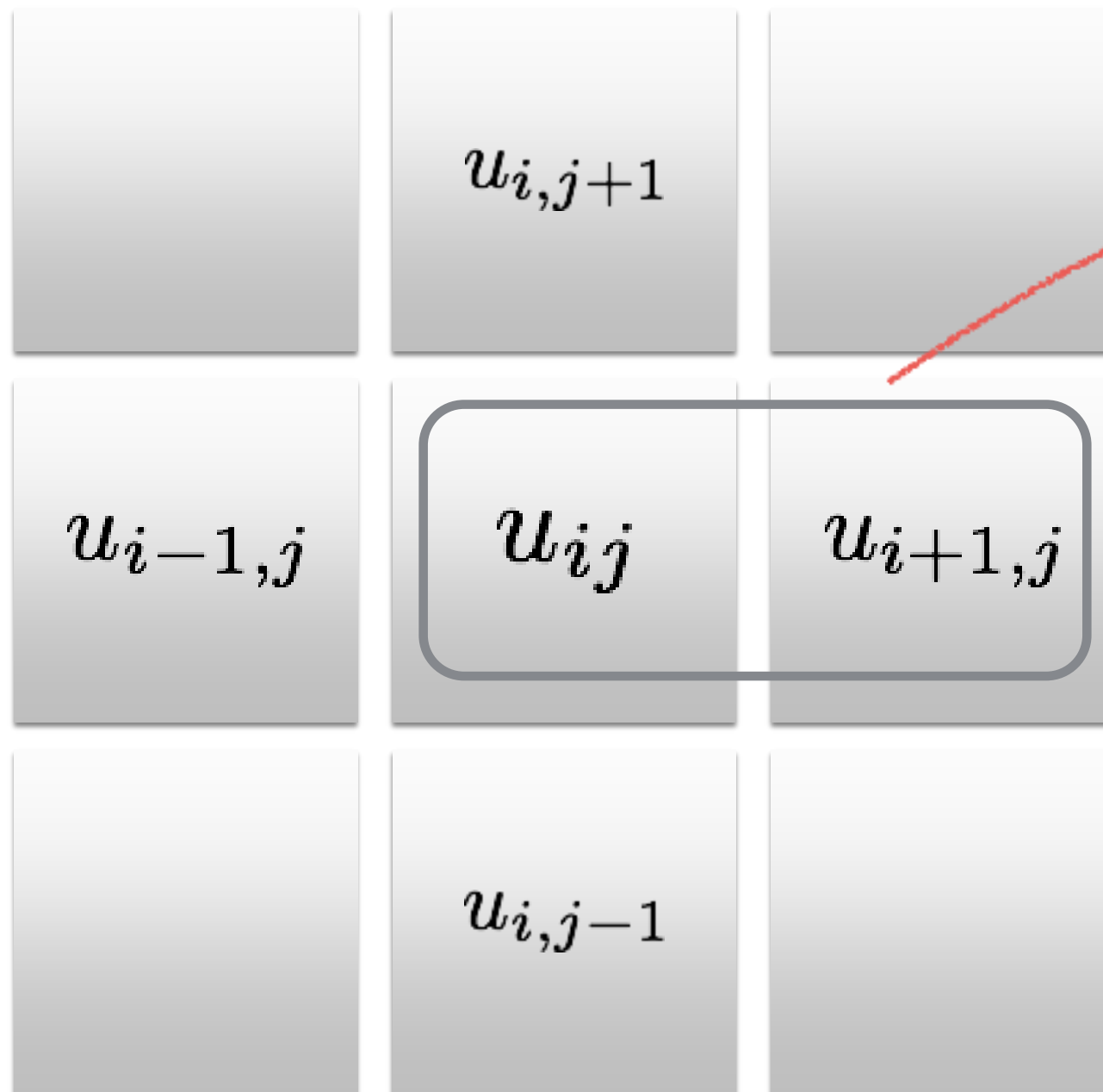


$$\min_{\boldsymbol{u}}(u_{i,j} - u_{i+1,j})^2$$

u-component of flow

*Which flow field optimizes the objective?* $\min_{\boldsymbol{u}}(u_{i,j} - u_{i+1,j})^2$
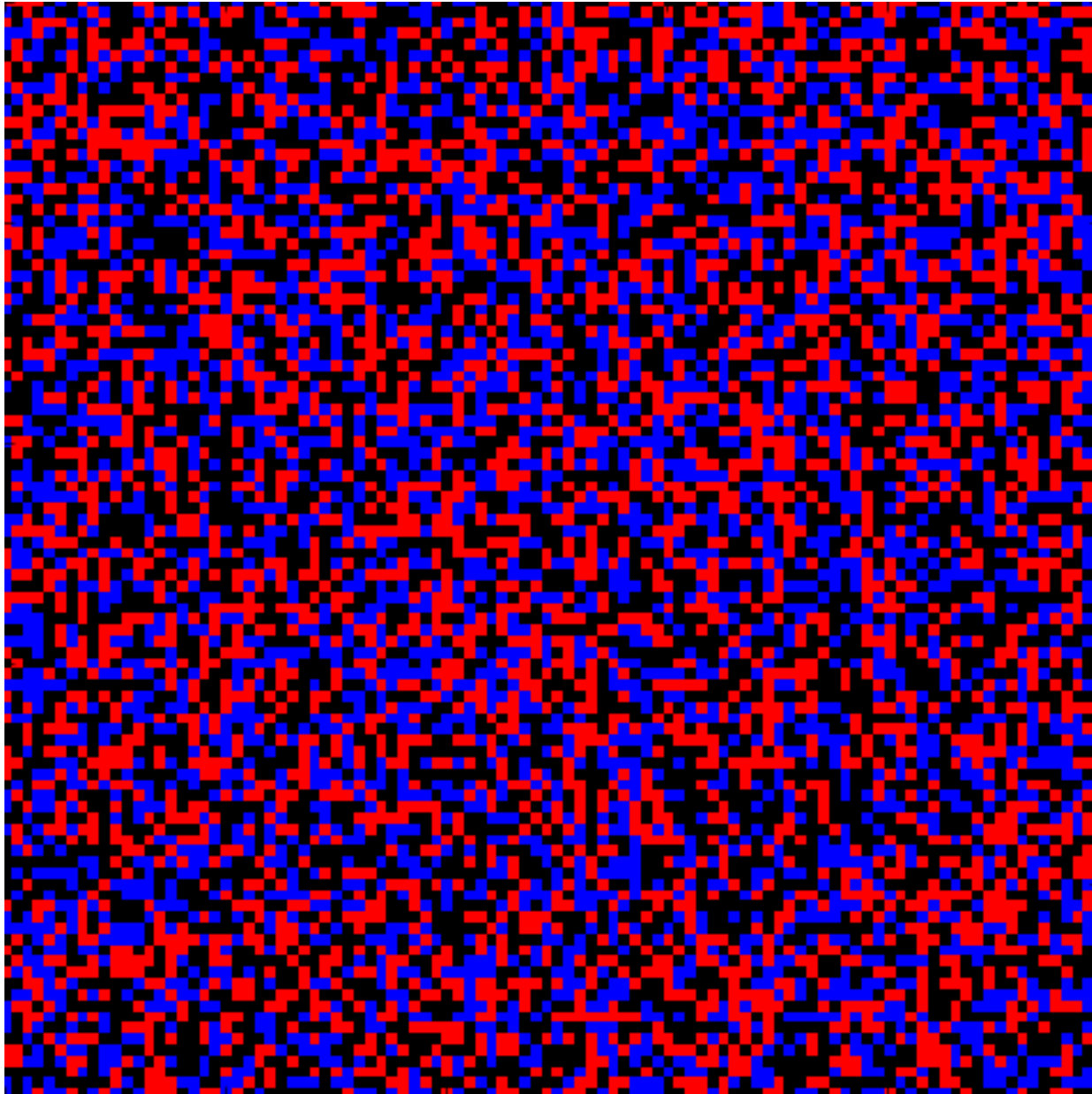


$$\sum_{ij}(u_{ij} - u_{i+1,j})^2 \quad ? \quad \sum_{ij}(u_{ij} - u_{i+1,j})^2$$

*Which flow field optimizes the objective?* $\min_{\boldsymbol{u}}(u_{i,j} - u_{i+1,j})^2$
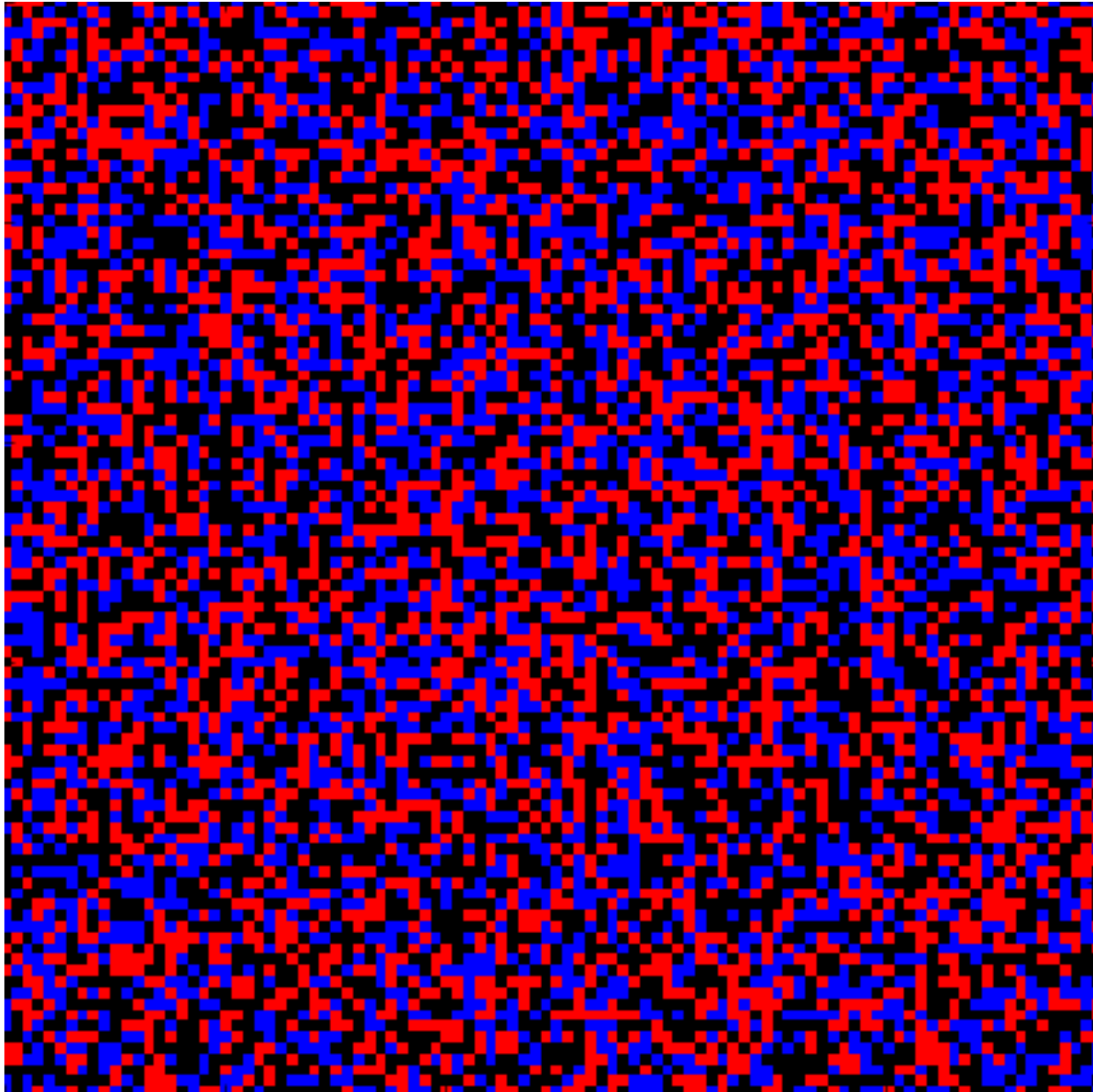


big
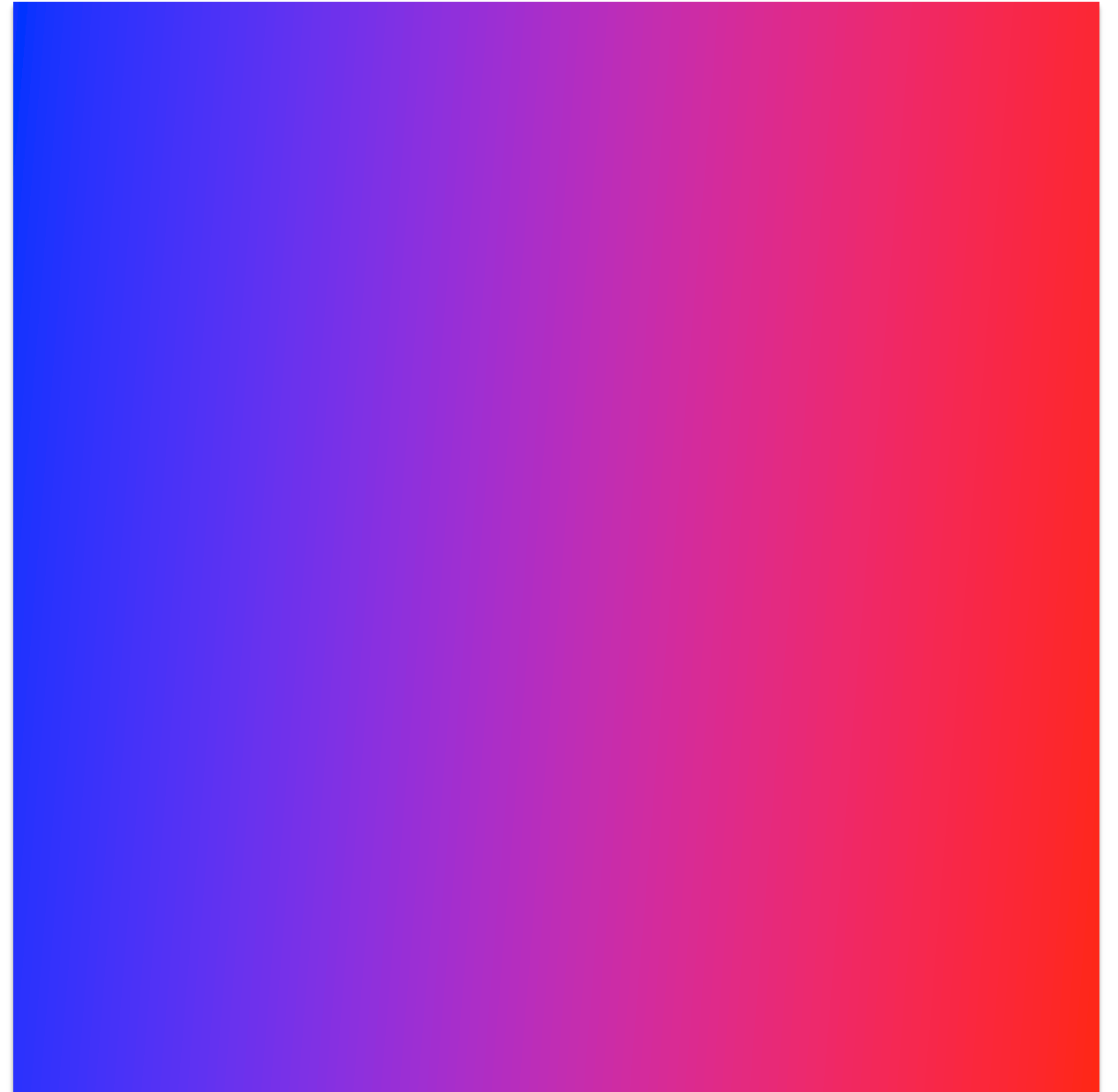


small

# Key idea

(of Horn-Schunck optical flow)

Enforce
**brightness constancy**

Enforce
**smooth flow field**

to compute optical flow

bringing it all together…

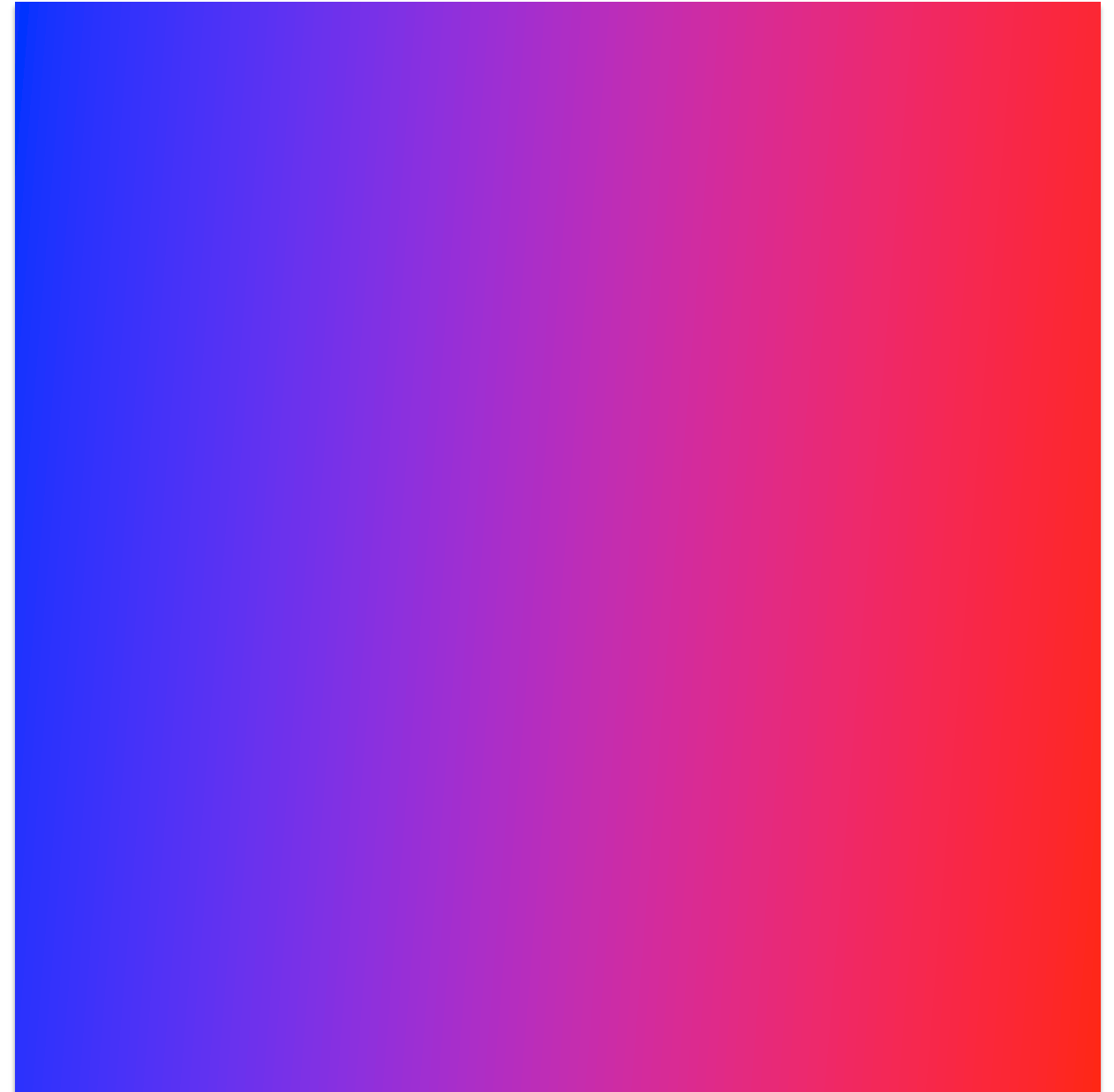# Horn-Schunck optical flow

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ \overset{\text{smoothness}}{E_s(i,j)} + \lambda \overset{\text{brightness constancy}}{E_d(i,j)} \right\}$$

weight

# HS optical flow objective function

**Brightness constancy**

$$E_d(i,j) = \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2$$

## Smoothness

$$E_s(i,j) = \frac{1}{4}\left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right]$$



$(u_{ij} - u_{i+1,j})$

$(u_{ij} - u_{i,j+1})$

$(v_{ij} - v_{i+1,j})$

$(v_{ij} - v_{i,j+1})$
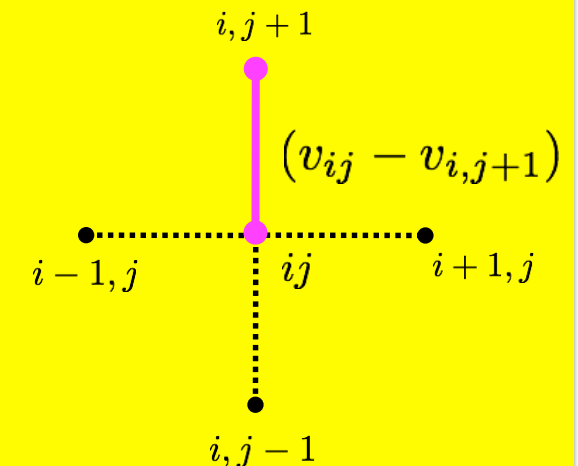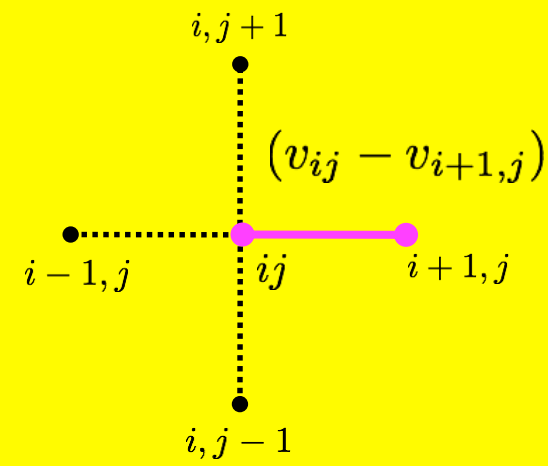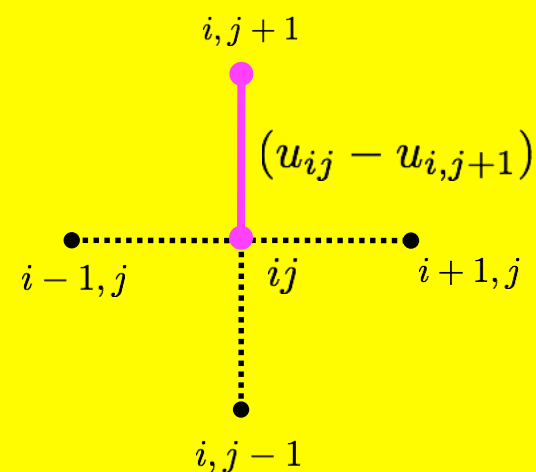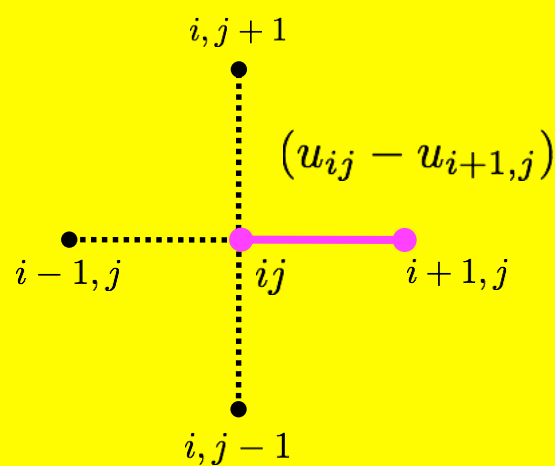
*why not all four neighbors?*

# How do we solve this minimization problem?

$$\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

# How do we solve this minimization problem?

$$\min_{\boldsymbol{u}, \boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$$

Compute partial derivative, derive update equations

**(gradient decent!)**

# Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_xu_{ij}+I_yv_{ij}+I_t\right]^2\right\}$$

smoothness term

brightness constancy

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}} =$$

*how many u terms depend on k and l?*

Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_xu_{ij}+I_yv_{ij}+I_t\right]^2\right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}}=$$

*how many u terms depend on k and l?*

**FOUR** from smoothness                    **ONE** from brightness constancy

Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

it's not so bad…

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

*how many u terms depend on k and l?*

**FOUR** from smoothness                    **ONE** from brightness constancy

# Compute the partial derivatives of this huge sum!

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

$$(u_{ij}^2 - 2u_{ij}u_{i+1,j} + u_{i+1,j}^2) \qquad (u_{ij}^2 - 2u_{ij}u_{i,j+1} + u_{i,j+1}^2)$$

(variable will appear four times in sum)

$i, j+1$

$(u_{ij} - u_{i+1,j})$

$i-1, j$     $ij$     $i+1, j$

$i, j-1$

# Compute the partial derivatives of this huge sum!

$$\sum_{ij}\left\{\frac{1}{4}\left[(u_{ij}-u_{i+1,j})^2+(u_{ij}-u_{i,j+1})^2+(v_{ij}-v_{i+1,j})^2+(v_{ij}-v_{i,j+1})^2\right]+\lambda\left[I_xu_{ij}+I_yv_{ij}+I_t\right]^2\right\}$$

$$(u_{ij}^2-2u_{ij}u_{i+1,j}+u_{i+1,j}^2)\qquad(u_{ij}^2-2u_{ij}u_{i,j+1}+u_{i,j+1}^2)$$

(variable will appear four times in sum)

$i,j+1$

$(u_{ij}-u_{i+1,j})$

$i-1,j$    $ij$    $i+1,j$

$i,j-1$
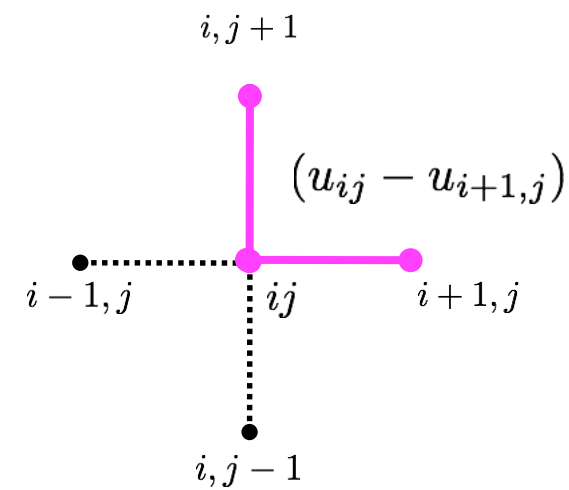
$$\frac{\partial E}{\partial u_{kl}}=2(u_{kl}-\bar{u}_{kl})+2\lambda(I_xu_{kl}+I_yv_{kl}+I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}}=2(v_{kl}-\bar{v}_{kl})+2\lambda(I_xu_{kl}+I_yv_{kl}+I_t)I_y$$

short hand for local average

$$\bar{u}_{ij}=\frac{1}{4}\left\{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\right\}$$

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

*Where are the extrema of E?*

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

*Where are the extrema of E?*

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

### *Where are the extrema of E?*

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

*this is a linear system* $$\mathbf{Ax = b}$$ *how do you solve this?*

ok, take a step back, why are we doing all this math?

We are solving for the optical flow (u,v) given two constraints

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

smoothness           brightness constancy

We need the math to minimize this
(back to the math)

Partial derivatives of Horn-Schunck objective function E:

$$\frac{\partial E}{\partial u_{kl}} = 2(u_{kl} - \bar{u}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_x$$

$$\frac{\partial E}{\partial v_{kl}} = 2(v_{kl} - \bar{v}_{kl}) + 2\lambda(I_x u_{kl} + I_y v_{kl} + I_t)I_y$$

*Where are the extrema of E?*

(set derivatives to zero and solve for unknowns u and v)

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

$$\mathbf{A x = b}$$  *how do you solve this?*

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

Recall $\quad \boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b} = \dfrac{\mathrm{adj}\mathbf{A}}{\det \mathbf{A}}\boldsymbol{b}$

$$(1 + \lambda I_x^2)u_{kl} + \lambda I_x I_y v_{kl} = \bar{u}_{kl} - \lambda I_x I_t$$

$$\lambda I_x I_y u_{kl} + (1 + \lambda I_y^2)v_{kl} = \bar{v}_{kl} - \lambda I_y I_t$$

Recall $\quad \boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b} = \dfrac{\mathrm{adj}\mathbf{A}}{\det \mathbf{A}}\boldsymbol{b}$

Same as the linear system:

$$\{1 + \lambda(I_x^2 + I_y^2)\}u_{kl} = (1 + \lambda I_x^2)\bar{u}_{kl} - \lambda I_x I_y \bar{v}_{kl} - \lambda I_x I_t$$

(det A)

$$\{1 + \lambda(I_x^2 + I_y^2)\}v_{kl} = (1 + \lambda I_y^2)\bar{v}_{kl} - \lambda I_x I_y \bar{u}_{kl} - \lambda I_y I_t$$

(det A)

$$\{1 + \lambda(I_x^2 + I_y^2)\}u_{kl} = (1 + \lambda I_x^2)\bar{u}_{kl} - \lambda I_x I_y \bar{v}_{kl} - \lambda I_x I_t$$

$$\{1 + \lambda(I_x^2 + I_y^2)\}v_{kl} = (1 + \lambda I_y^2)\bar{v}_{kl} - \lambda I_x I_y \bar{u}_{kl} - \lambda I_y I_t$$

Rearrange to get update equations:

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

new value / old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

**Recall:** $\displaystyle\min_{\boldsymbol{u},\boldsymbol{v}} \sum_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$

When lambda is small (lambda inverse is big)…

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

new value    old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

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new value    old average    goes to zero

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

goes to zero

**Recall:** $\min\limits_{\boldsymbol{u},\boldsymbol{v}} \sum\limits_{i,j} \left\{ E_s(i,j) + \lambda E_d(i,j) \right\}$

When lambda is small (lambda inverse is big)…

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x$$

goes to zero

new value    old average

$$\hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

goes to zero

…we only care about smoothness.

ok, take a step back, why did we do all this math?

We are solving for the optical flow (u,v) given two constraints

$$\sum_{ij} \left\{ \frac{1}{4} \left[ (u_{ij} - u_{i+1,j})^2 + (u_{ij} - u_{i,j+1})^2 + (v_{ij} - v_{i+1,j})^2 + (v_{ij} - v_{i,j+1})^2 \right] + \lambda \left[ I_x u_{ij} + I_y v_{ij} + I_t \right]^2 \right\}$$

smoothness                                                                                          brightness constancy

We needed the math to minimize this
(now to the algorithm)

# Horn-Schunck
# Optical Flow Algorithm

1. Precompute image gradients $\quad I_y \quad I_x$

2. Precompute temporal gradients $\quad I_t$

3. Initialize flow field

$$\boldsymbol{u} = \boldsymbol{0}$$
$$\boldsymbol{v} = \boldsymbol{0}$$

4. While not converged

   Compute flow field updates for each pixel:

$$\hat{u}_{kl} = \bar{u}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_x \qquad \hat{v}_{kl} = \bar{v}_{kl} - \frac{I_x \bar{u}_{kl} + I_y \bar{v}_{kl} + I_t}{\lambda^{-1} + I_x^2 + I_y^2} I_y$$

**Just 8 lines of code!**

# References

Basic reading:
- Szeliski, Section 8.4.