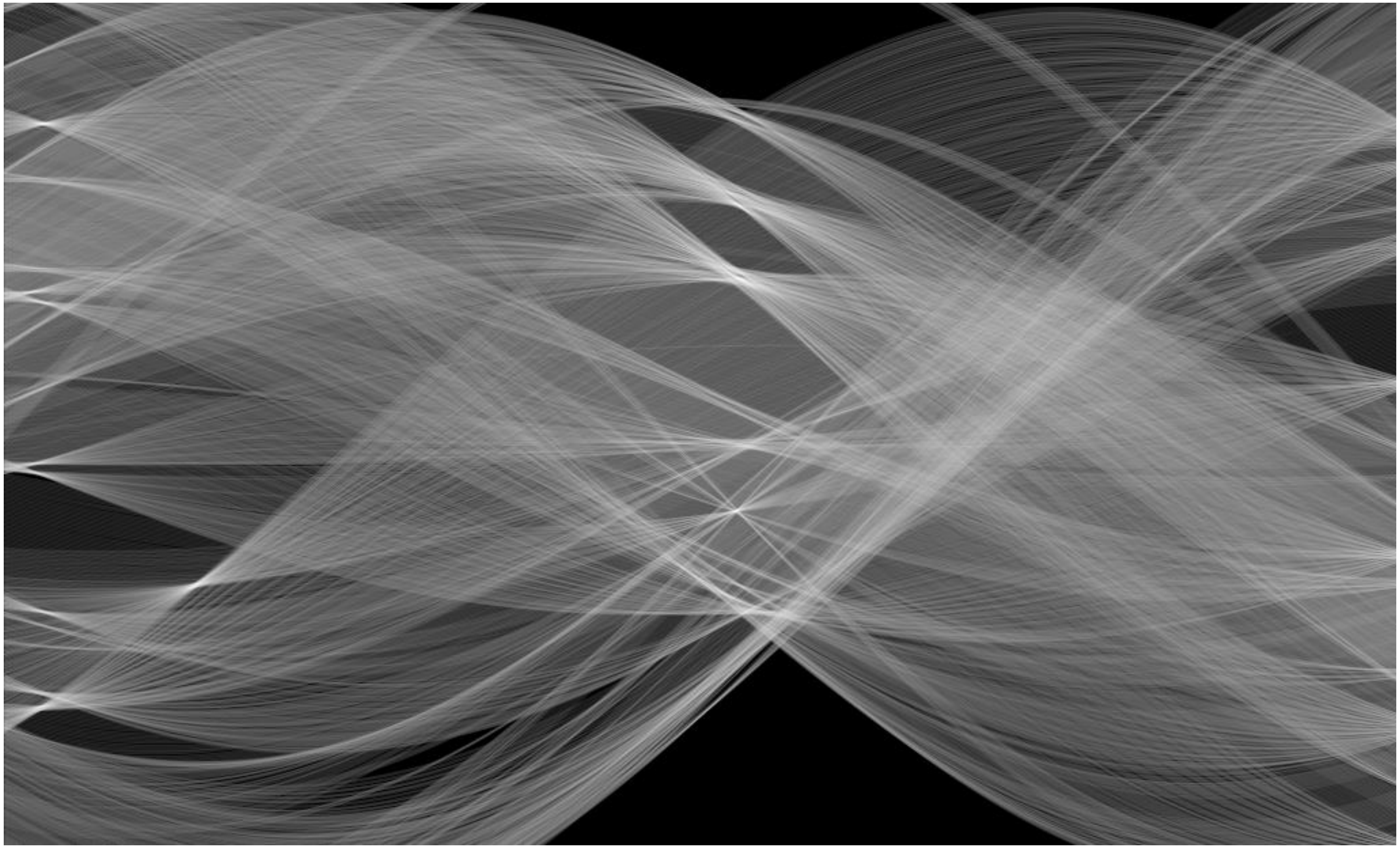
Hough transform



16-385 Computer Vision
Spring 2019, Lecture 4

http://www.cs.cmu.edu/~16385/

Course announcements

- Homework 1 is now due on **Monday February 11th!**
  - Any questions about the homework?
  - How many of you have looked at/started/finished homework 1?

- Some changes to office hours for this week only:
  - Tuesday’s office hours will be 5:30-7:30 pm and will be covered by Yannis.
  - Friday’s 3-5 pm office hours will be covered by Anshuman.
  - There will be an extra set of office hours this week to make up for the change.

Overview of today’s lecture

Leftover from Lecture 3:

- Frequency-domain filtering.
- Revisiting sampling.

New in lecture 4:

- Finding boundaries.
- Line fitting.
- Line parameterizations.
- Hough transform.
- Hough circles.
- Some applications.

Slide credits

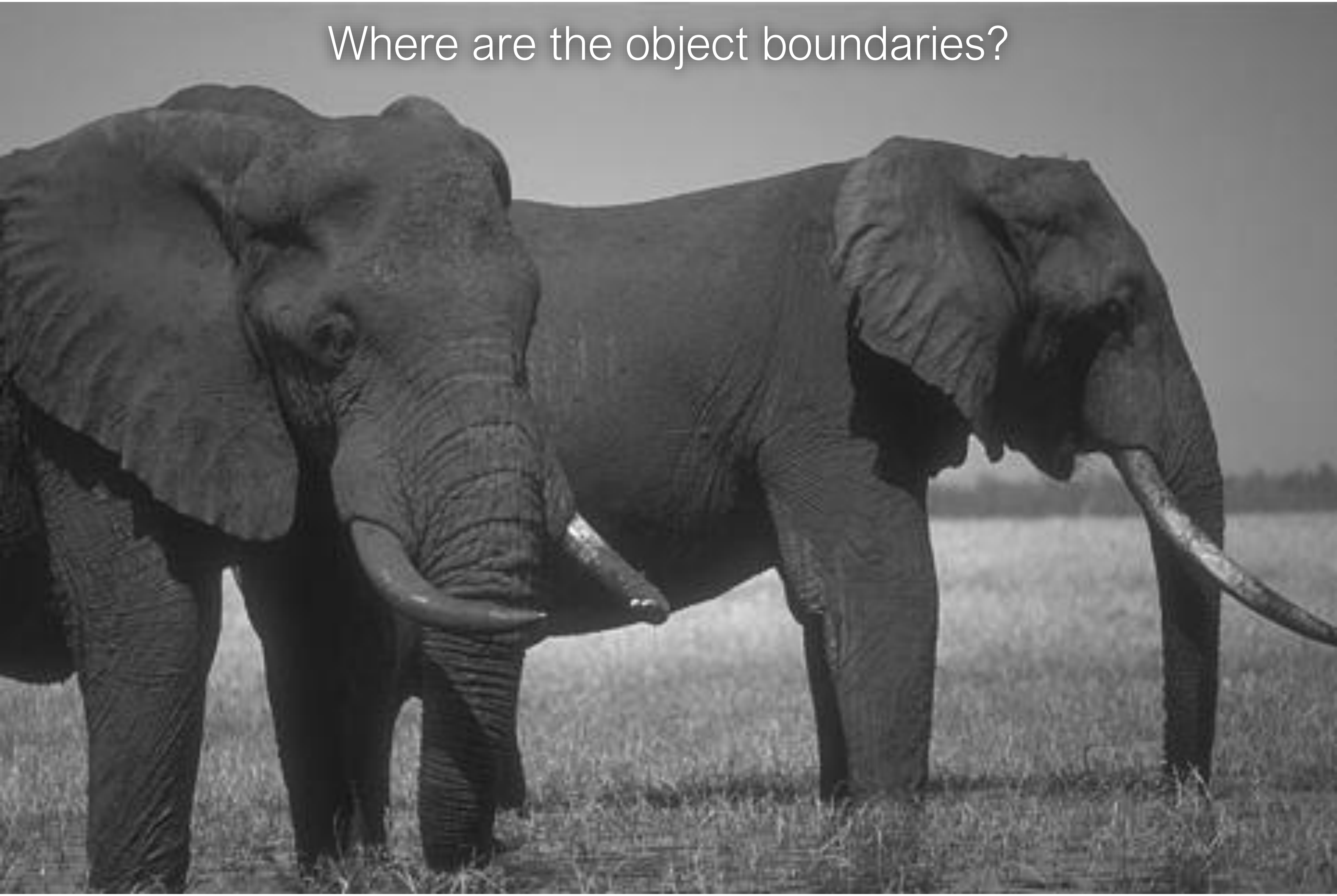
Most of these slides were adapted from:

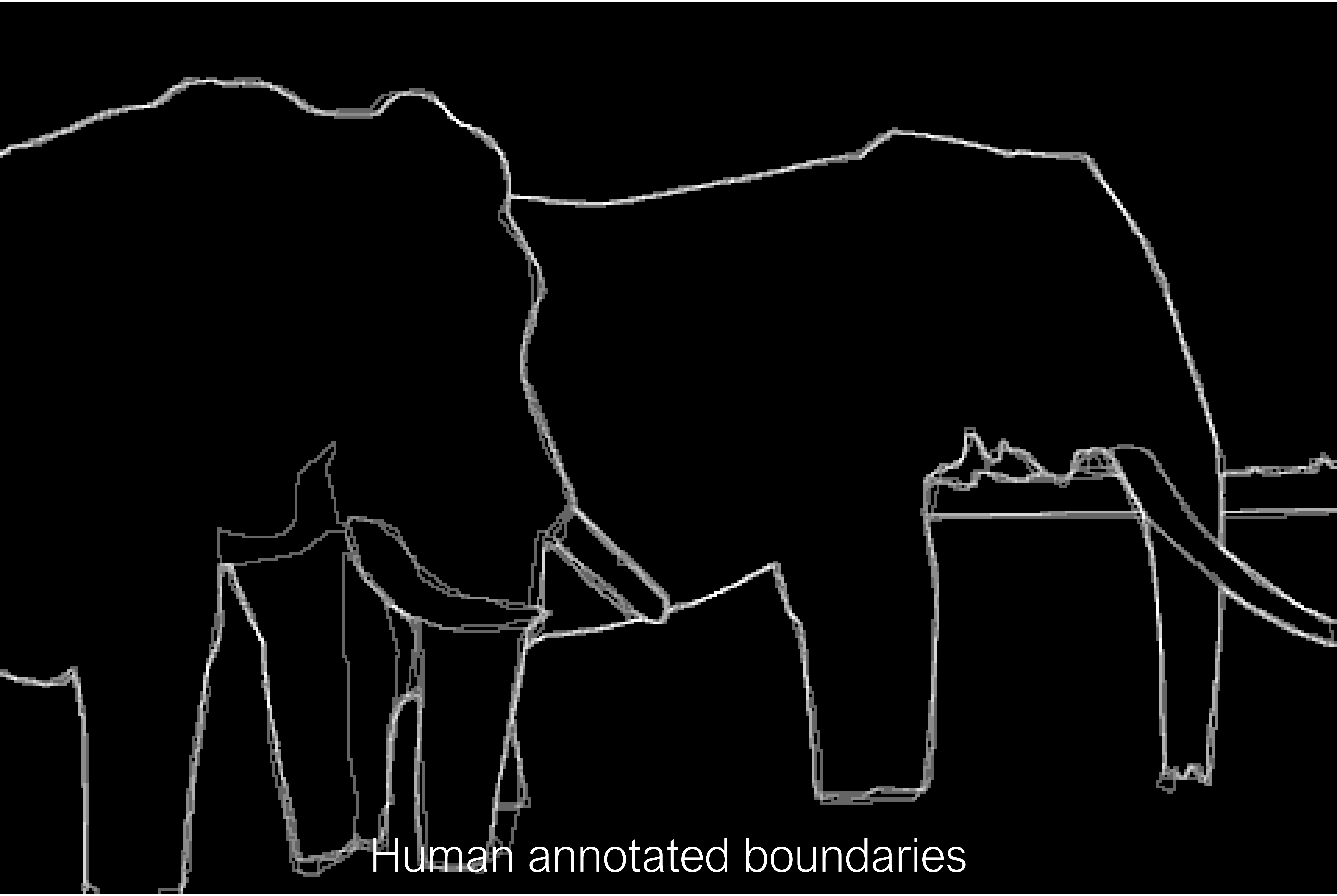
- Kris Kitani (15-463, Fall 2016).

Some slides were inspired or taken from:

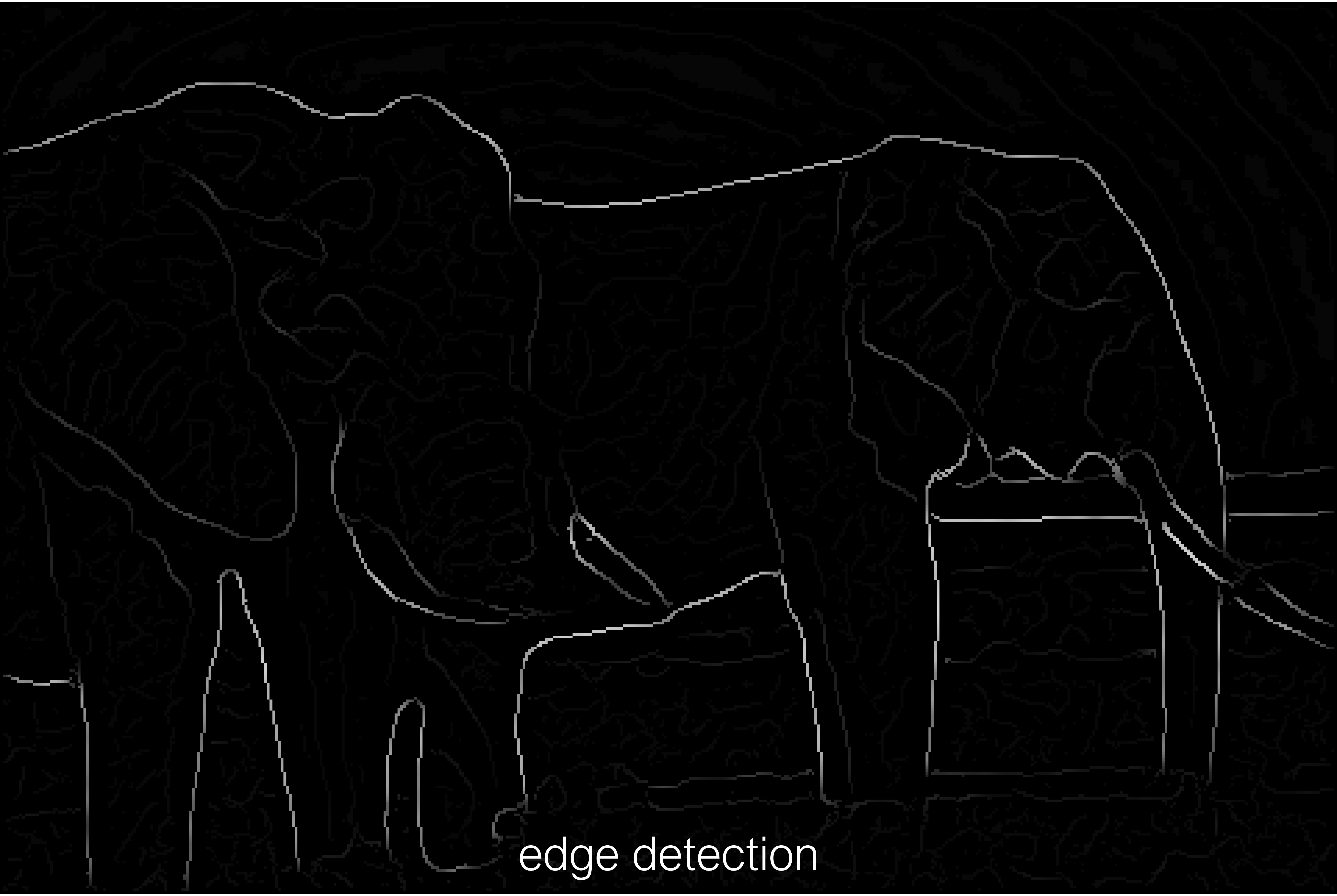
- Fredo Durand (MIT).
- James Hays (Georgia Tech).

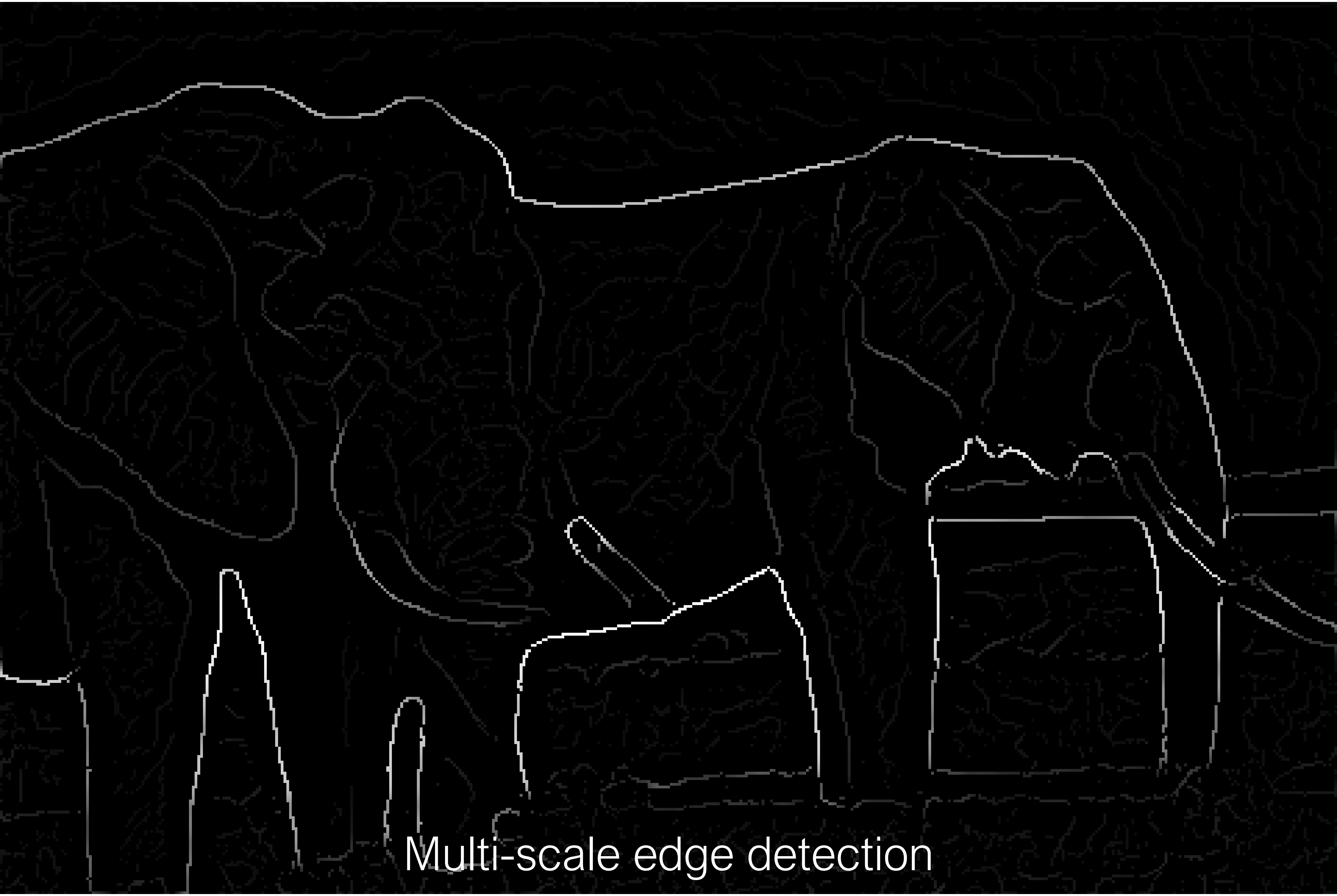
Finding boundaries

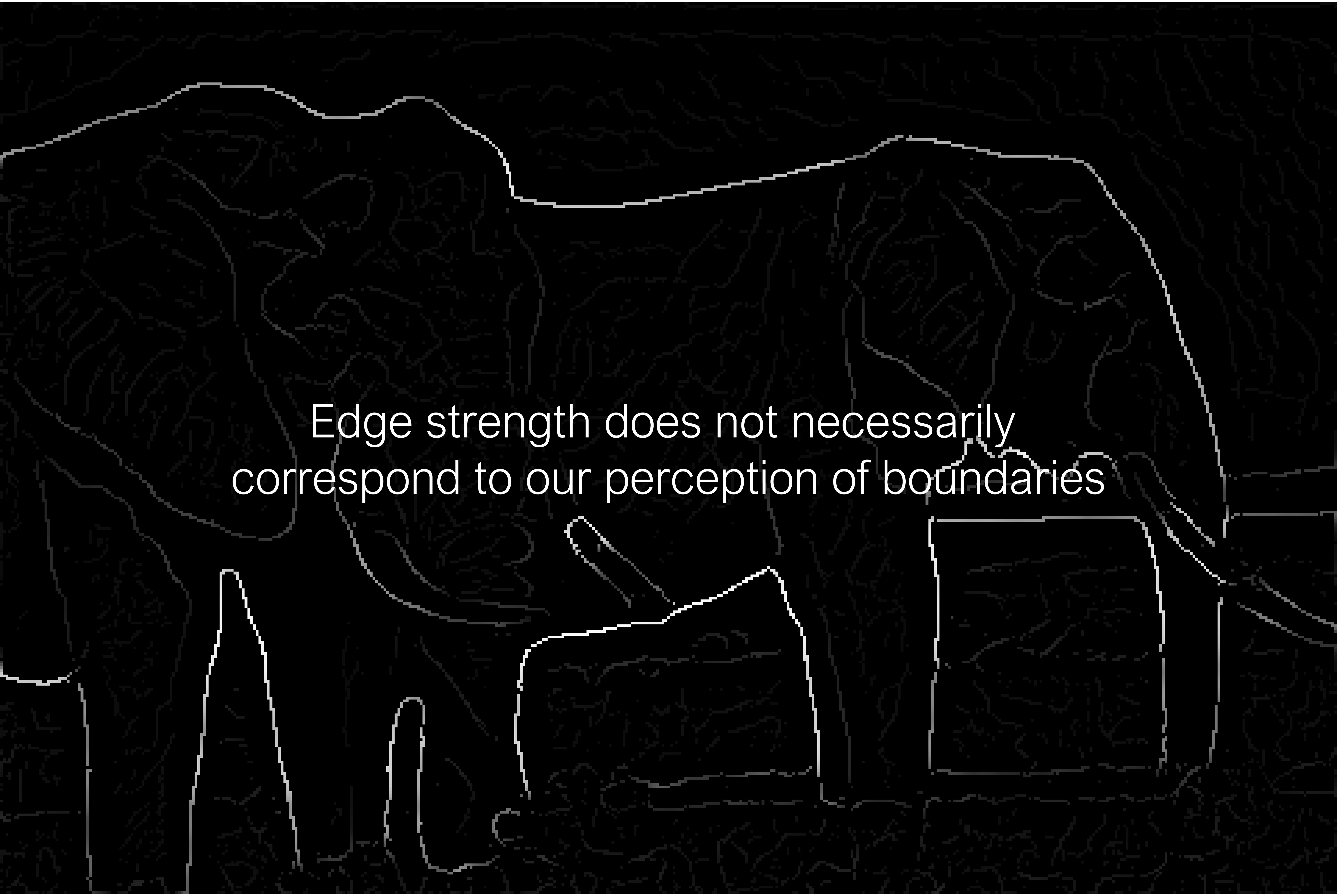

Where are the object boundaries?



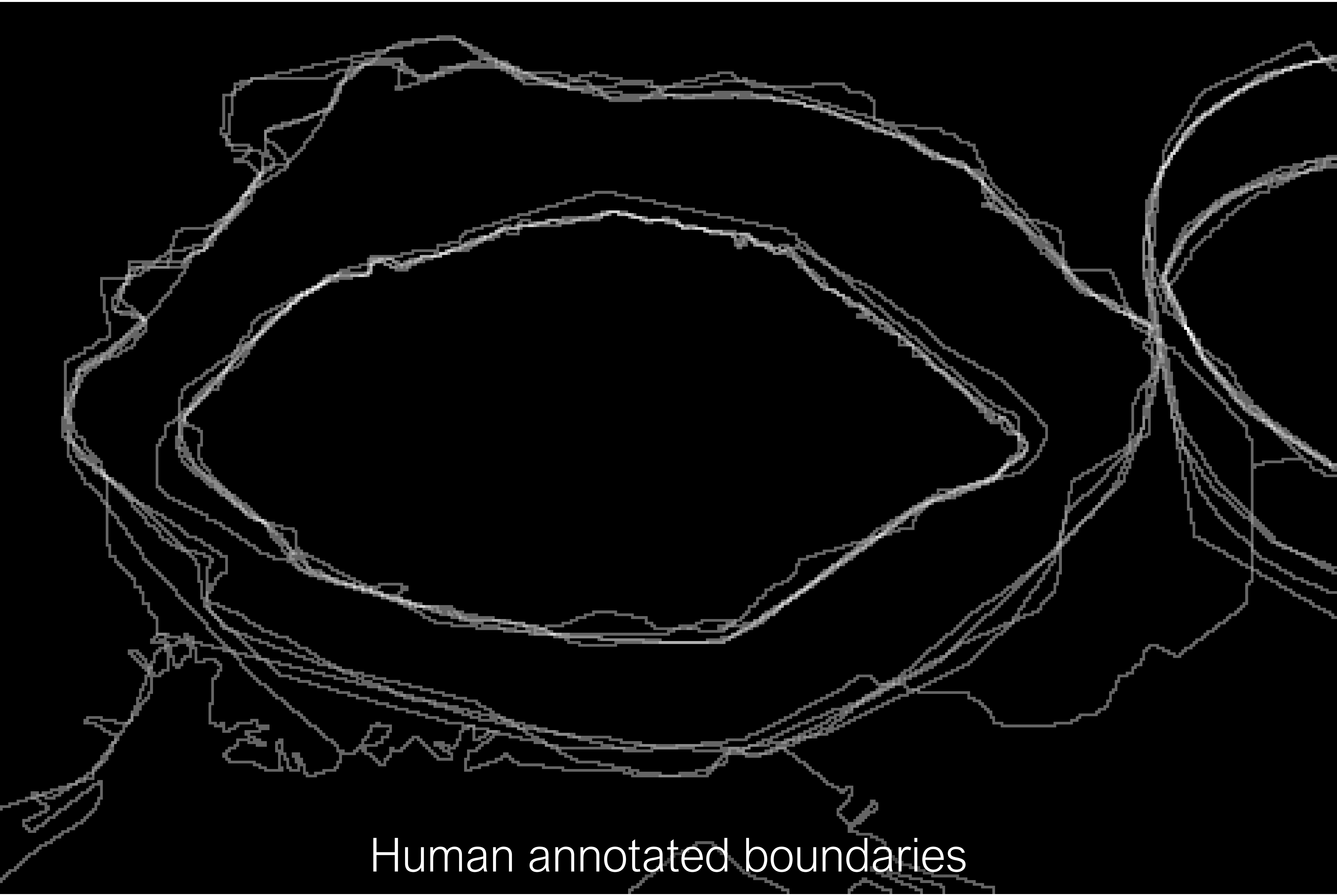
Human annotated boundaries


edge detection


Multi-scale edge detection


Edge strength does not necessarily
correspond to our perception of boundaries


Where are the object boundaries?

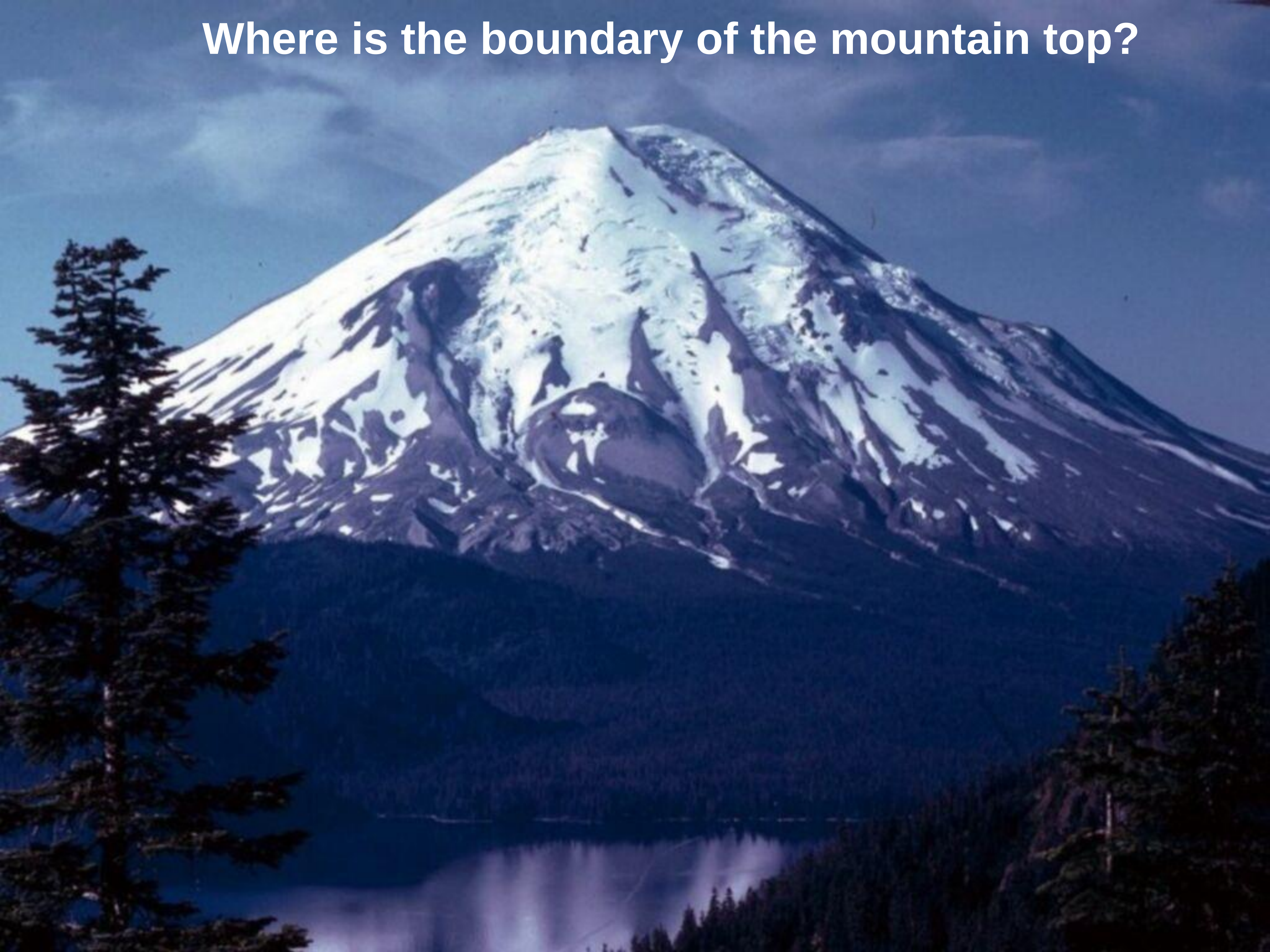


Human annotated boundaries


edge detection

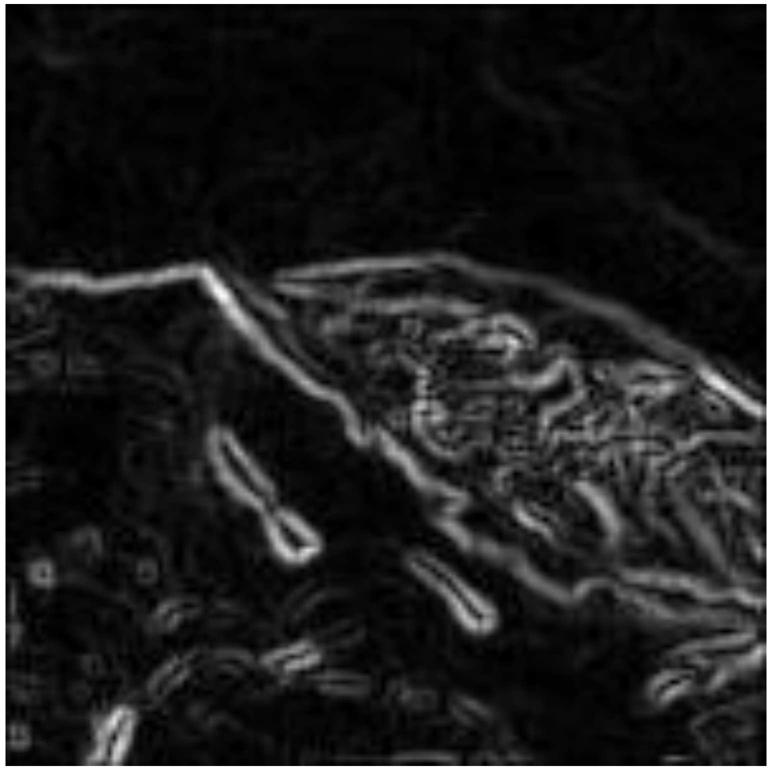

Defining boundaries are hard for us too

Where is the boundary of the mountain top?

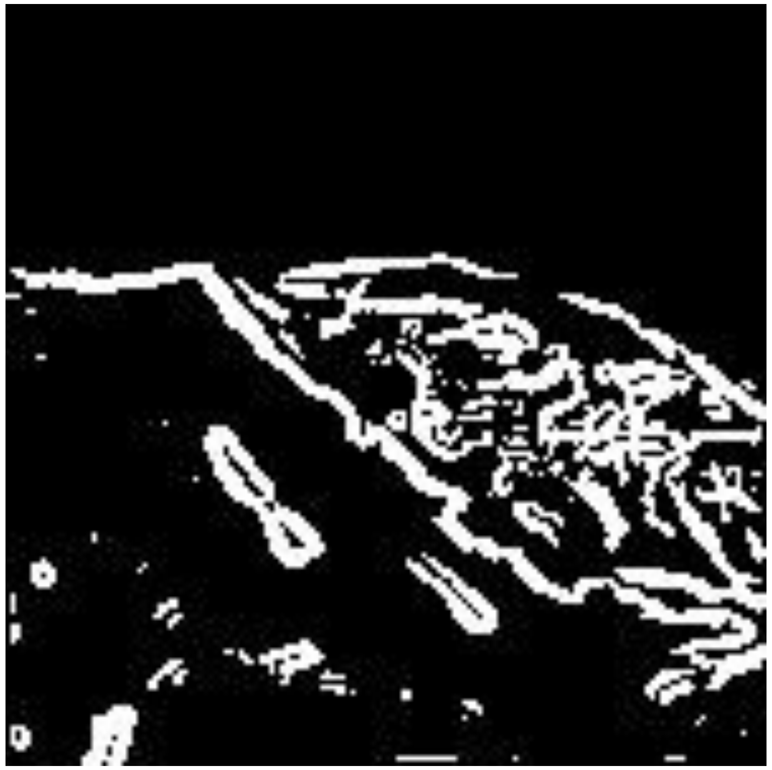


Lines are hard to find

Original image



Edge detection

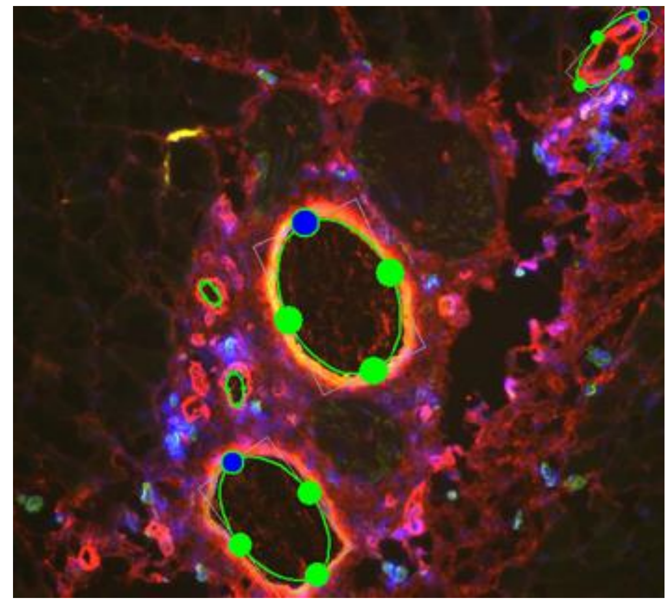


Thresholding

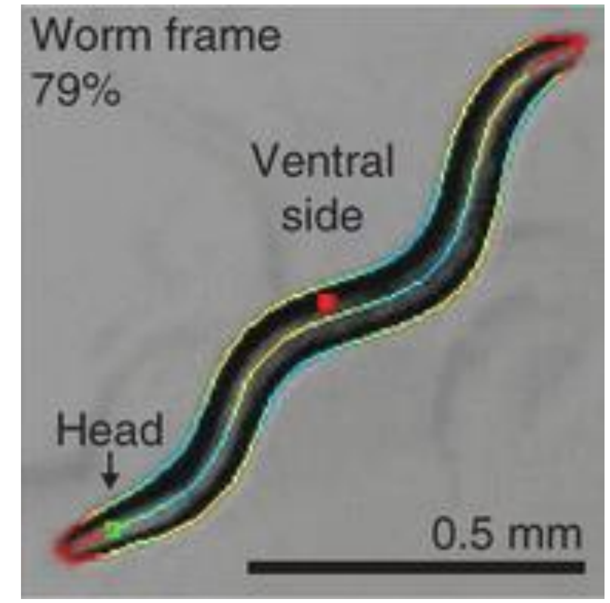
Noisy edge image
Incomplete boundaries

Applications

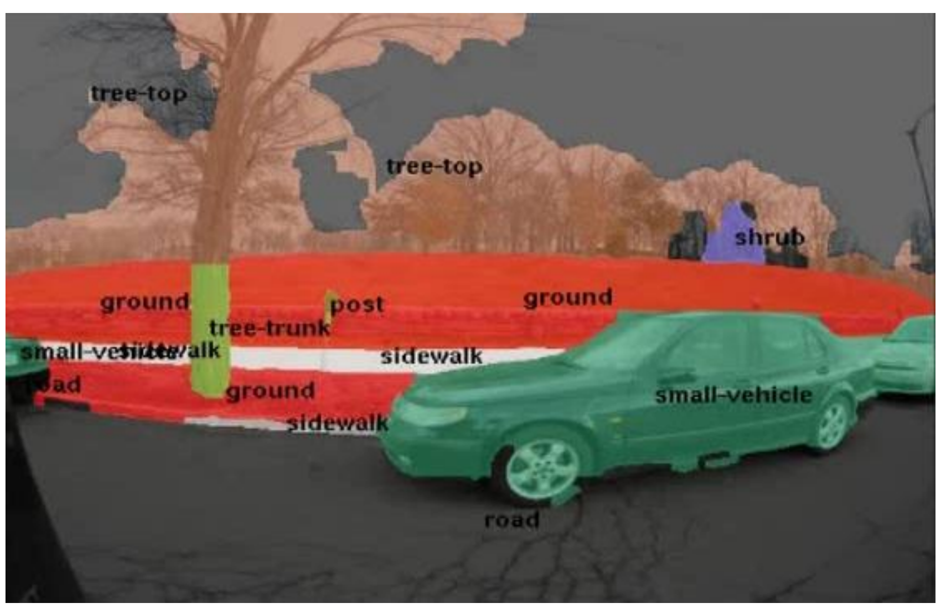
Autonomous Vehicles
(lane line detection)



tissue engineering
(blood vessel counting)

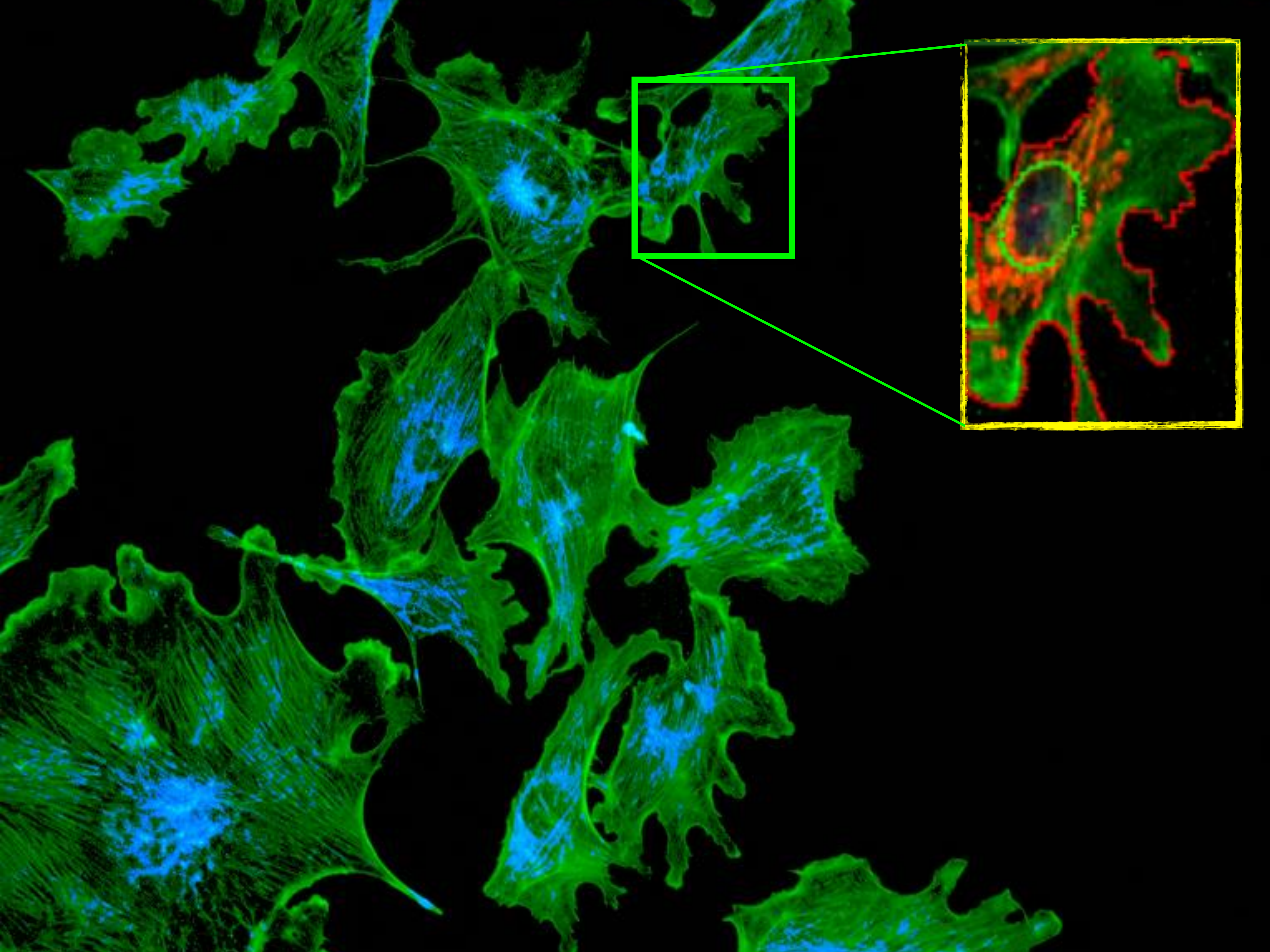


behavioral genetics
(earthworm contours)



Autonomous Vehicles
(semantic scene segmentation)

Computational Photography
(image inpainting)



Line fitting

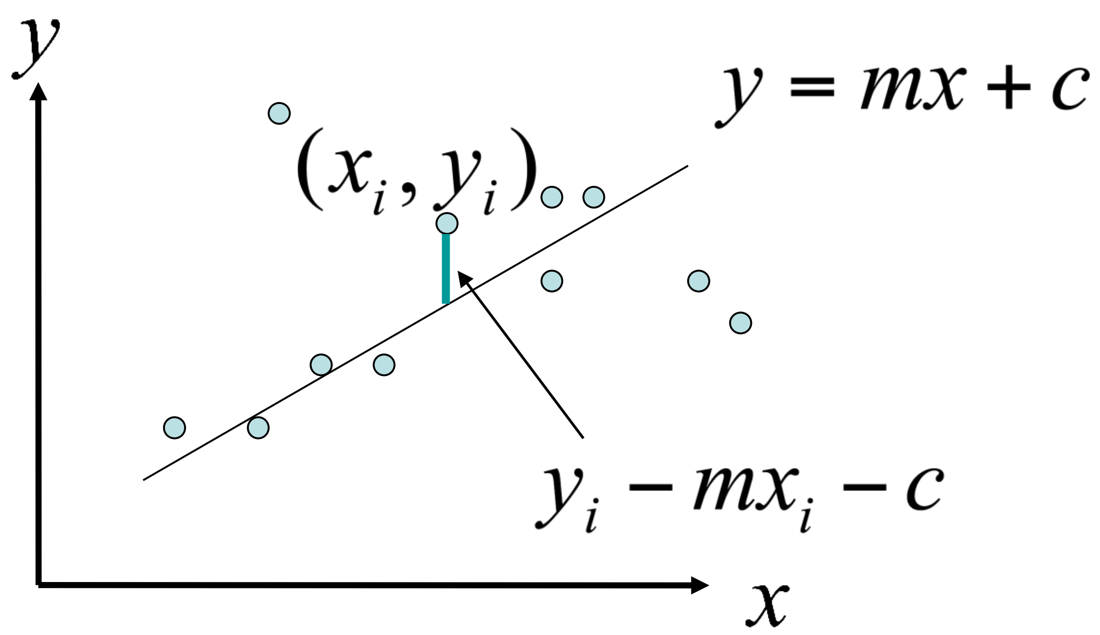
Line fitting

Given: Many (x_i, y_i) pairs

Find: Parameters (m, c)

Minimize: Average square distance:

$$E = \sum_i \frac{(y_i - mx_i - c)^2}{N}$$



Line fitting

Given: Many (x_i, y_i) pairs

Find: Parameters (m, c)

Minimize: Average square distance:

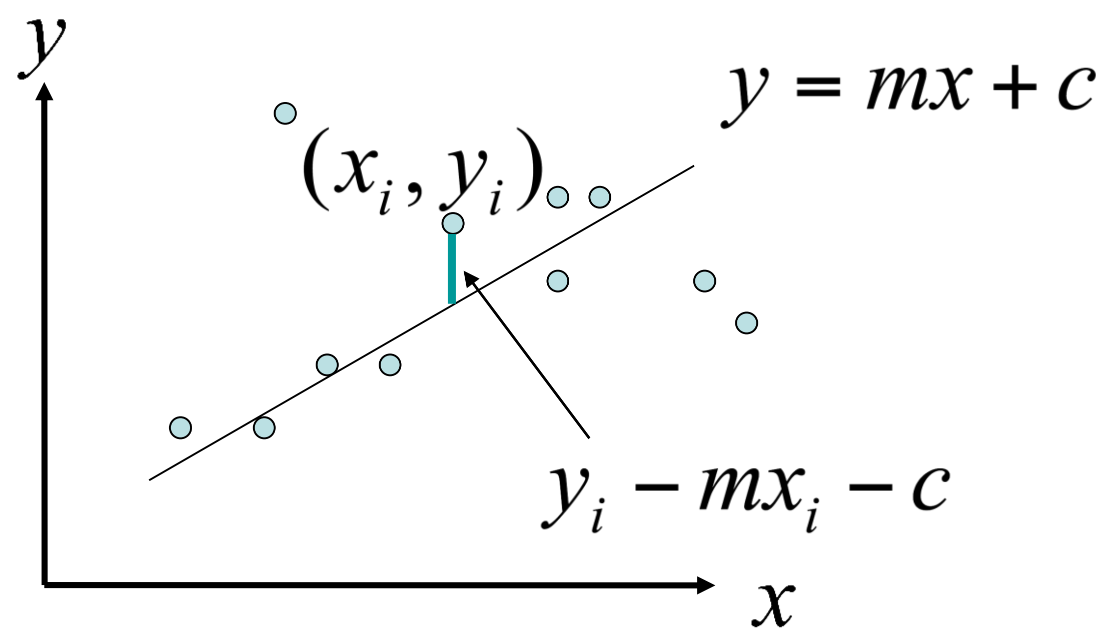
$$E = \sum_i \frac{(y_i - mx_i - c)^2}{N}$$

Using:

$$\frac{\partial E}{\partial m} = 0 \quad \& \quad \frac{\partial E}{\partial c} = 0$$

Note:

$$\bar{y} = \frac{\sum_i y_i}{N} \qquad \bar{x} = \frac{\sum_i x_i}{N}$$



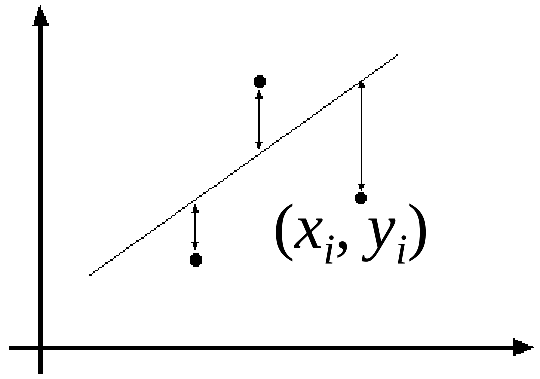
$$c = \bar{y} - m\bar{x}$$

$$m = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$$

What are some problems with the approach?

Data: $(x_1, y_1), \ldots, (x_n, y_n)$

Line equation: $y_i = m\,x_i + b$



Find (m, b) to minimize $E = \sum_{i=1}^{n} (y_i - m x_i - b)^2$

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad B = \begin{bmatrix} m \\ b \end{bmatrix}$$

$$E = \|Y - XB\|^2 = (Y - XB)^T (Y - XB) = Y^T Y - 2(XB)^T Y + (XB)^T (XB)$$

$$\frac{dE}{dB} = 2X^T XB - 2X^T Y = 0 \qquad X^T XB = X^T Y$$

Normal equations: least squares solution to $XB=Y$

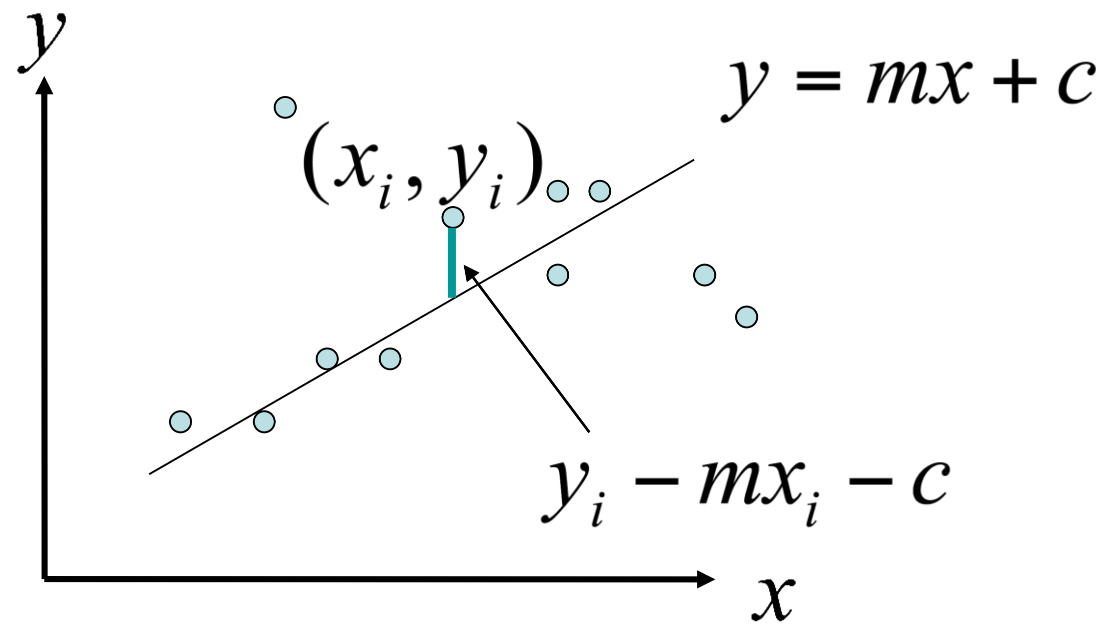
Line fitting

Given: Many (x_i, y_i) pairs

Find: Parameters (m, c)

Minimize: Average square distance:

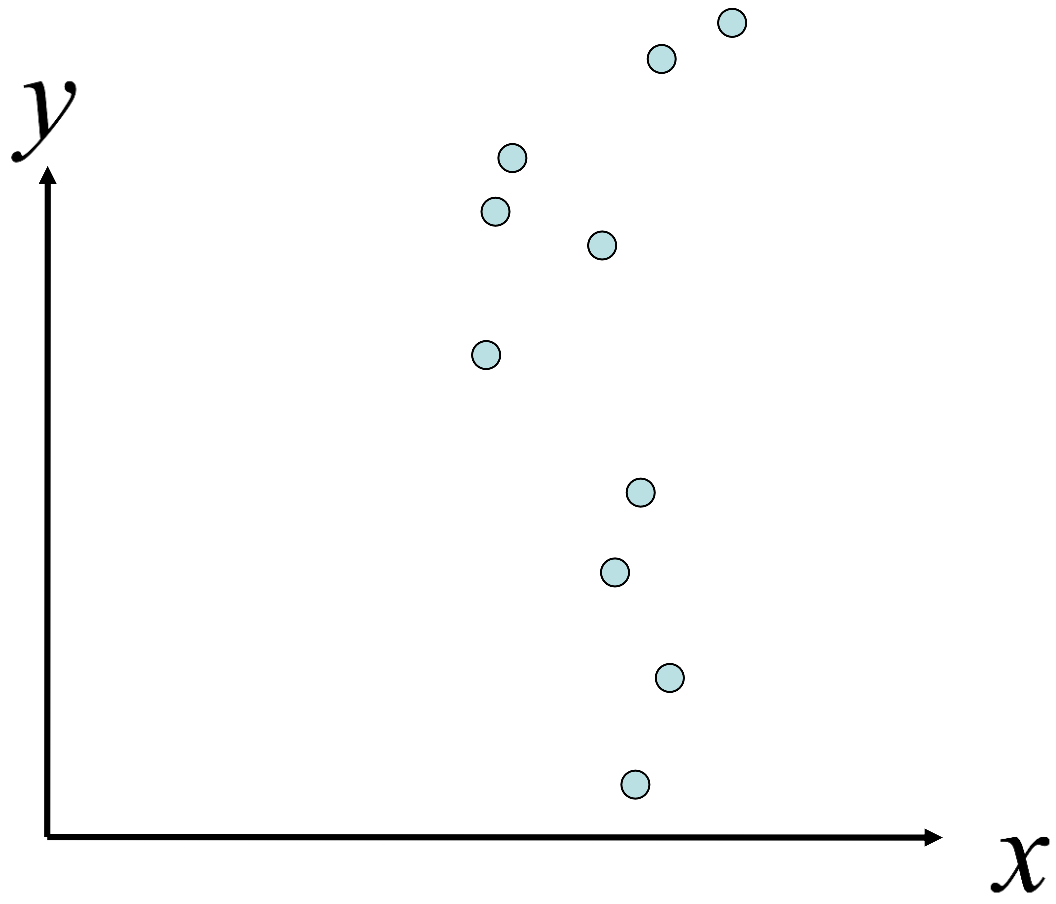
$$E = \sum_i \frac{(y_i - mx_i - c)^2}{N}$$



How can we solve this minimization?

Problems with parameterizations

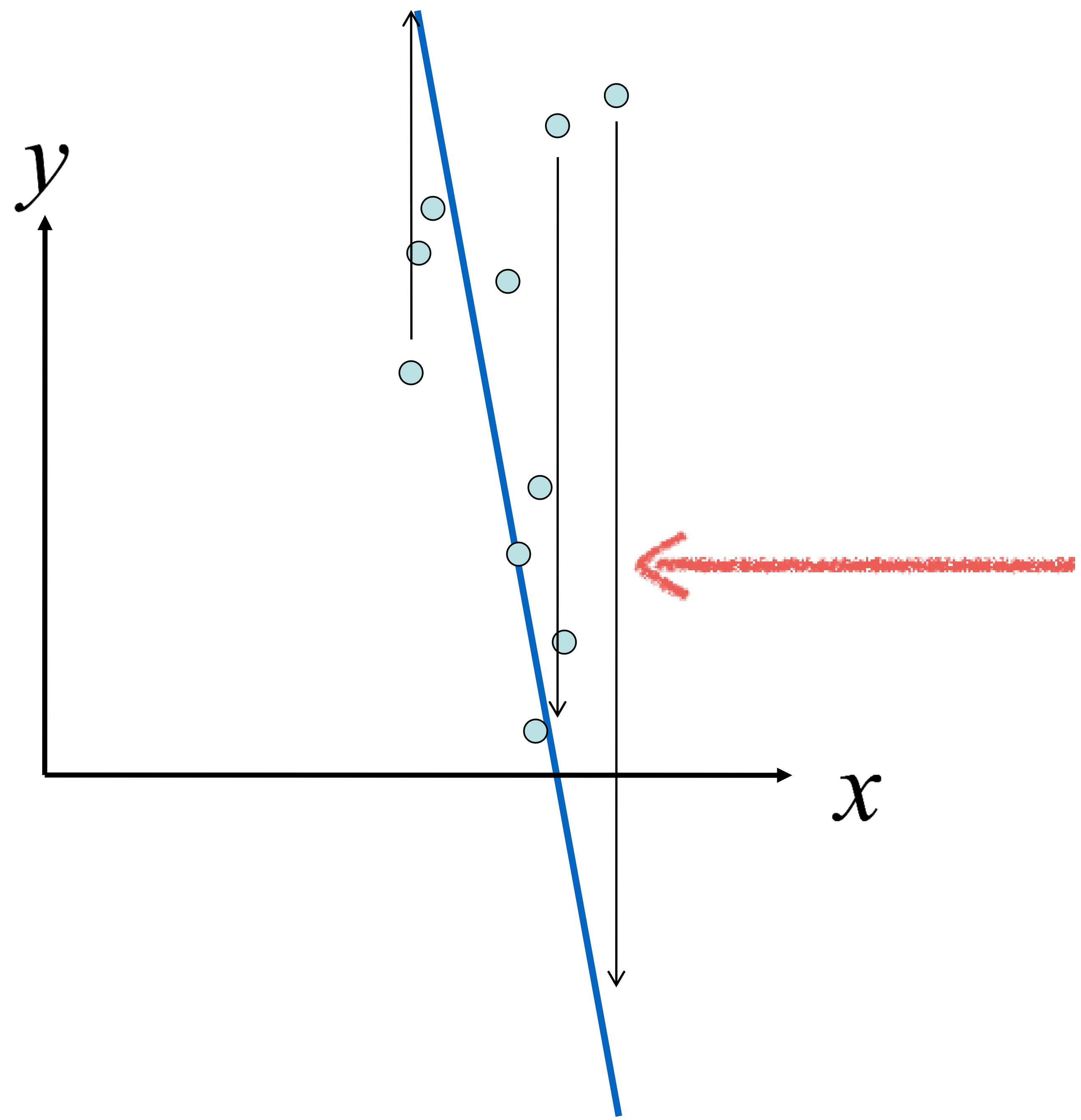
Where is the line that minimizes E? $E = \sum_{i=1}^{n} (y_i - m x_i - b)^2$



Problems with parameterizations

Where is the line that minimizes E?

$$E = \sum_{i=1}^{n} (y_i - m x_i - b)^2$$

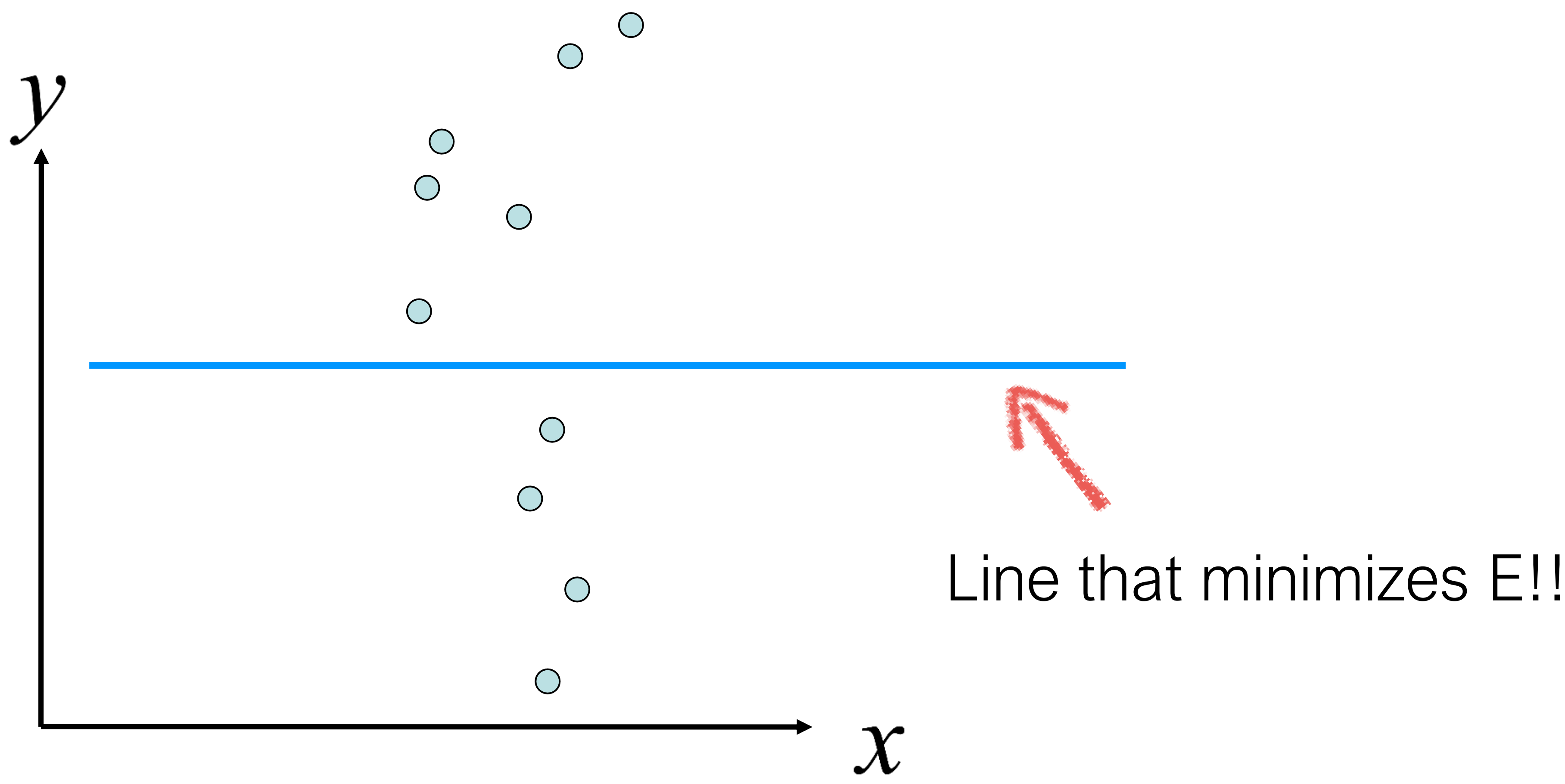


Huge E!

Problems with parameterizations

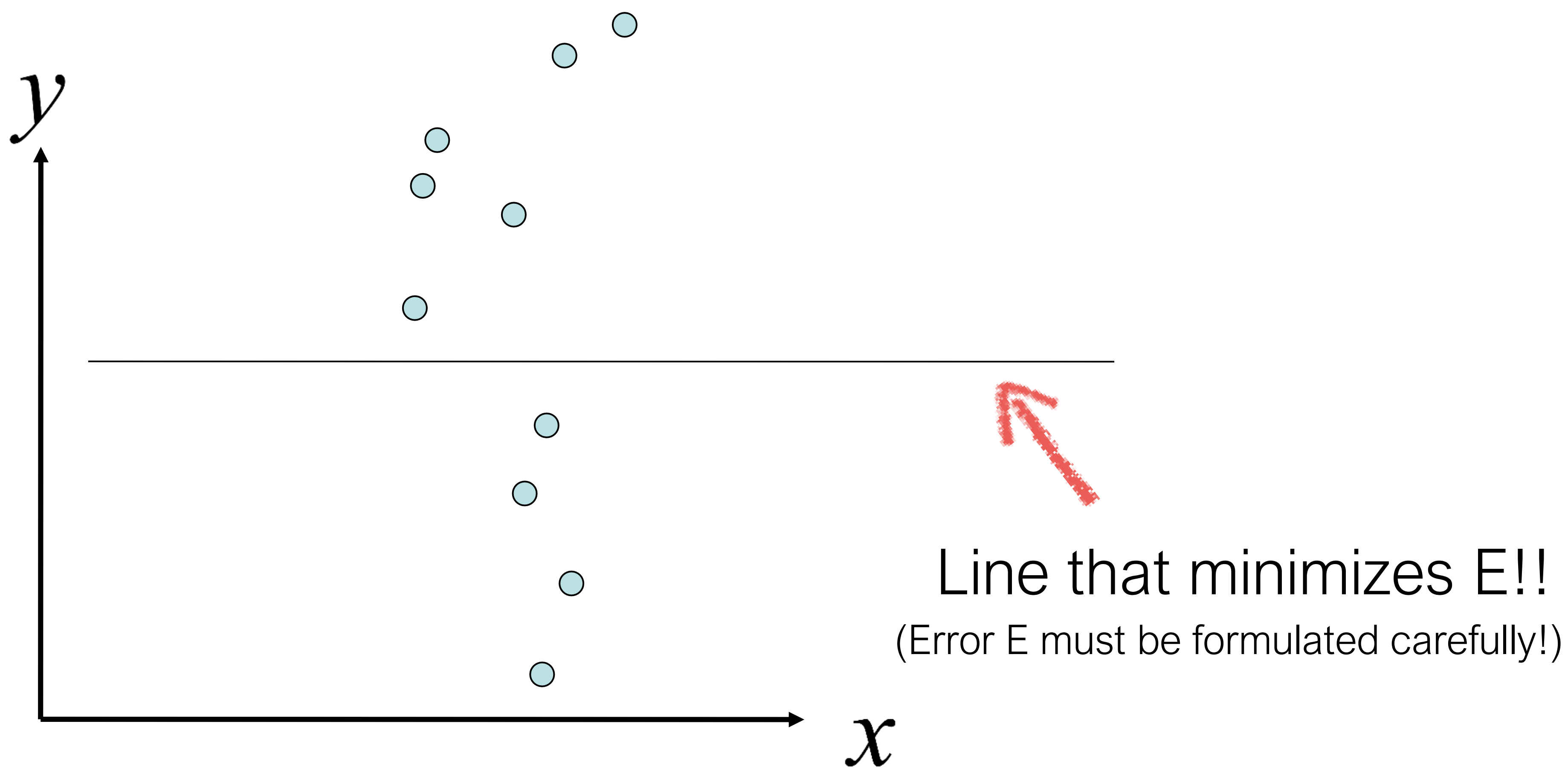
Where is the line that minimizes E?

$$E = \sum_{i=1}^{n}(y_i - mx_i - b)^2$$



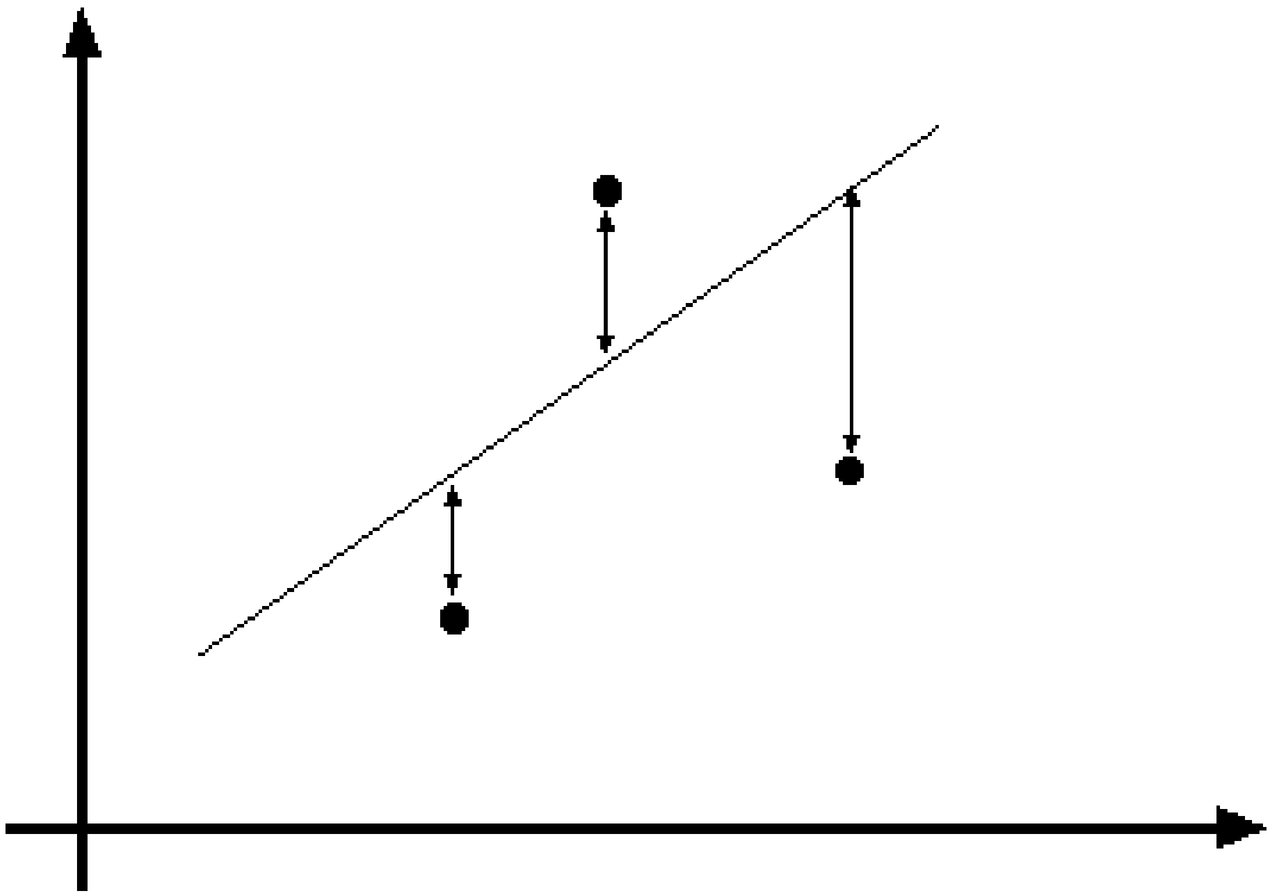
Problems with parameterizations

Where is the line that minimizes E?

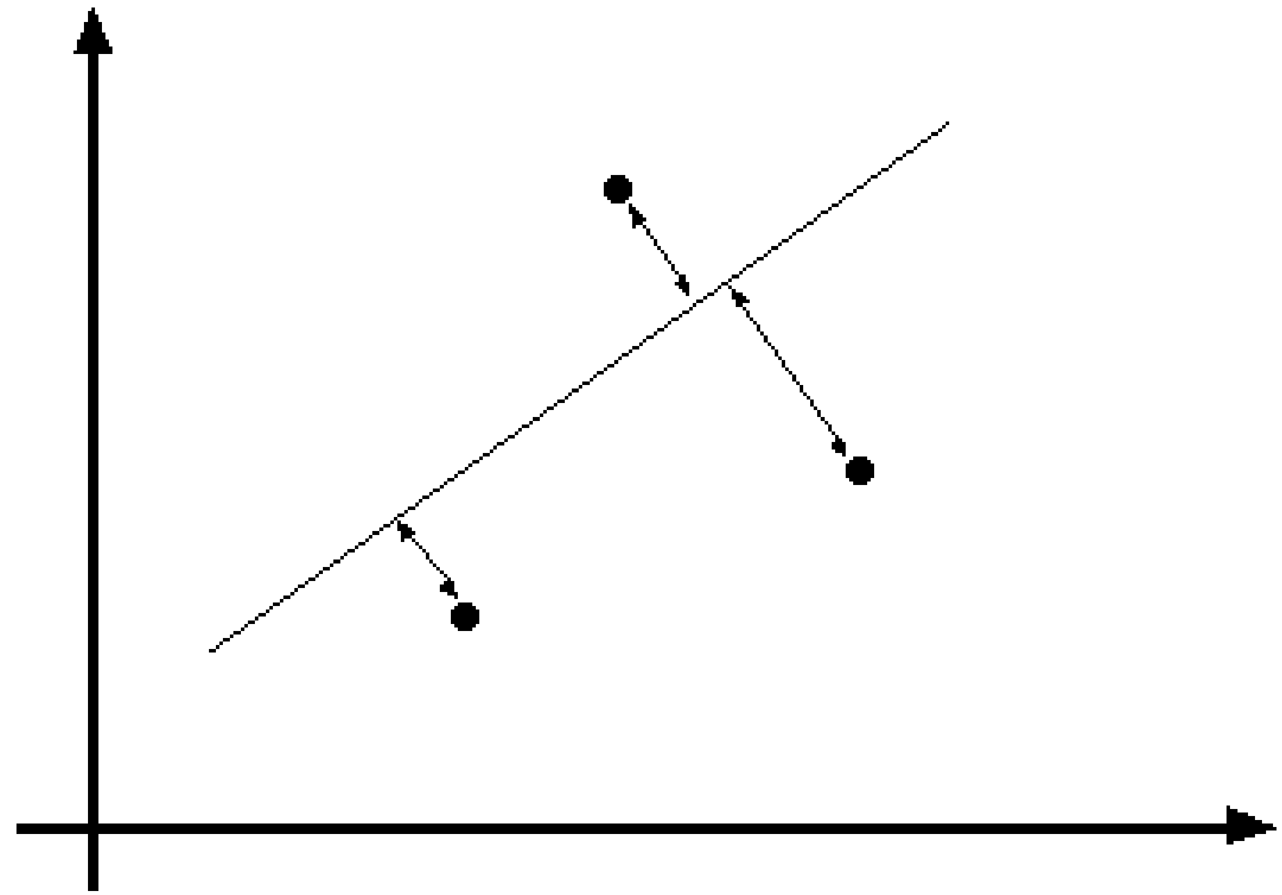


How can we deal with this?

Line fitting is easily setup as a maximum likelihood problem
… but choice of model is important

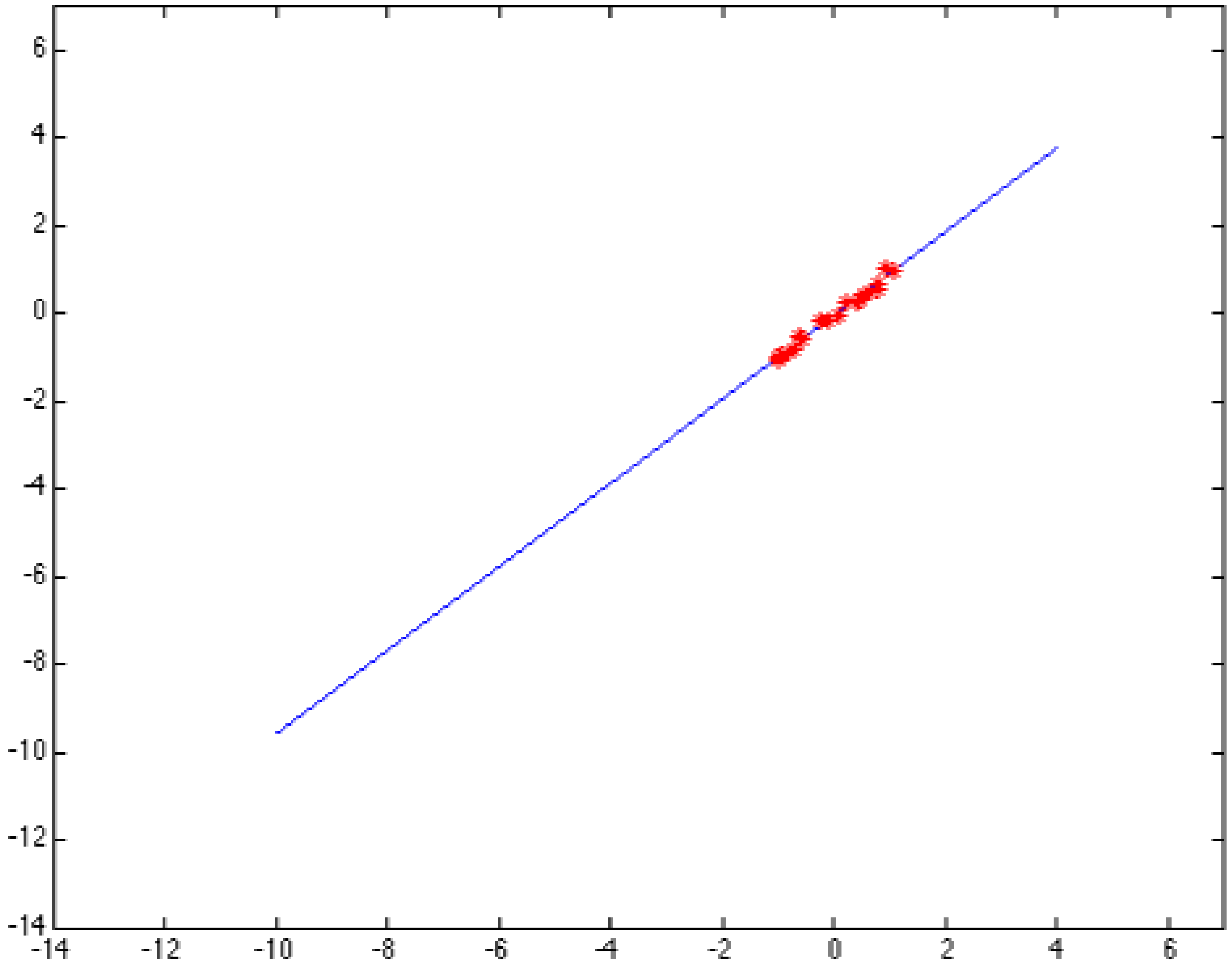


$$E = \sum_{i=1}^{n} (y_i - mx_i - b)^2$$

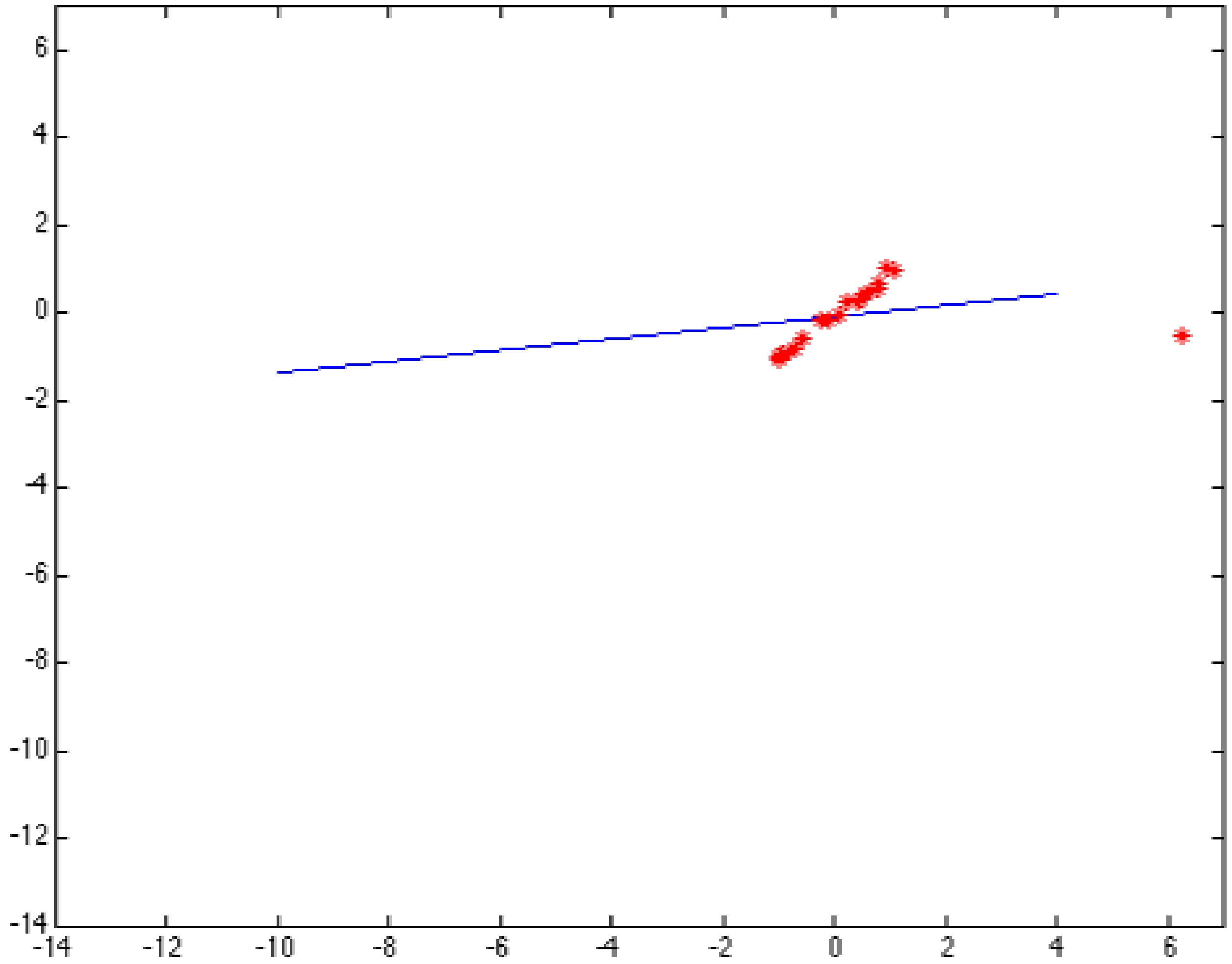


What optimization are we solving here?

Problems with noise



Least-squares error fit



Squared error heavily penalizes outliers

Model fitting is difficult because…

- **Extraneous data:** clutter or multiple models
  - We do not know what is part of the model?
  - Can we pull out models with a few parts from much larger amounts of background clutter?
- **Missing data:** only some parts of model are present
- **Noise**

- Cost:
  - It is not feasible to check all combinations of features by fitting a model to each possible subset

So what can we do?

Line parameterizations

Slope intercept form

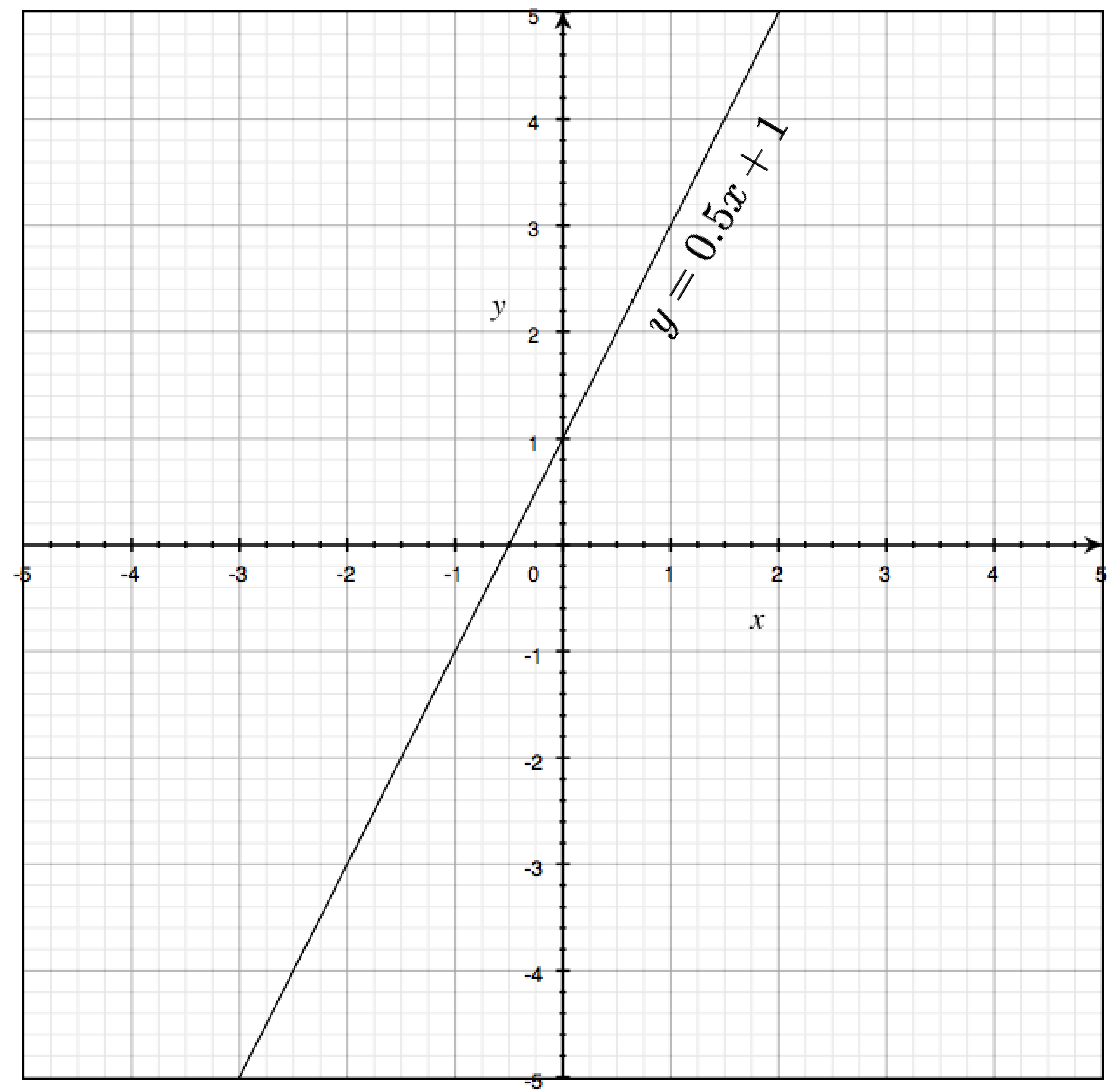
$$y = mx + b$$

slope y-intercept

What are m and b?

Slope intercept form

$$y = mx + b$$



Double intercept form

$$\frac{x}{a} + \frac{y}{b} = 1$$

x-intercept y-intercept

What are x and y?

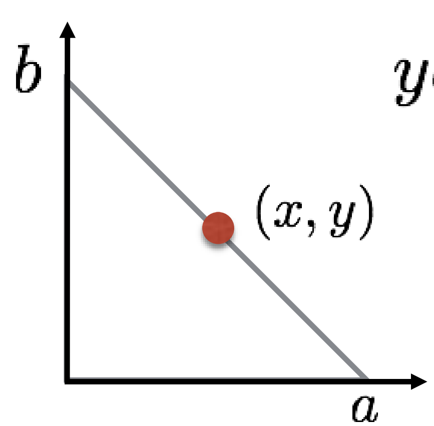
Double intercept form

$$\frac{x}{a}+\frac{y}{b}=1$$

x-intercept　　y-intercept

Derivation:

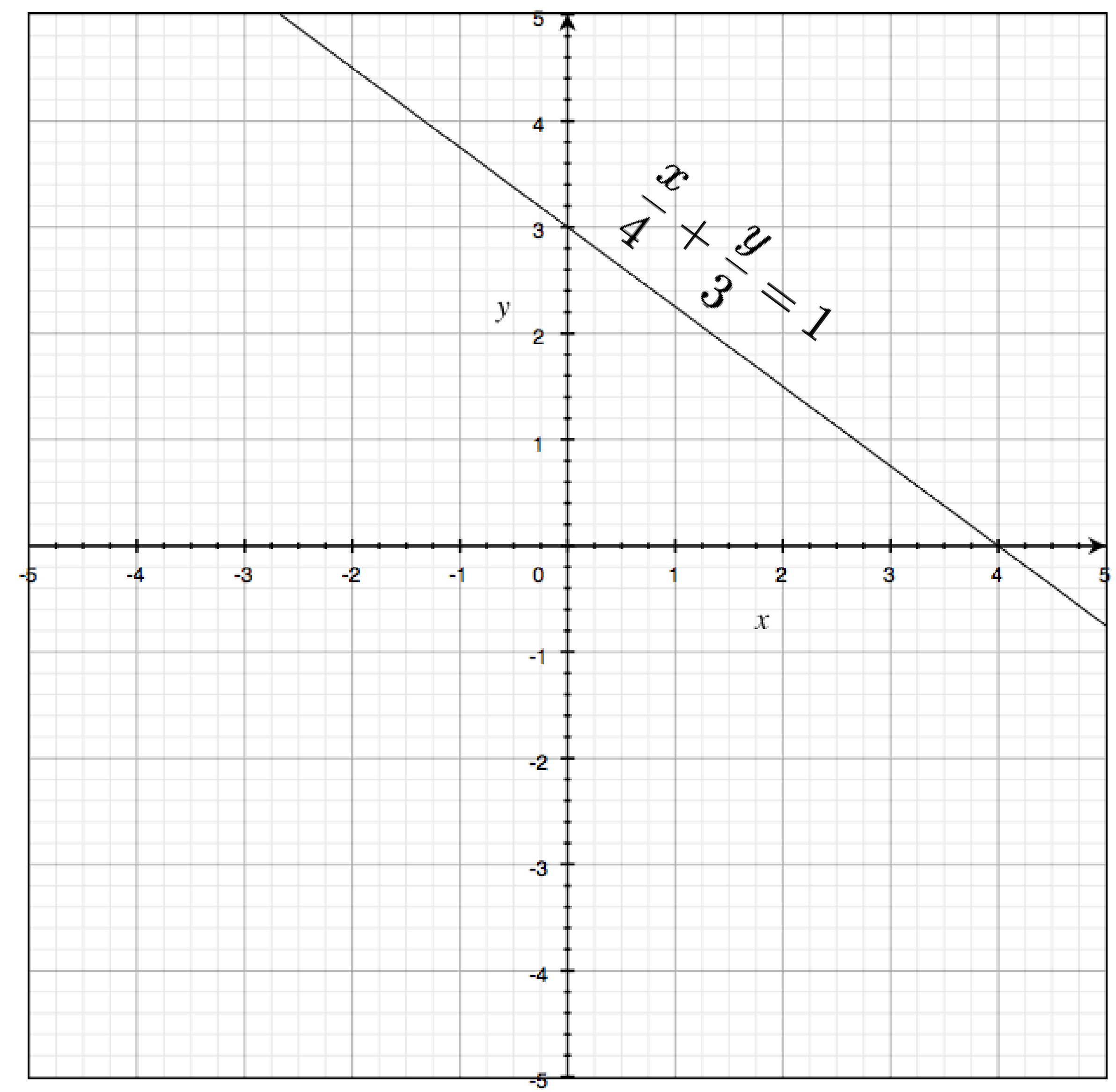
(Similar slope) $$\frac{y-b}{x-0}=\frac{0-y}{a-x}$$



$$ya+yx-ba+bx=-yx$$

$$ya+bx=ba$$

$$\frac{y}{b}+\frac{x}{a}=1$$



Normal Form

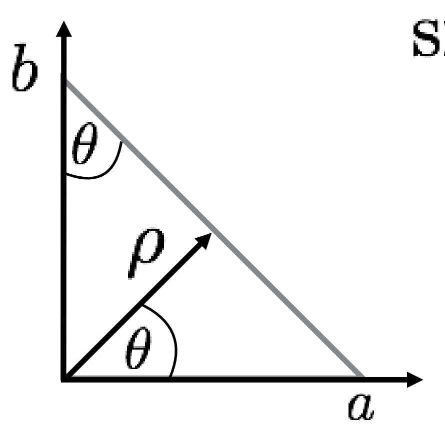
$$x \cos \theta + y \sin \theta = \rho$$

What are rho and theta?

Normal Form

$$x\cos\theta + y\sin\theta = \rho$$

Derivation:

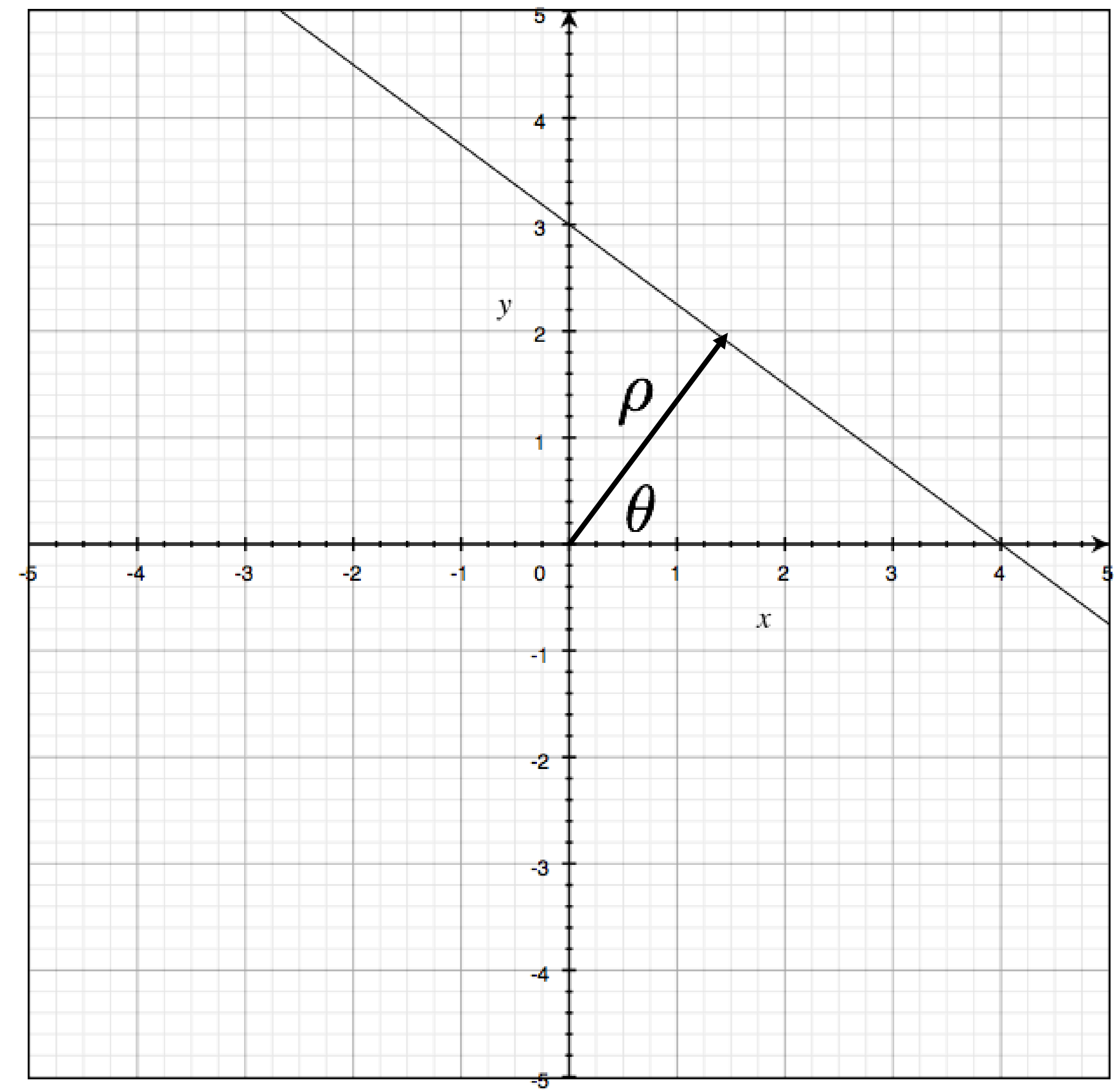


$$\cos\theta = \frac{\rho}{a} \rightarrow a = \frac{\rho}{\cos\theta}$$

$$\sin\theta = \frac{\rho}{b} \rightarrow b = \frac{\rho}{\sin\theta}$$

plug into: $\frac{x}{a} + \frac{y}{b} = 1$

$$x\cos\theta + y\sin\theta = \rho$$



Hough transform

Hough transform

- Generic framework for detecting a parametric model
- Edges don’t have to be connected
- Lines can be occluded
- Key idea: edges **vote** for the possible models

Image and parameter space

variables

$$y = mx + b$$

parameters

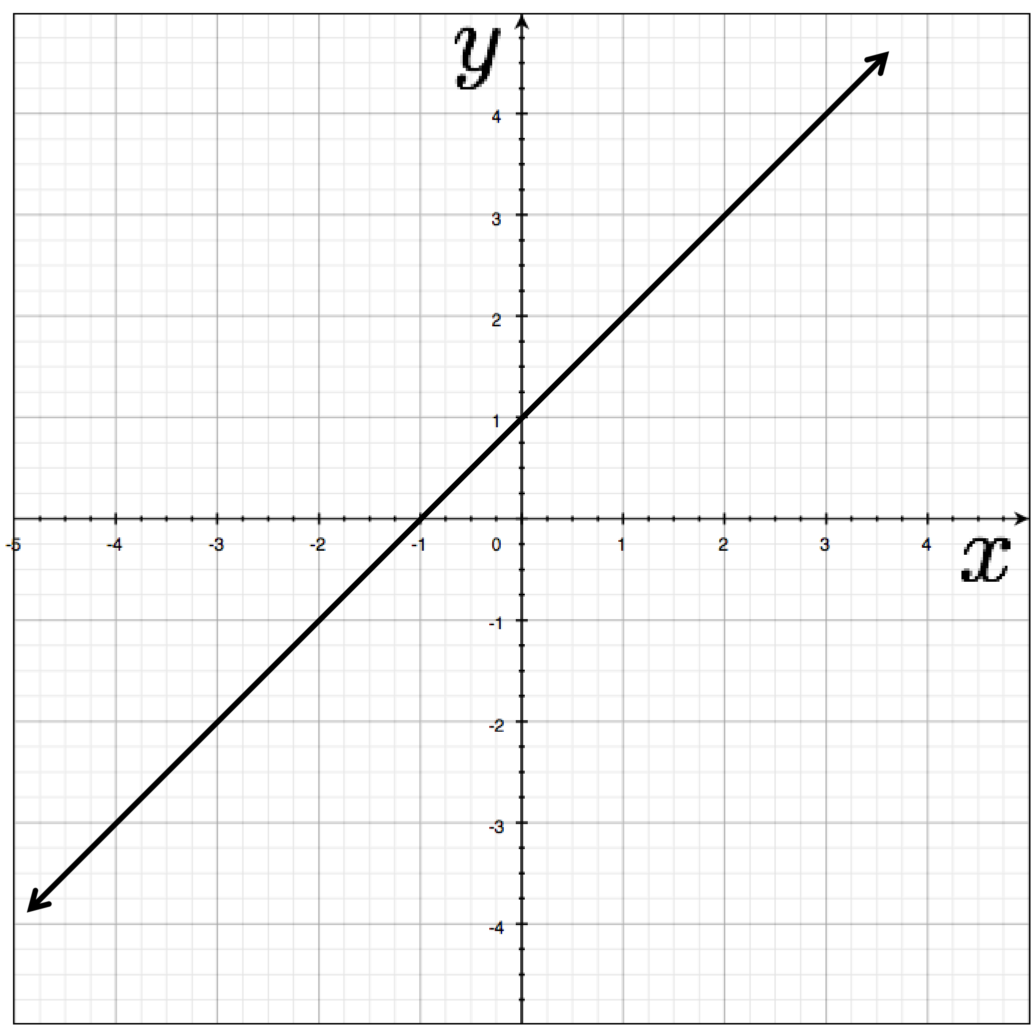


Image space

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

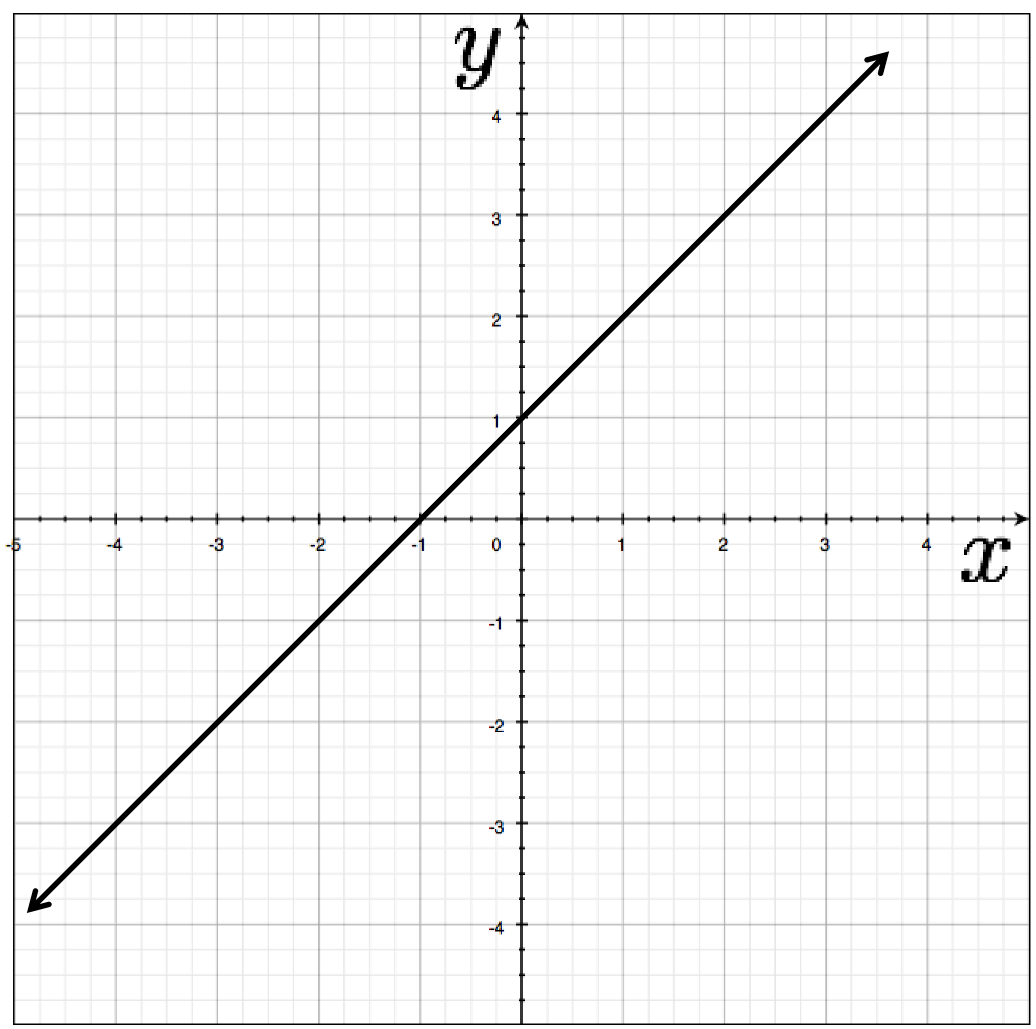
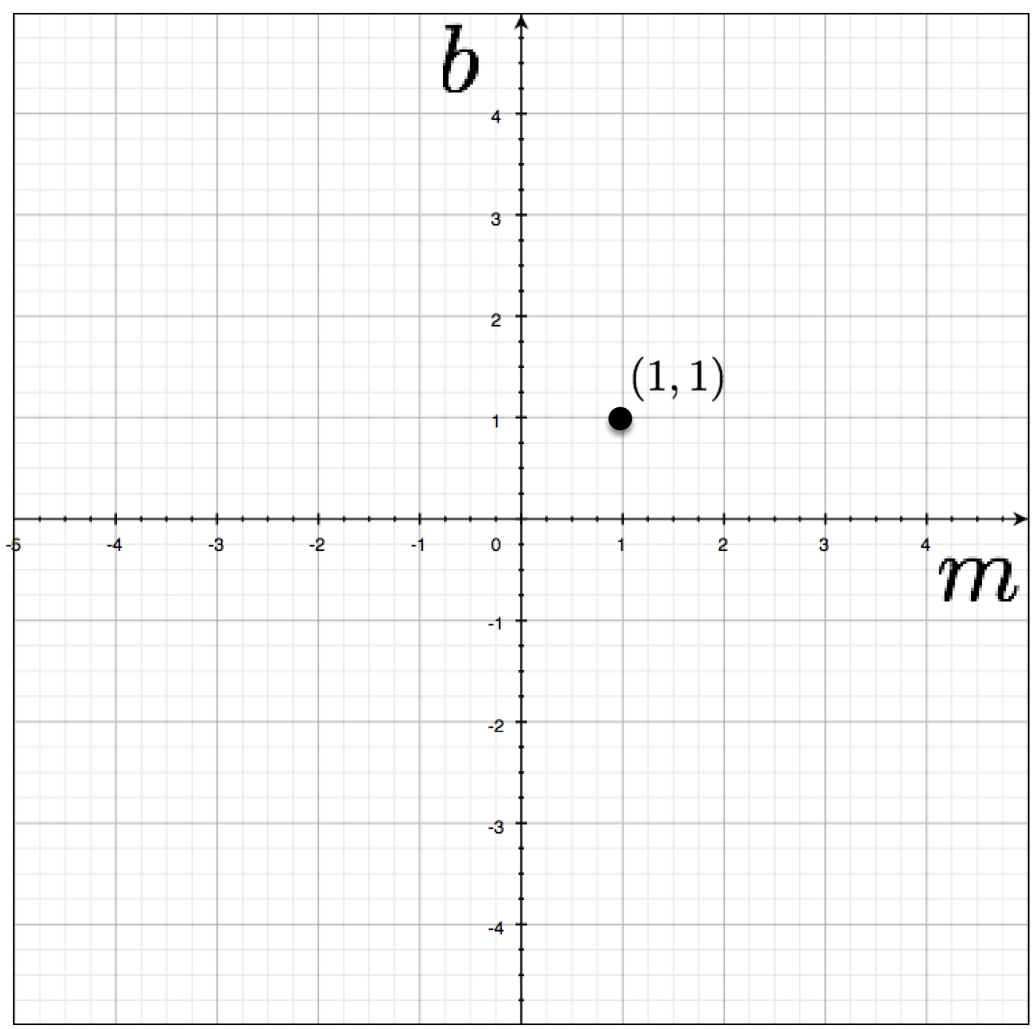


Image space

a line becomes a point



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

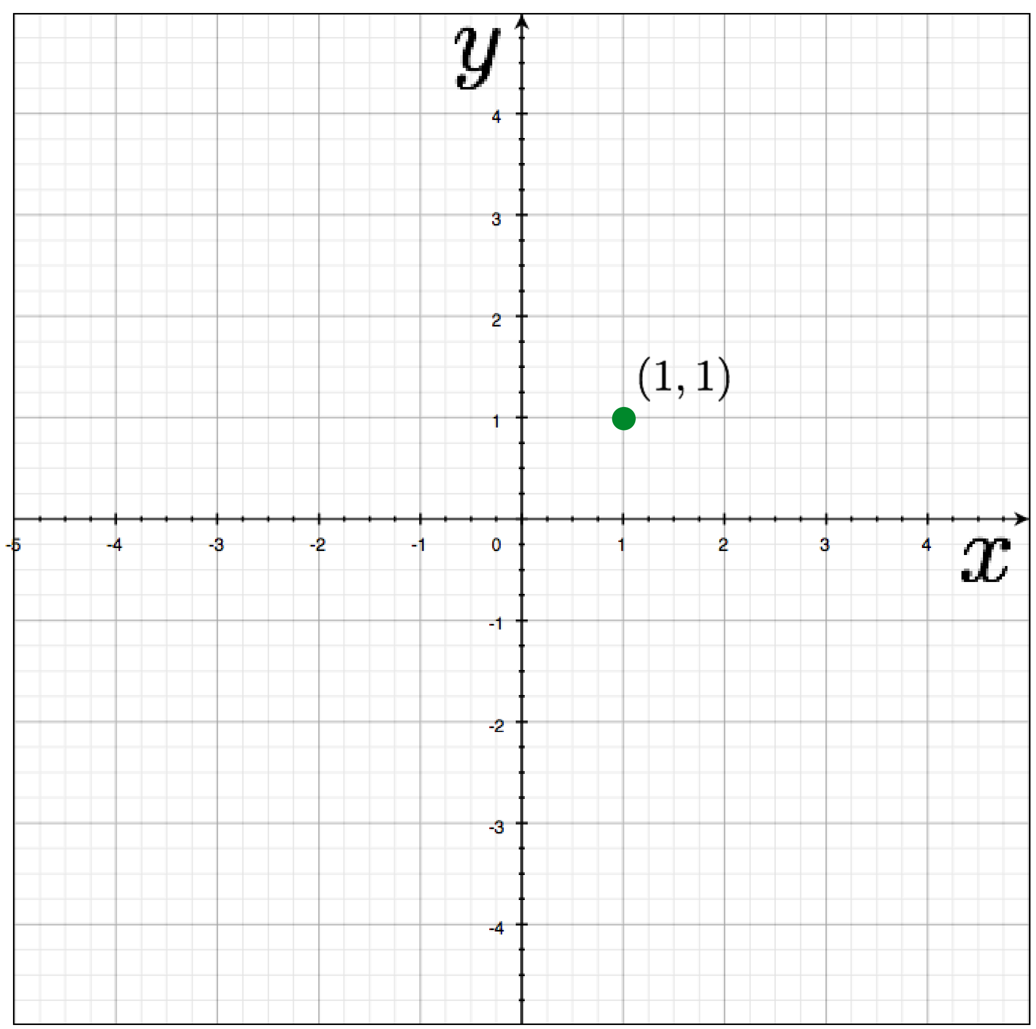


Image space

What would a point in image space become in parameter space?

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

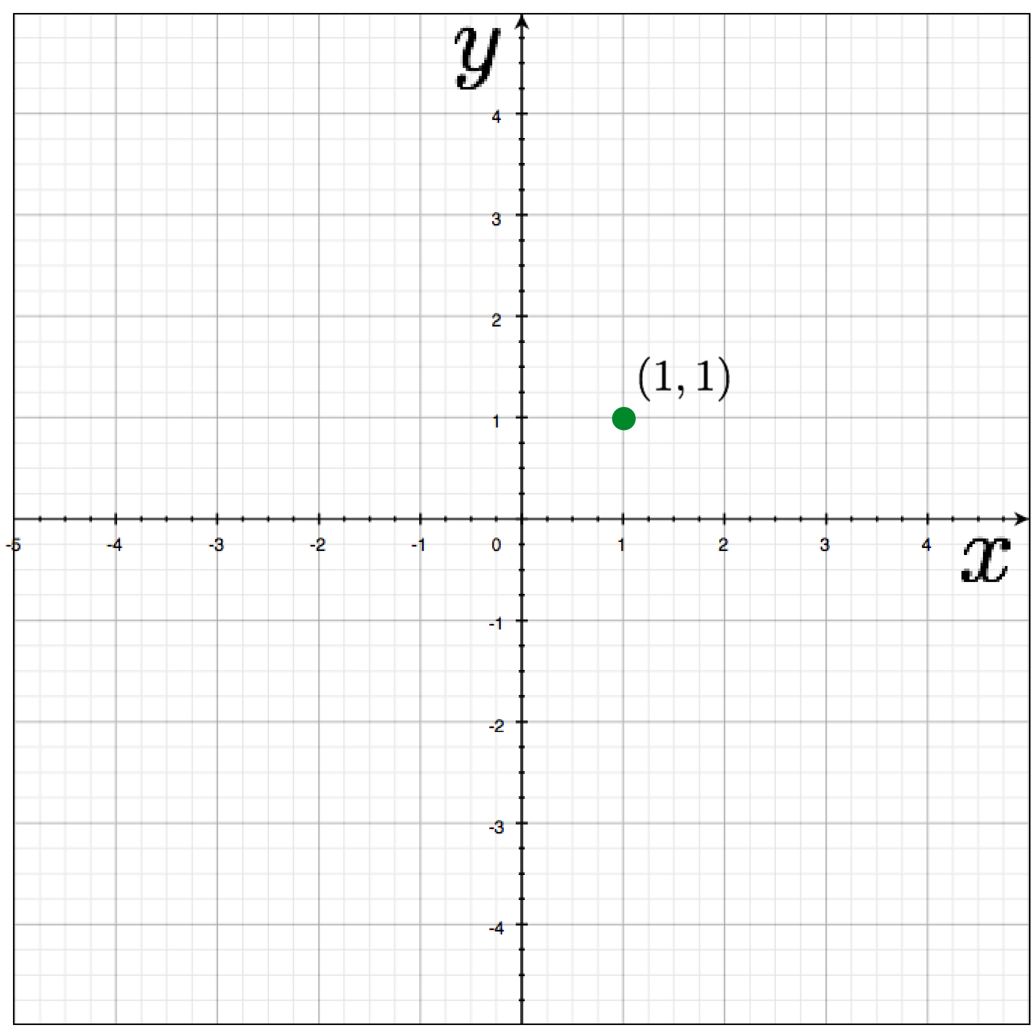
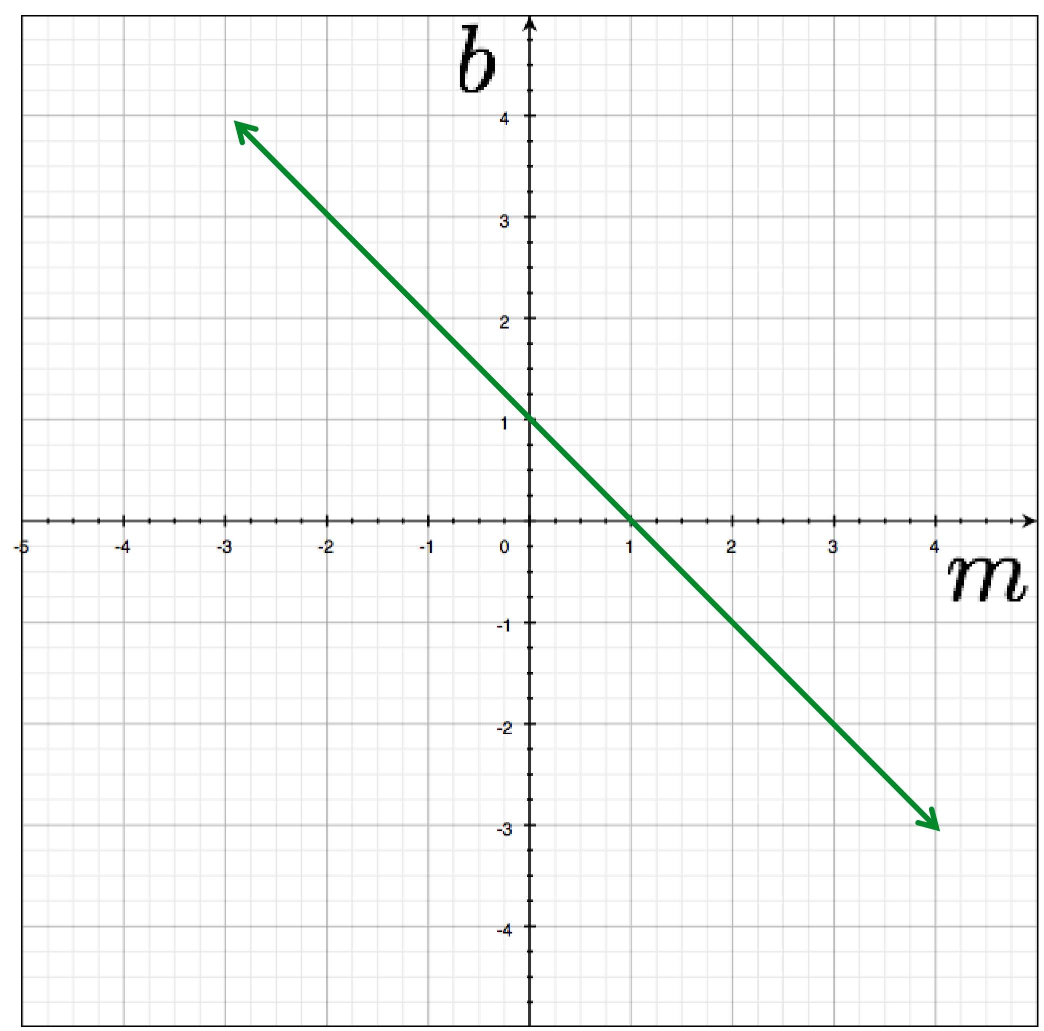


Image space

a point becomes a line



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

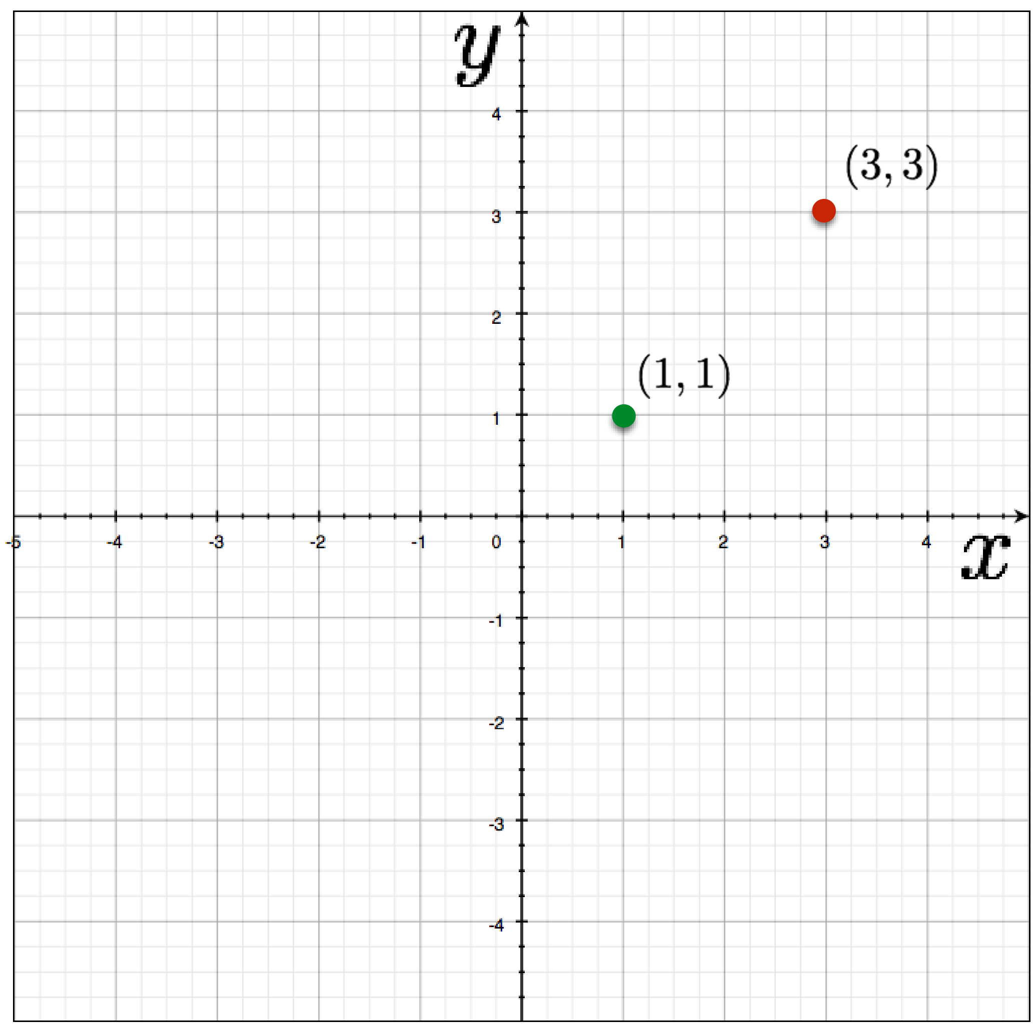
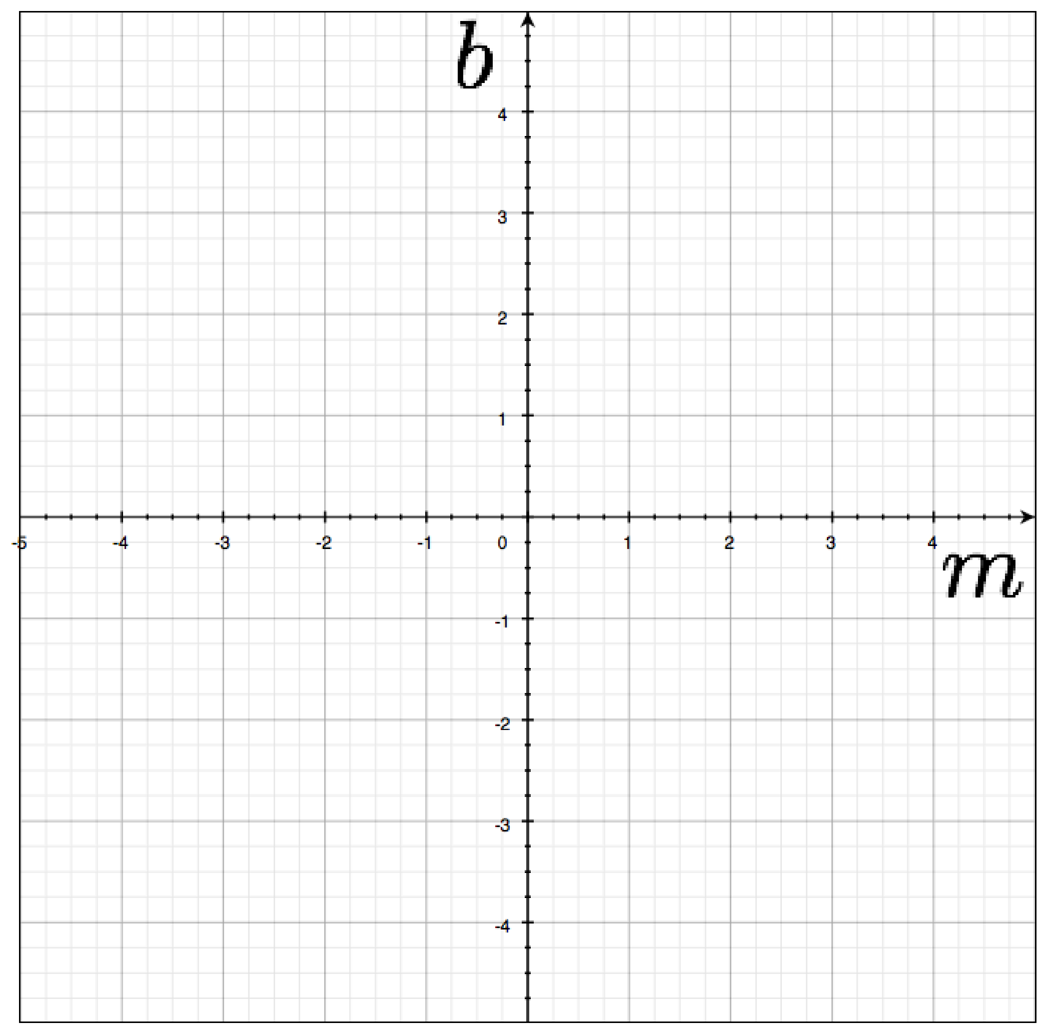


Image space

two points become ?



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

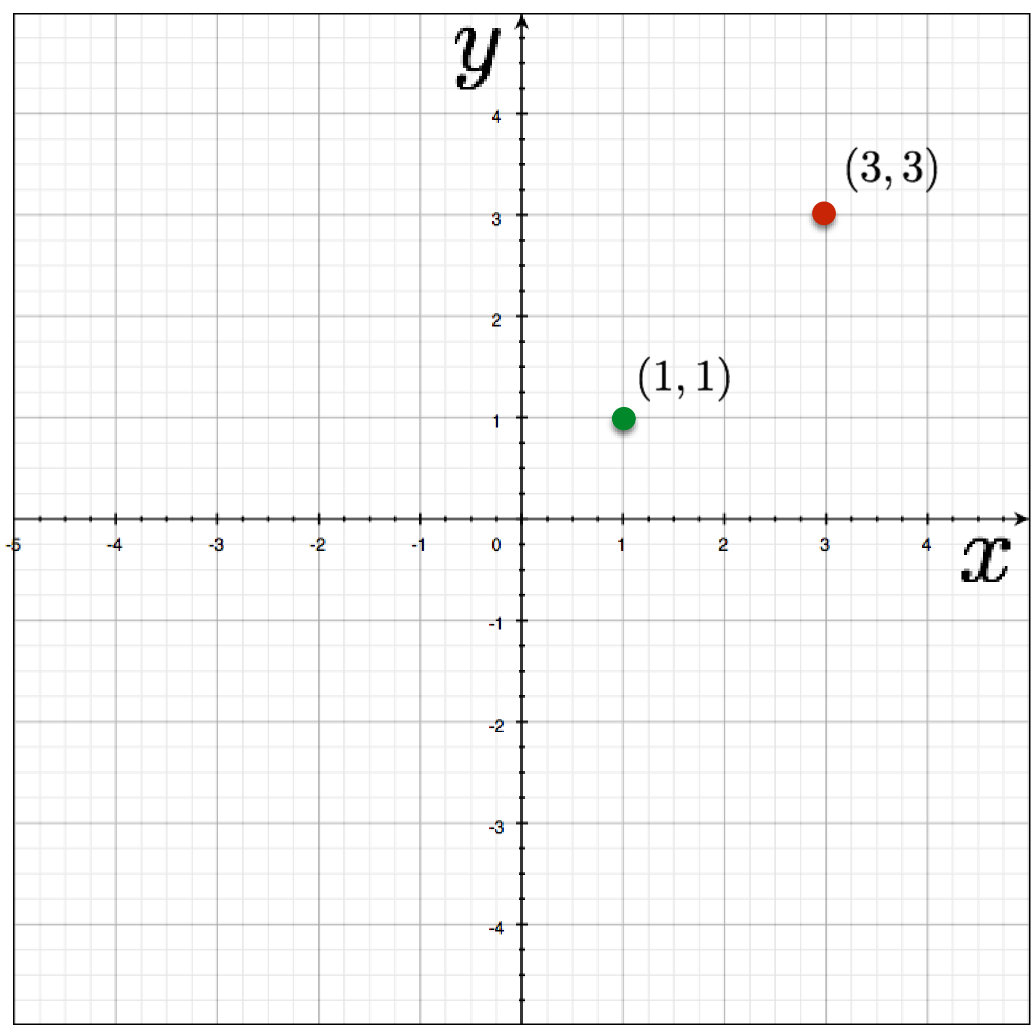
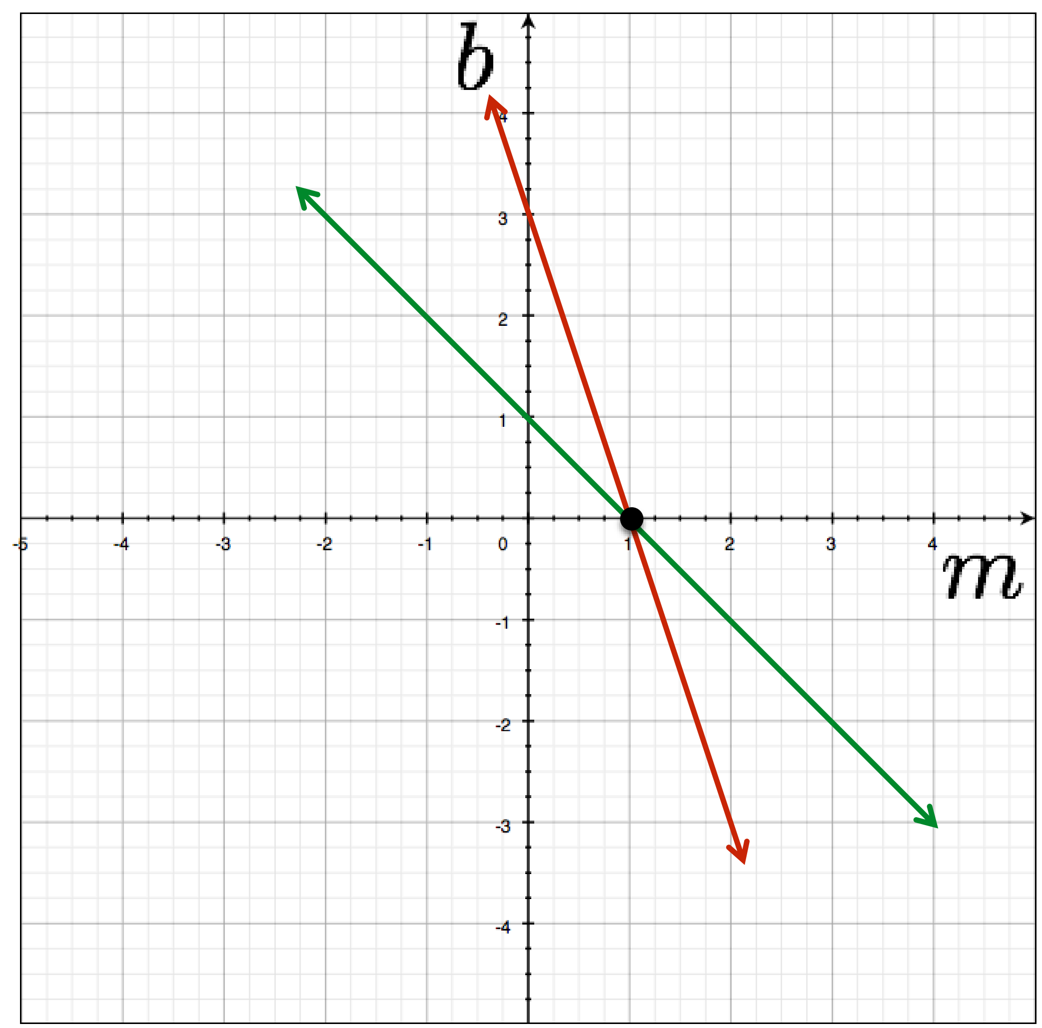


Image space

two points become ?



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

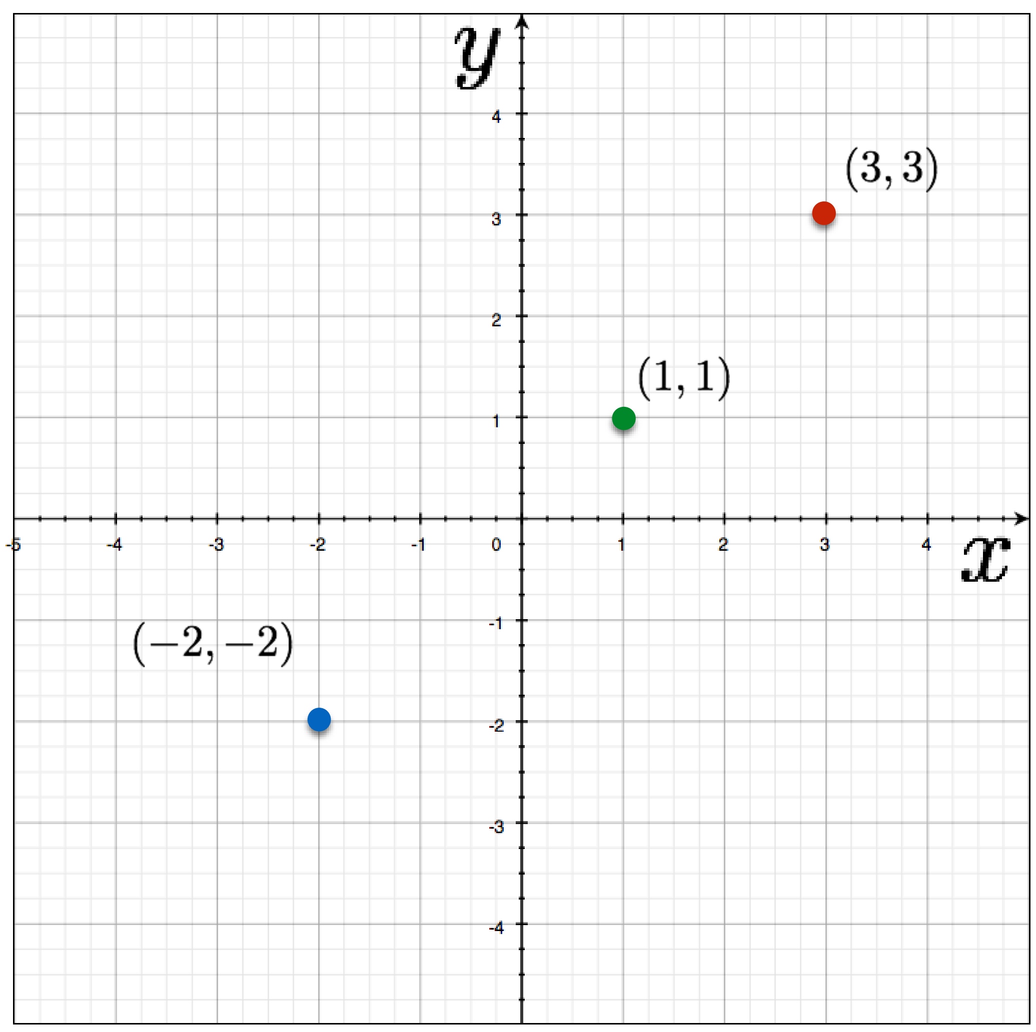
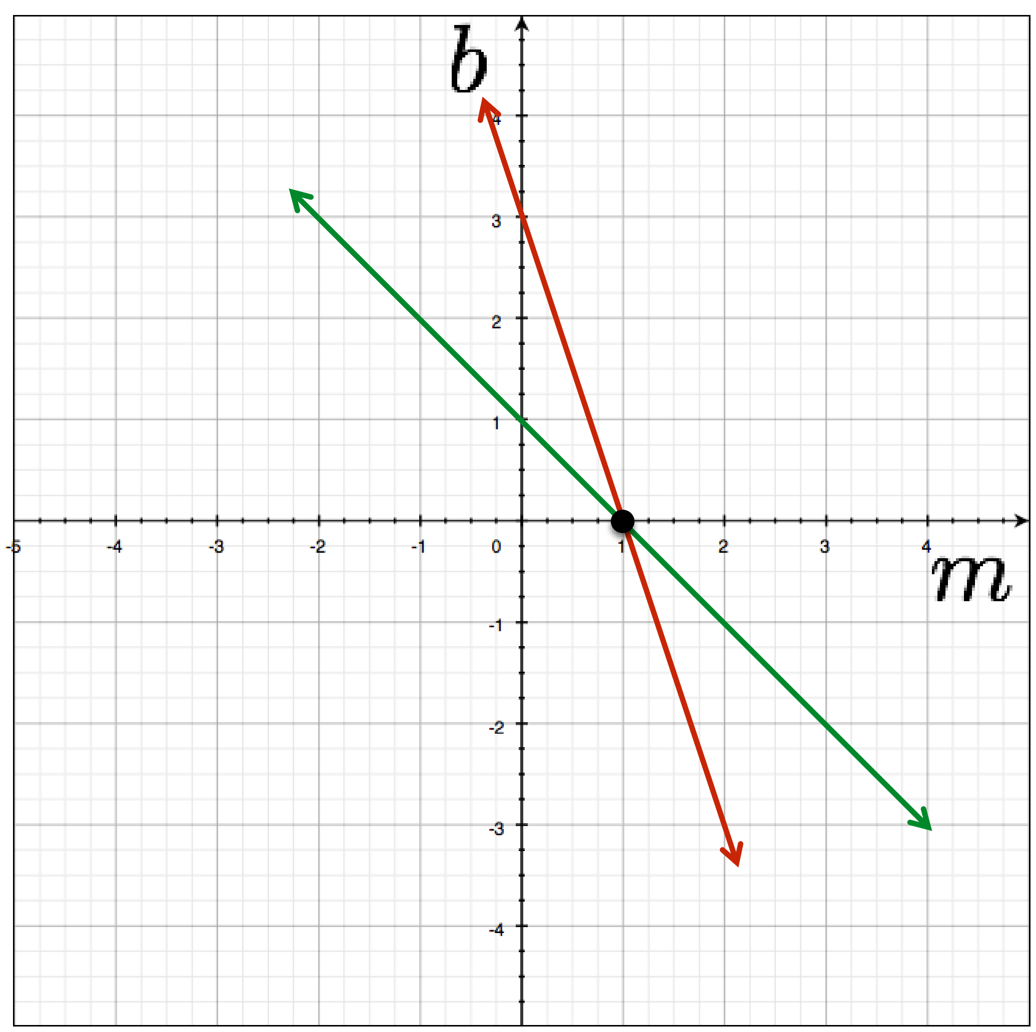


Image space



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

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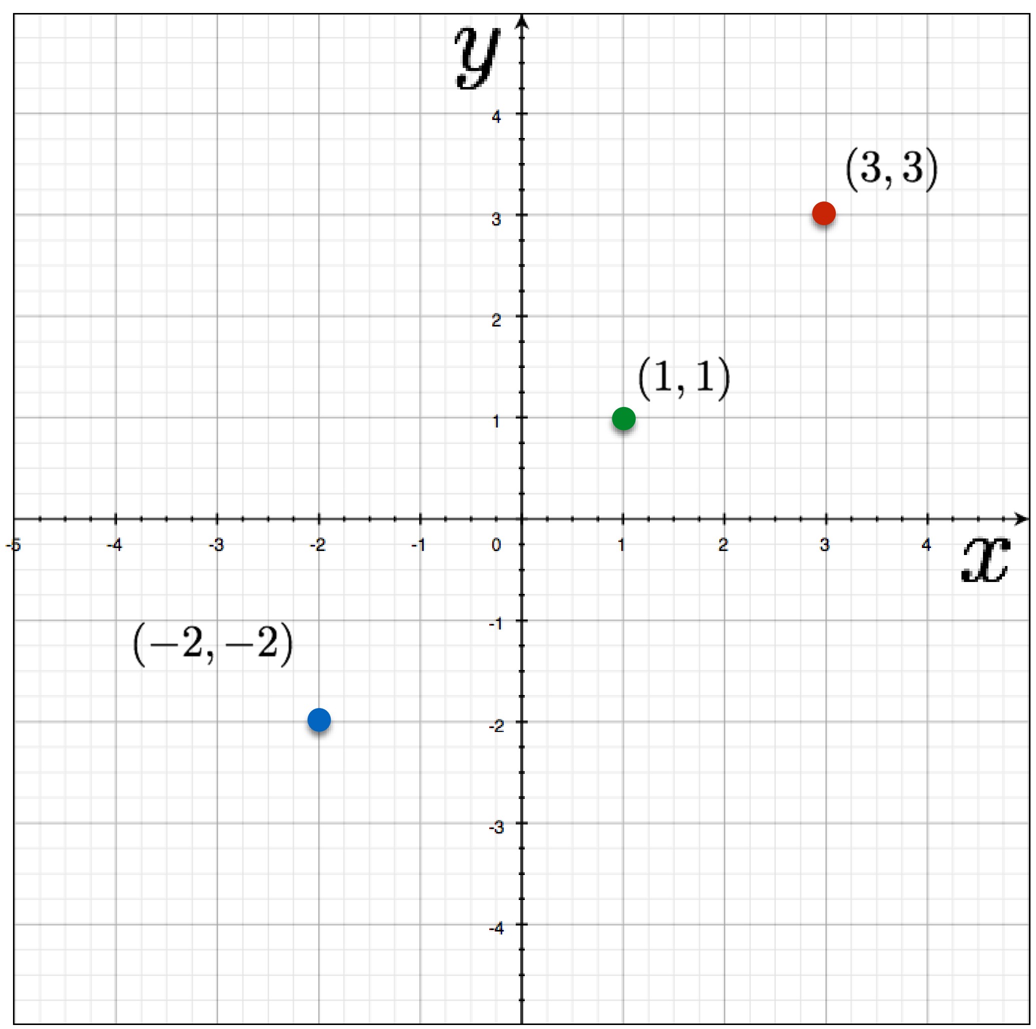
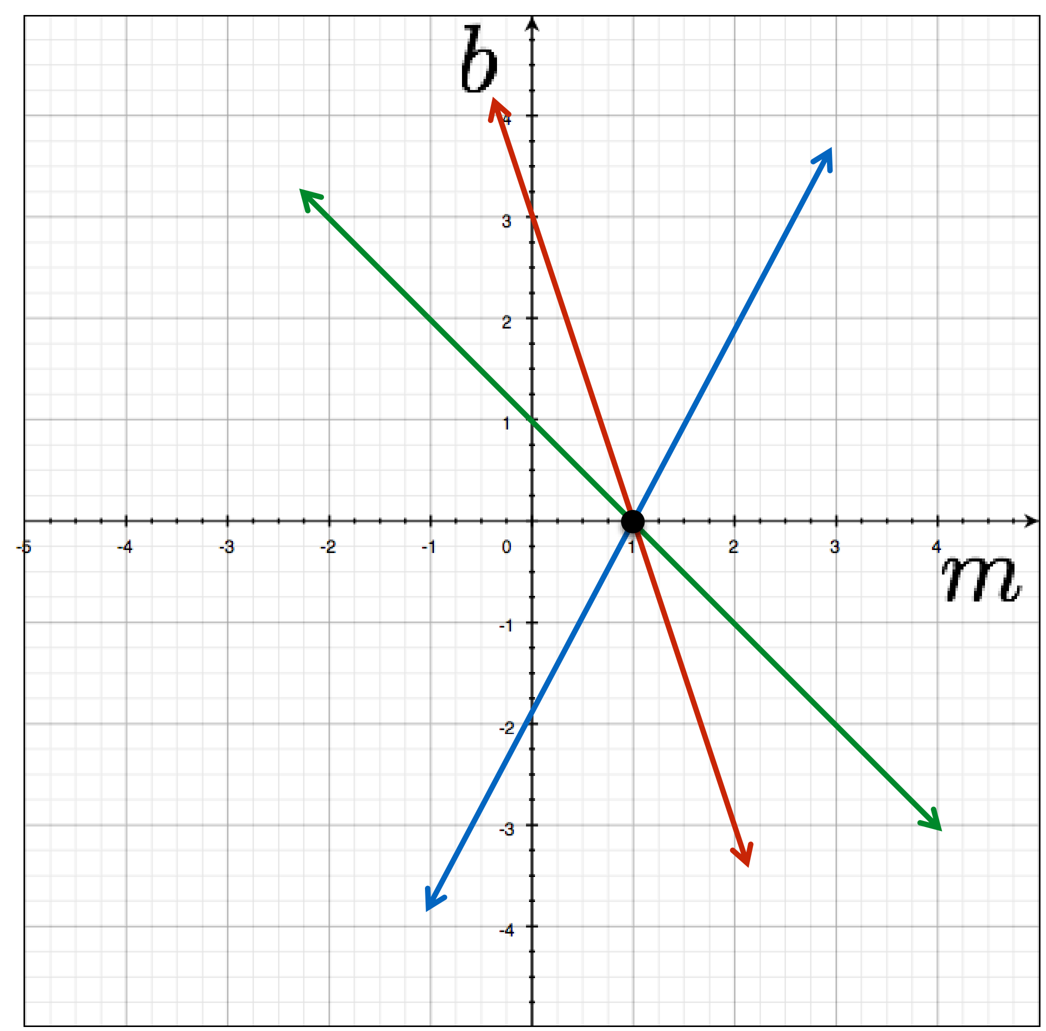


Image space

three points become ?



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

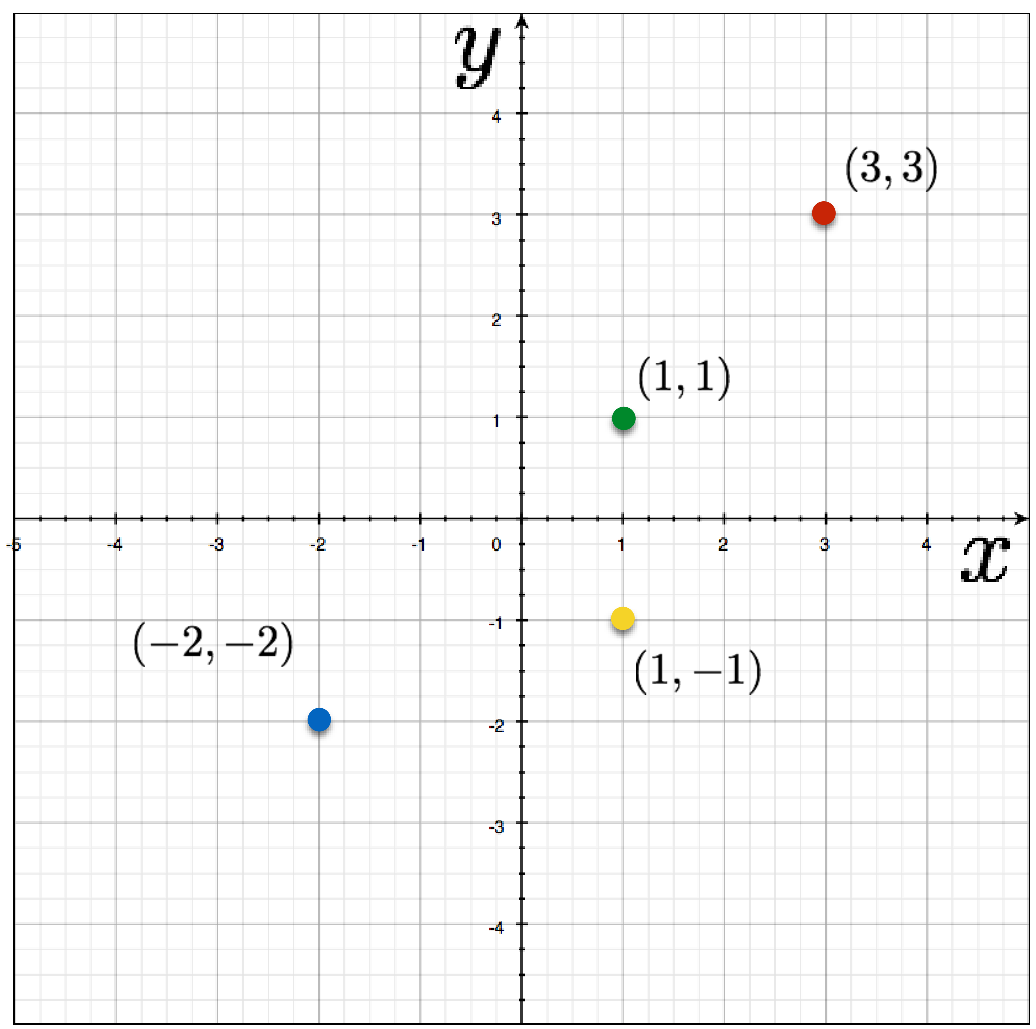
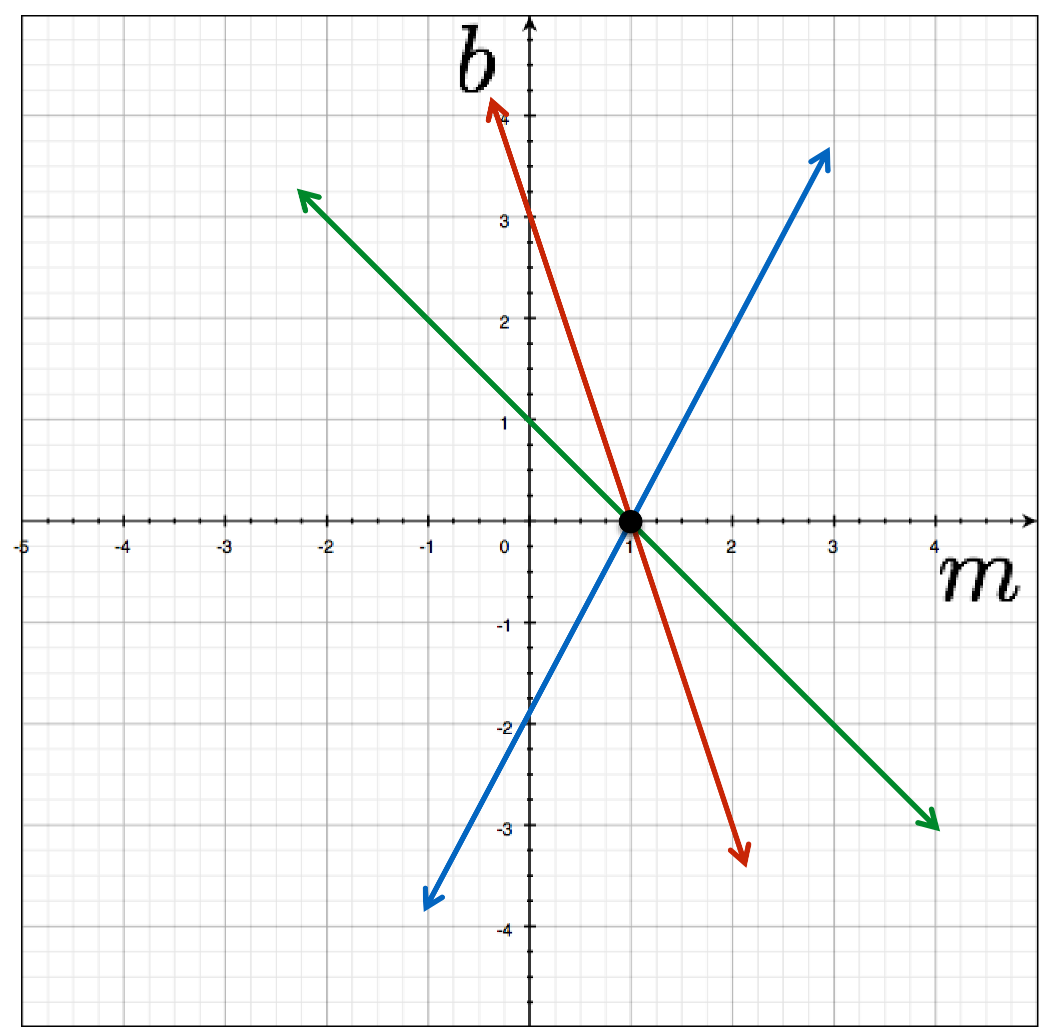


Image space

four points become ?



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

variables

$$y - mx = b$$

parameters

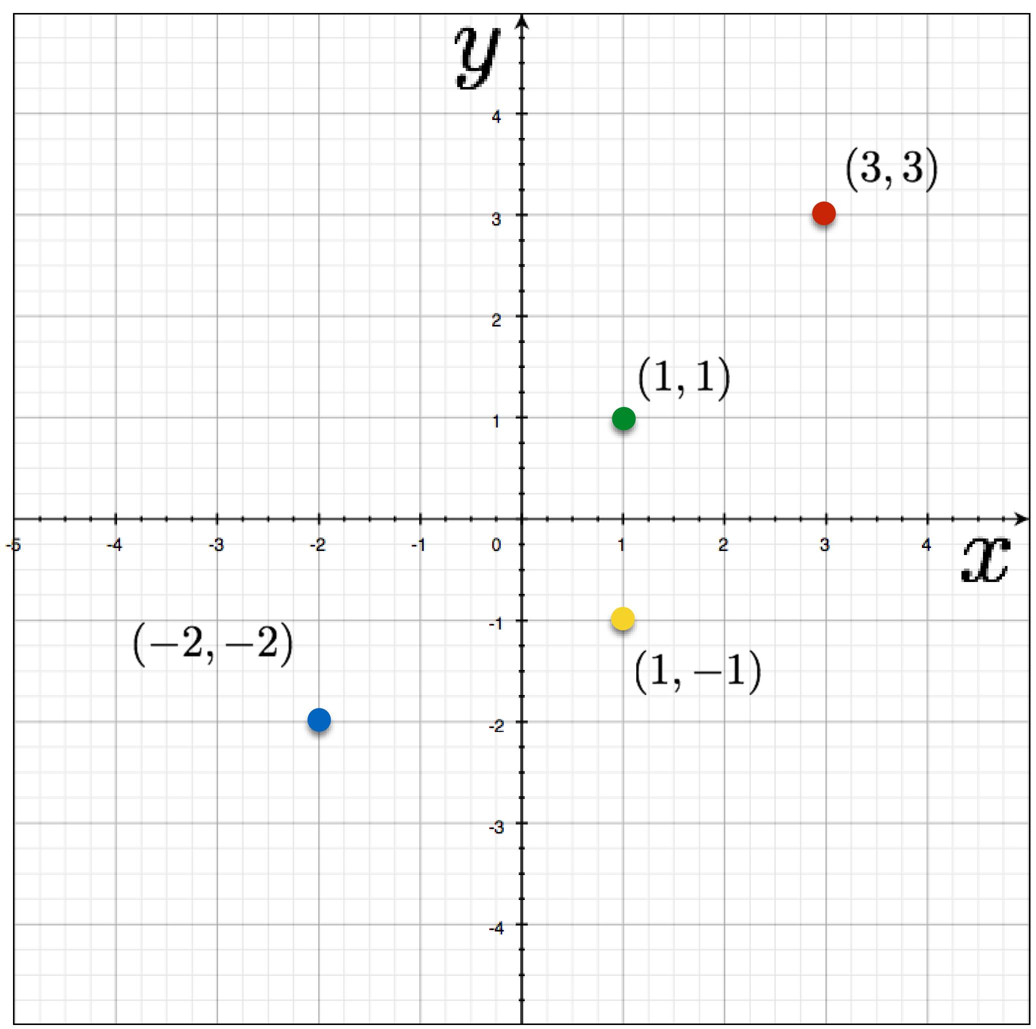
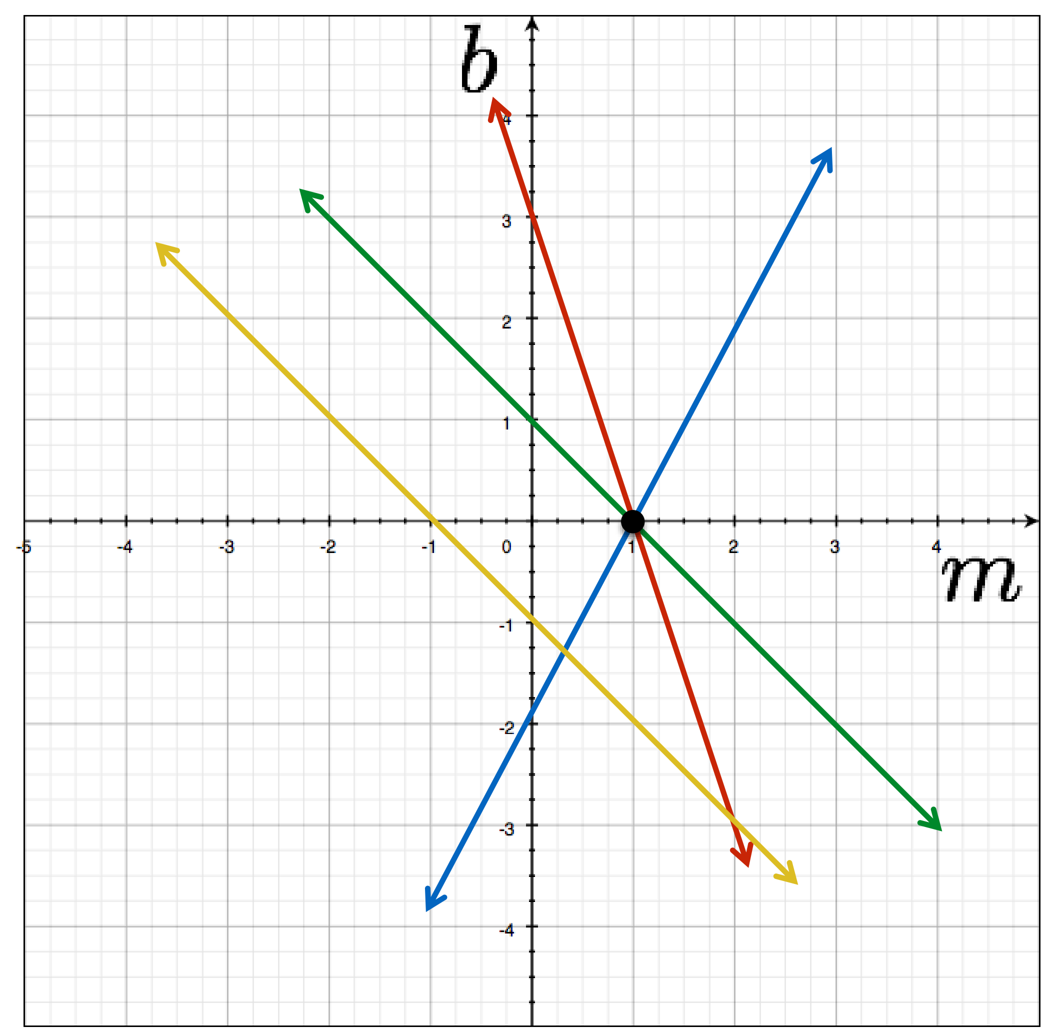


Image space

four points become ?



Parameter space

How would you find the best fitting line?

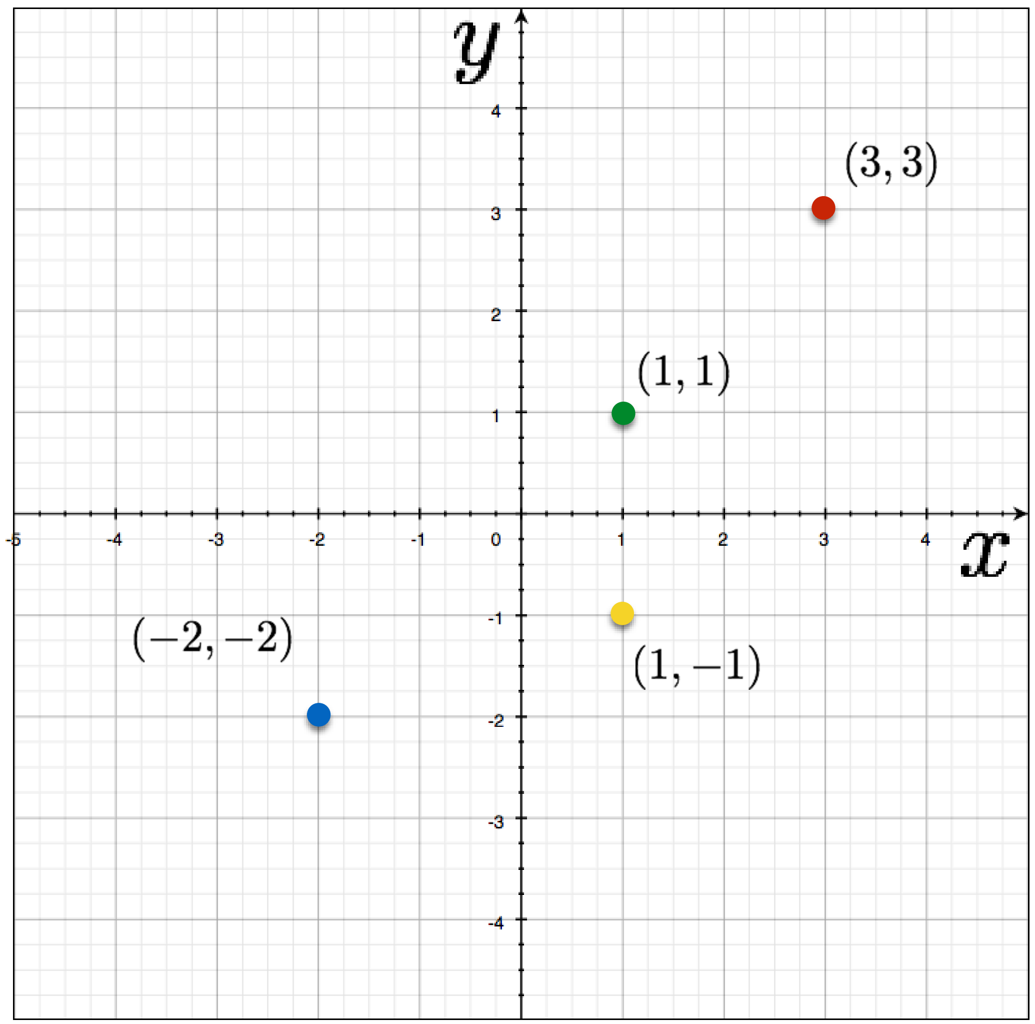
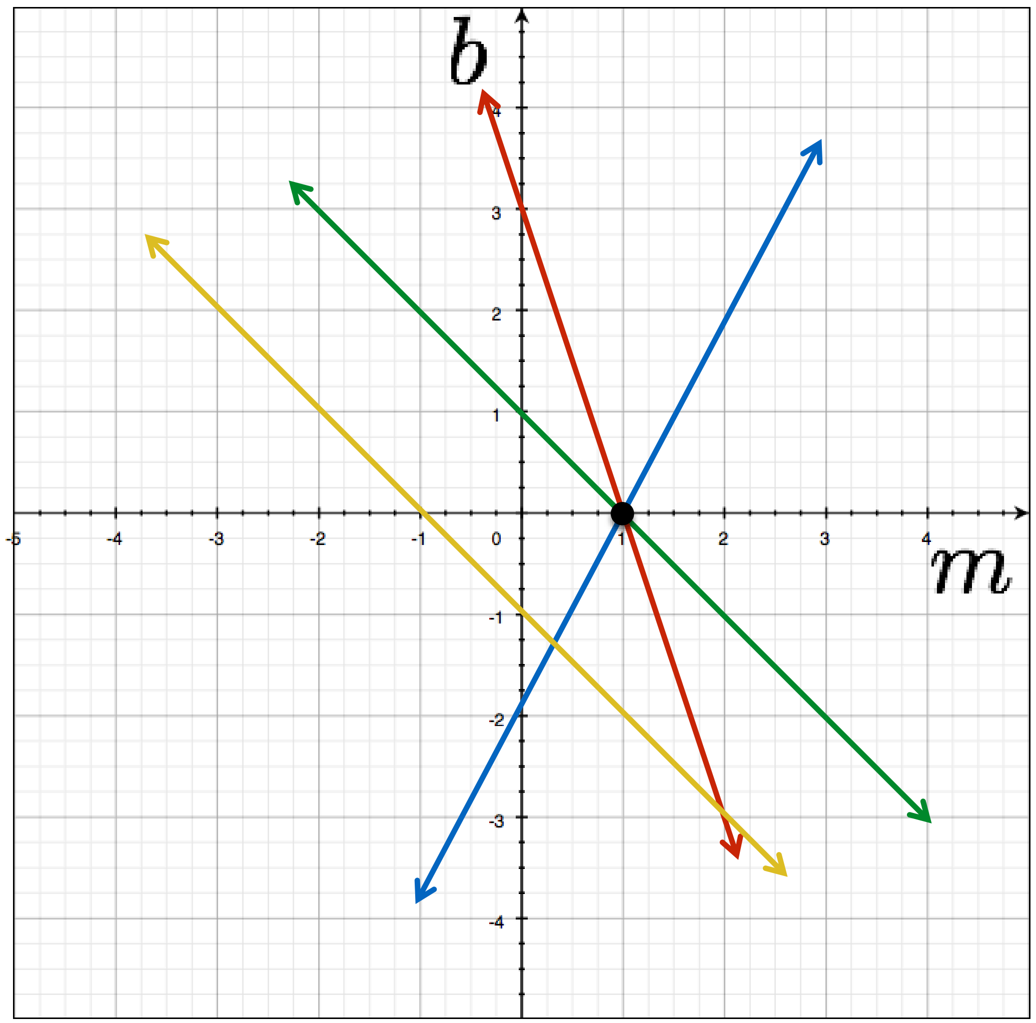


Image space



Parameter space

Is this method robust to measurement noise?

Is this method robust to outliers?

Line Detection by Hough Transform

Algorithm:

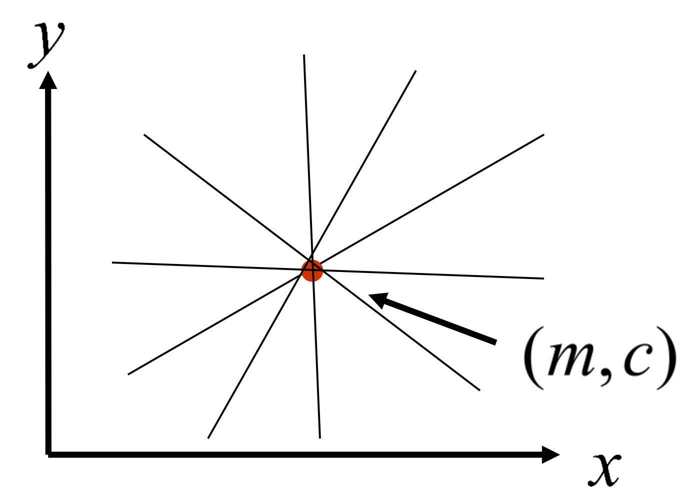
1. Quantize Parameter Space (m, c)

2. Create Accumulator Array $A(m, c)$

3. Set $A(m, c) = 0 \quad \forall m, c$

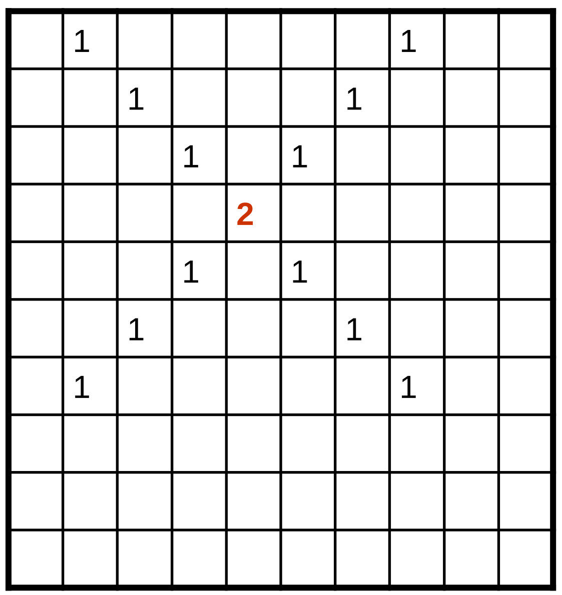
4. For each image edge (x_i, y_i)
    For each element in $A(m, c)$
        If (m, c) lies on the line: $c = -x_i m + y_i$
            Increment $A(m, c) = A(m, c) + 1$

5. Find local maxima in $A(m, c)$



Parameter Space

$A(m, c)$



Problems with parameterization

How big does the accumulator need to be for the parameterization (m, c)*?*

$A(m, c)$

	1						1		
		1				1			
			1		1				
				2					
			1		1				
		1				1			
	1						1		

Problems with parameterization

How big does the accumulator need to be for the parameterization (m,c)*?*

$A(m,c)$

	1						1		
		1				1			
			1		1				
				2					
			1		1				
		1				1			
	1						1		

The space of m is huge!

$$-\infty \leq m \leq \infty$$

The space of c is huge!

$$-\infty \leq c \leq \infty$$

Better Parameterization

Use normal form:

$$x \cos\theta + y \sin\theta = \rho$$

Given points (x_i, y_i) find (ρ, θ)

Hough Space Sinusoid

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \rho \leq \rho_{max}$$

(Finite Accumulator Array Size)

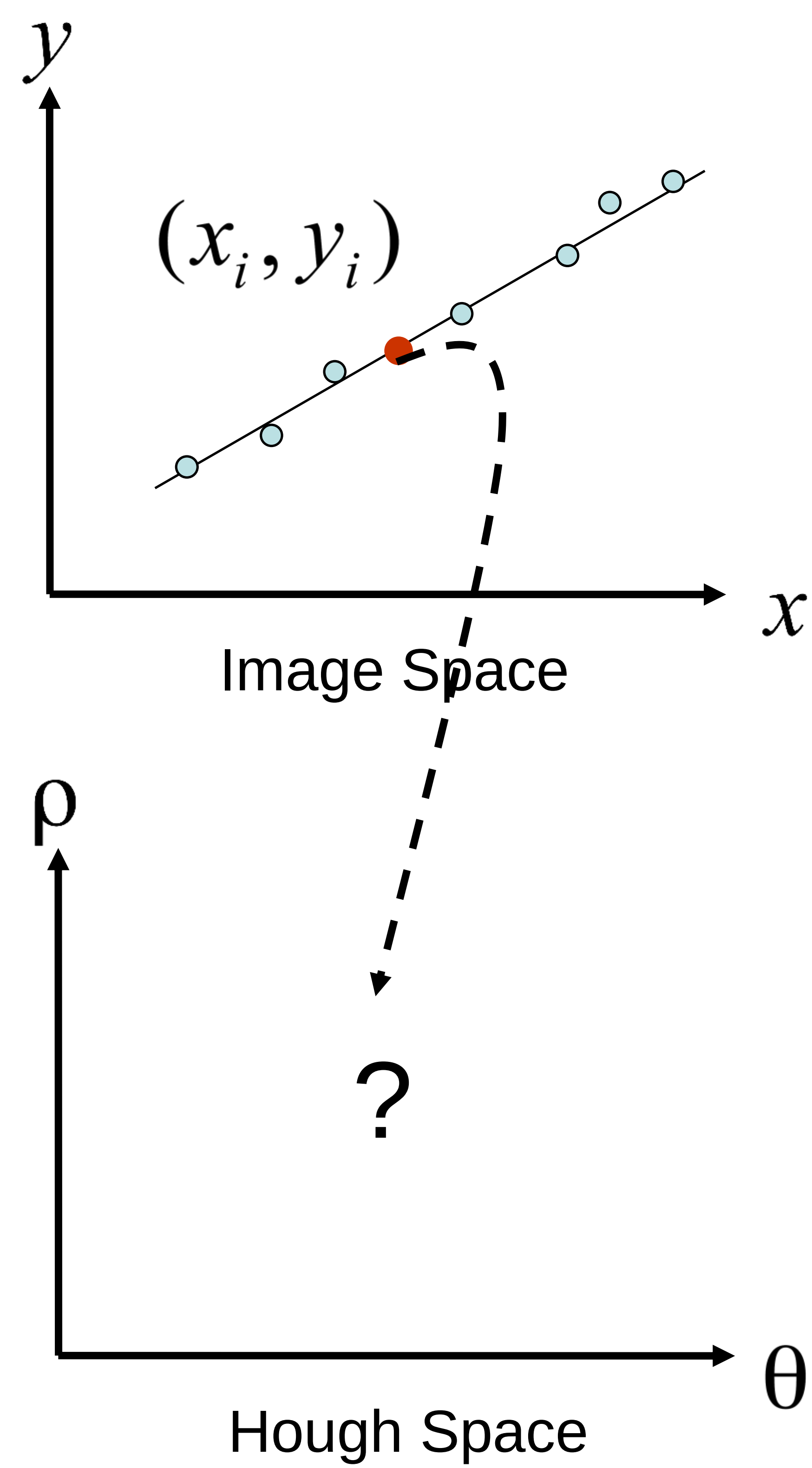


Image and parameter space

variables

$$y = mx + b$$

parameters

parameters

$$x\cos\theta + y\sin\theta = \rho$$

variables

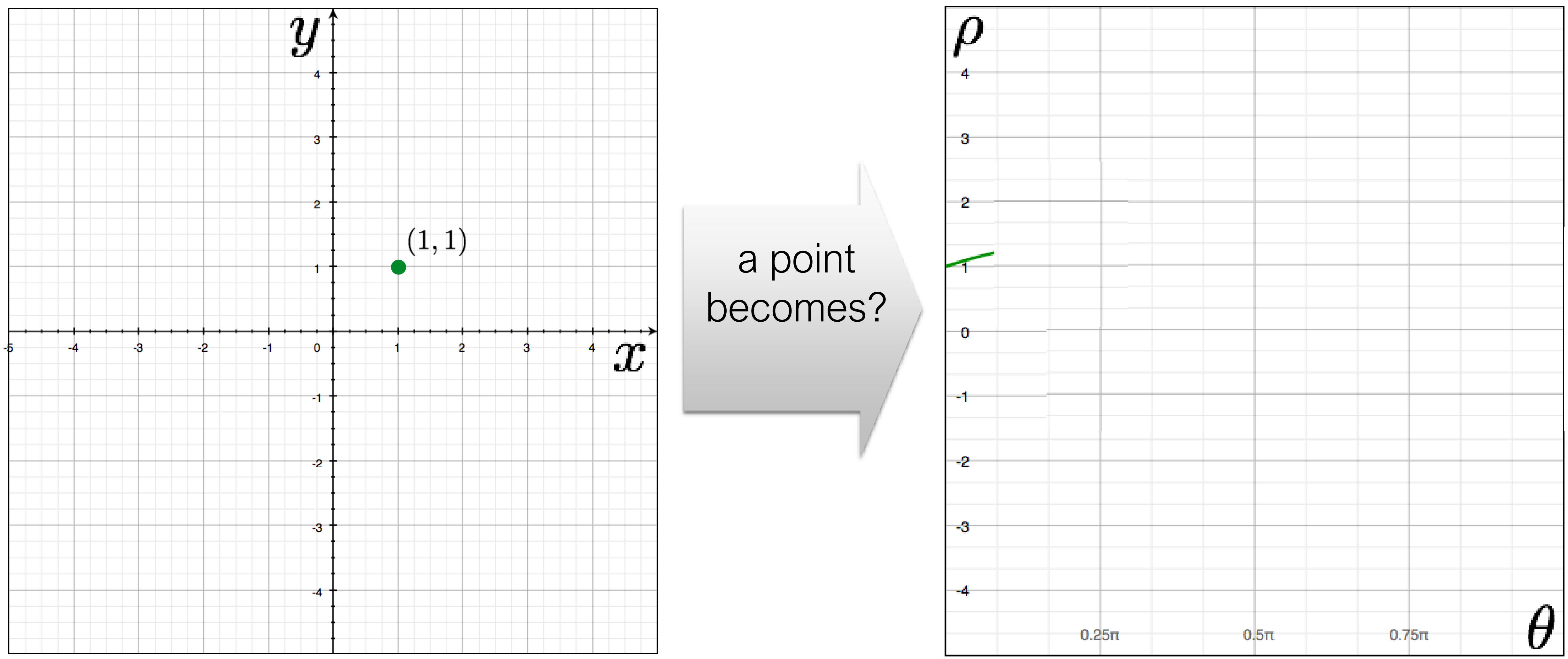


Image space

Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

parameters

$$x\cos\theta + y\sin\theta = \rho$$

variables

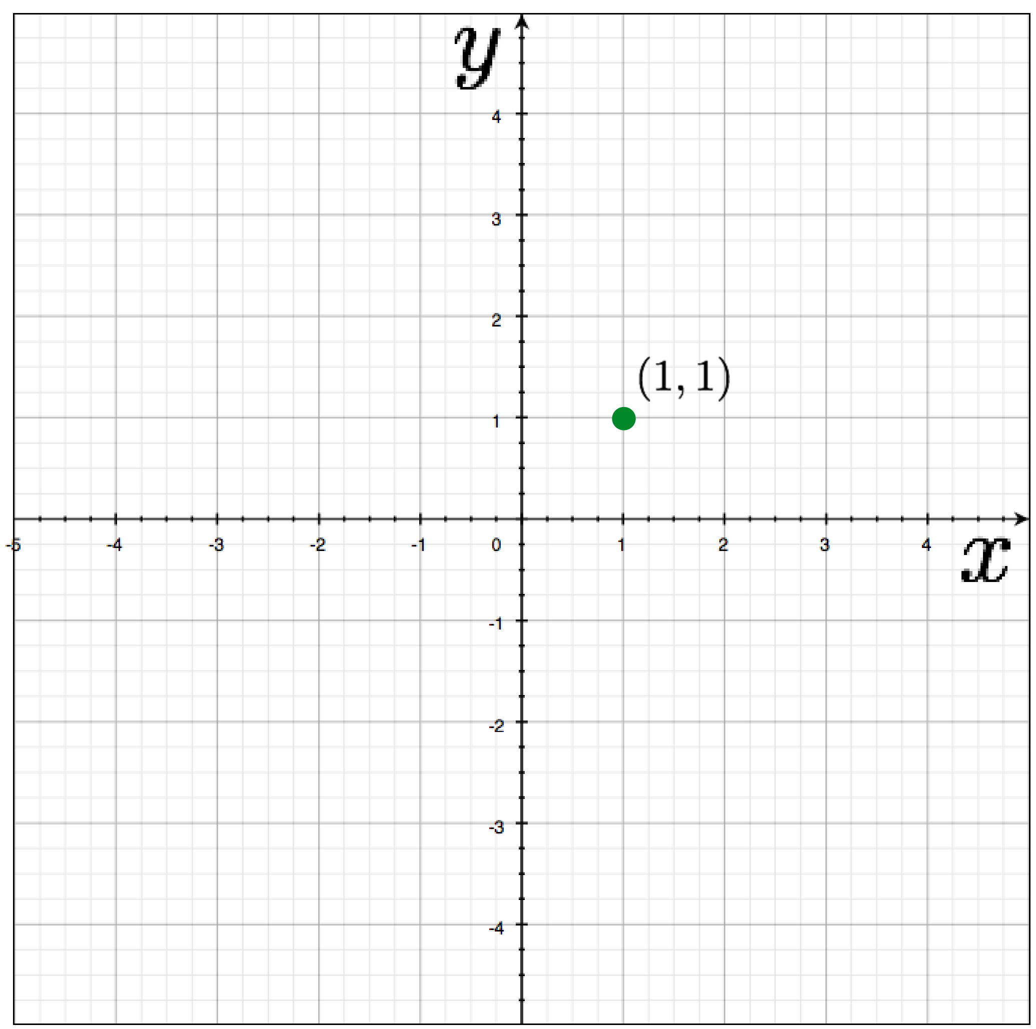


Image space

a point becomes a wave

Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

$$x\cos\theta + y\sin\theta = \rho$$

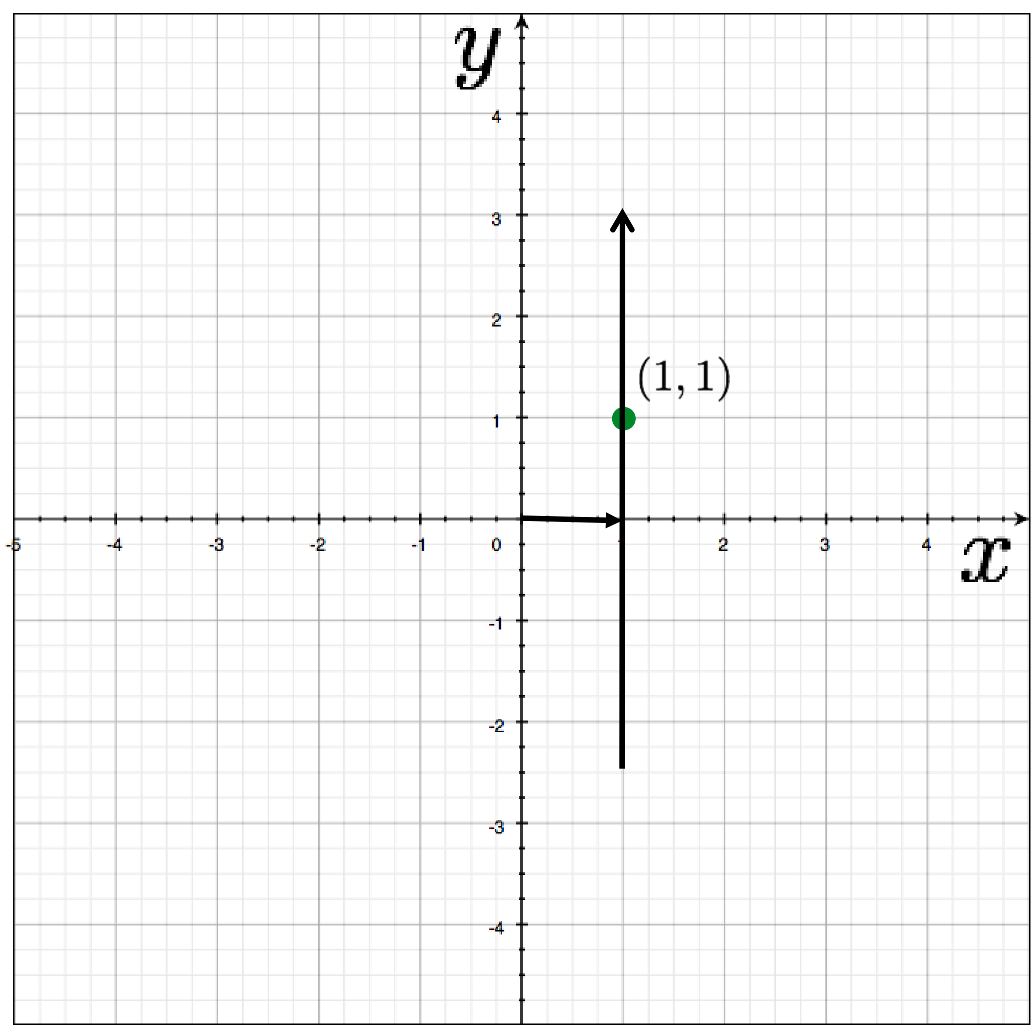
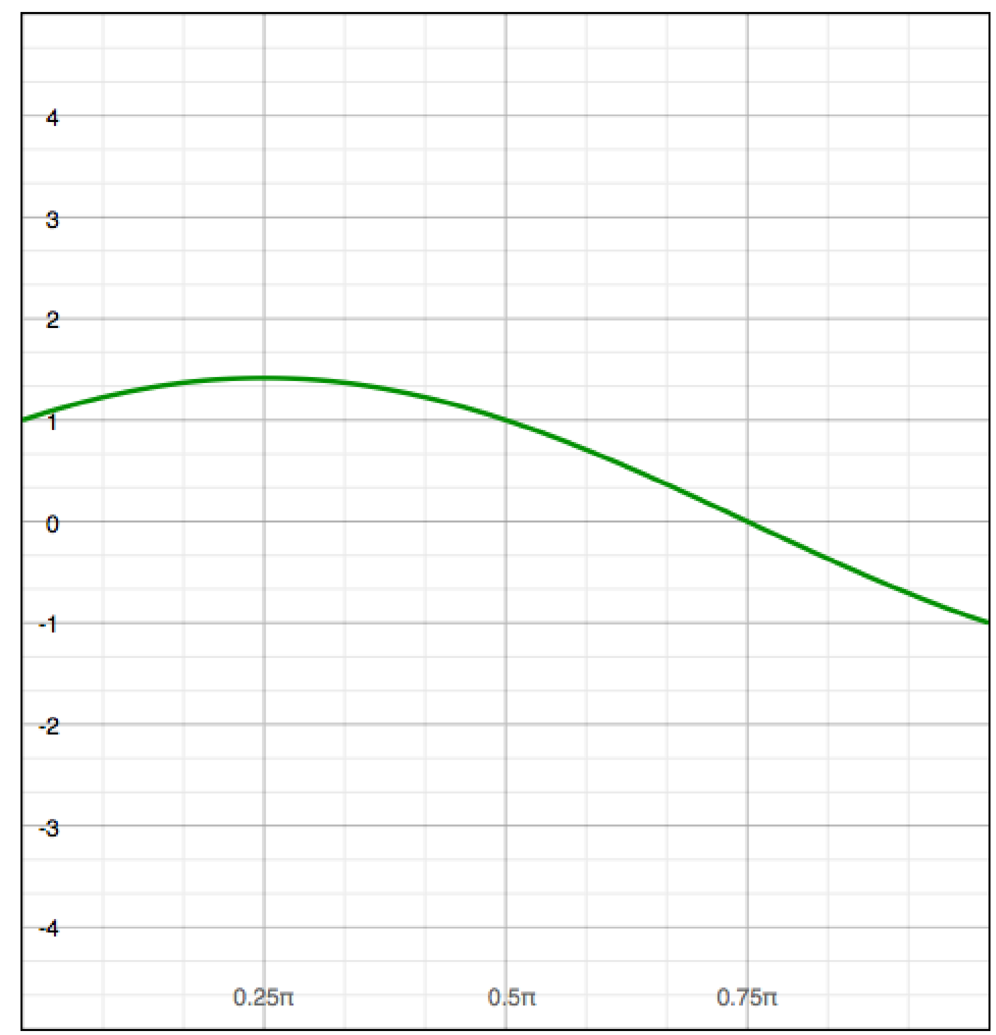


Image space

a line becomes?



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

$$x\cos\theta + y\sin\theta = \rho$$

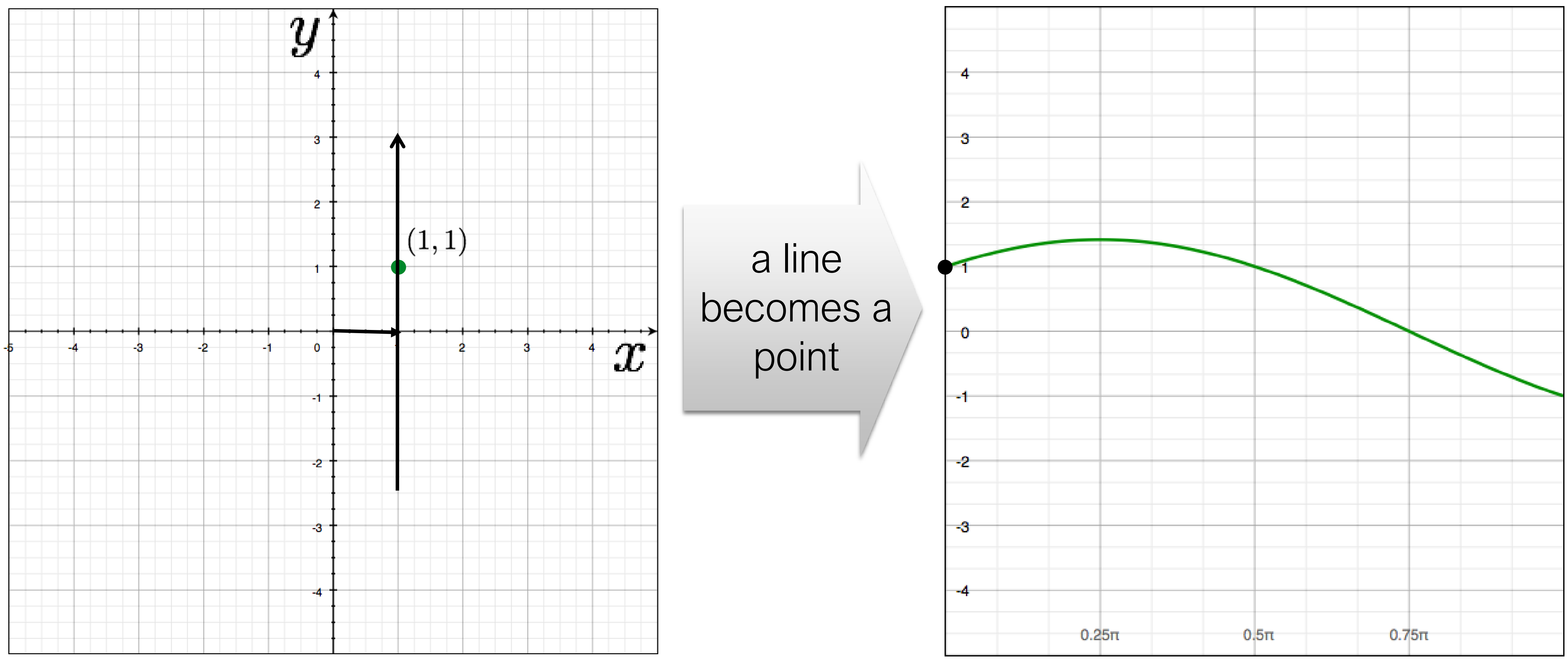


Image space

Parameter space

Image and parameter space

$$y = mx + b$$

$$x\cos\theta + y\sin\theta = \rho$$

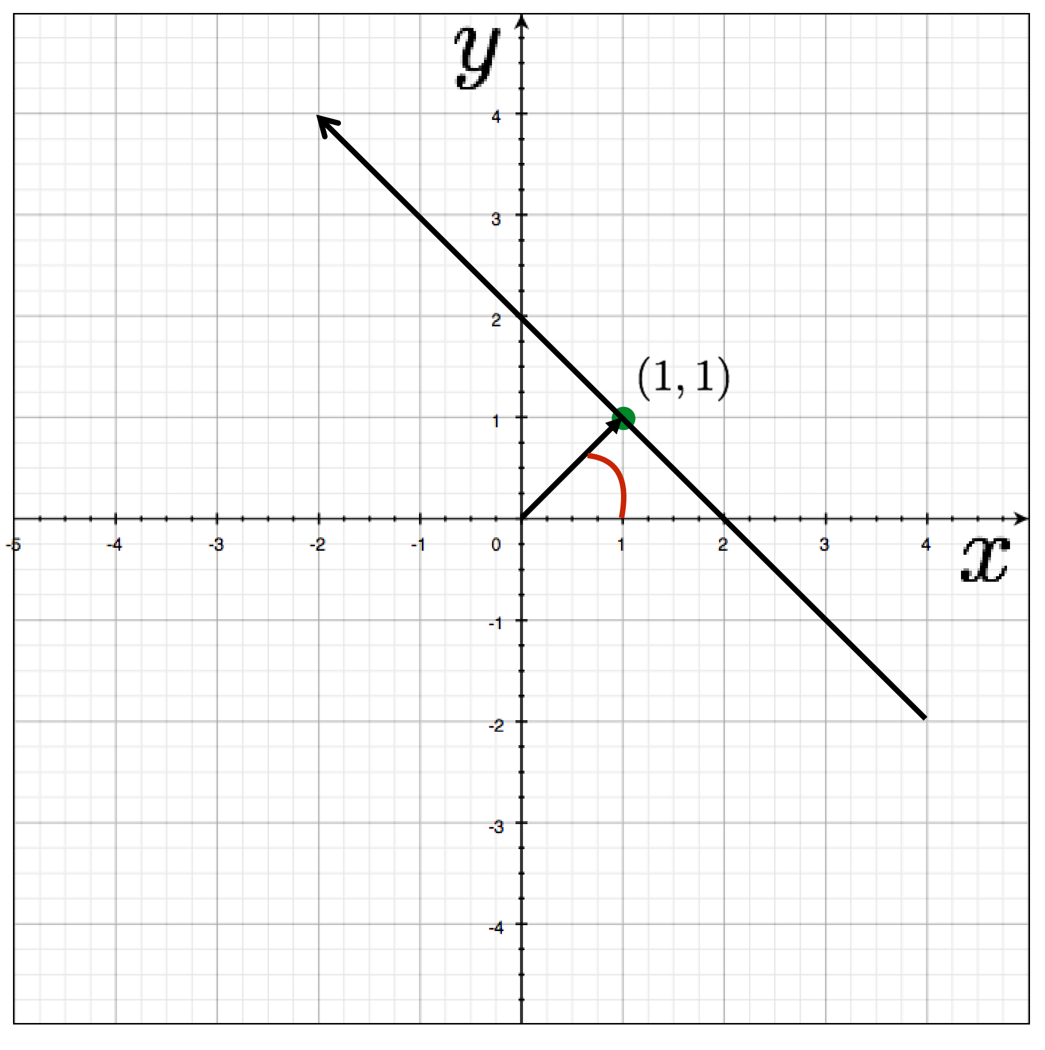
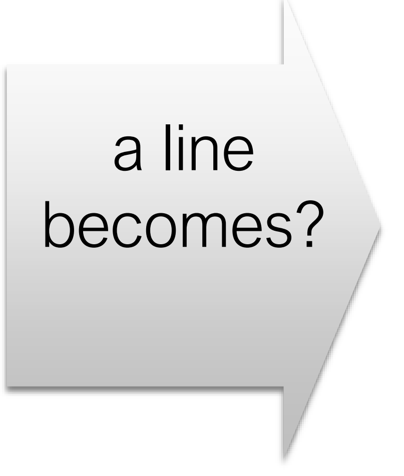
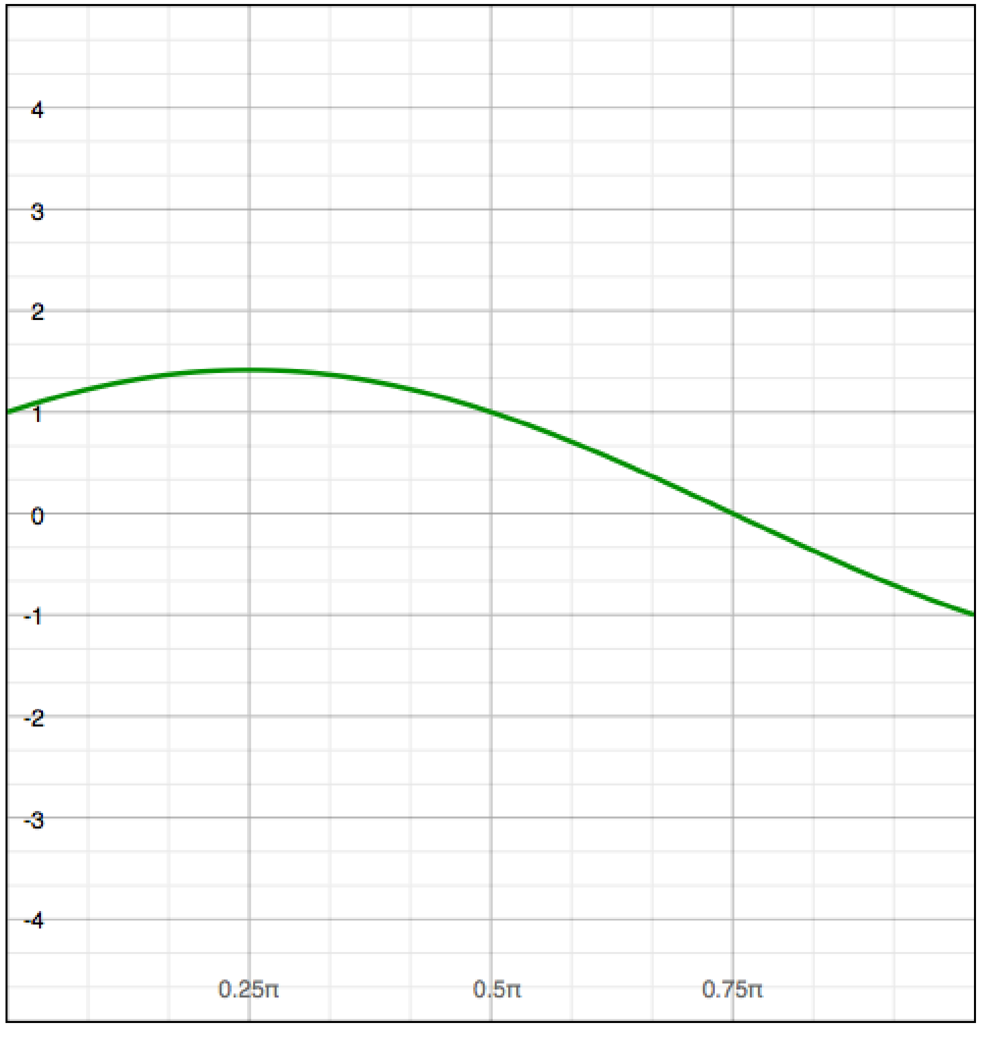


Image space





Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

$$x\cos\theta + y\sin\theta = \rho$$

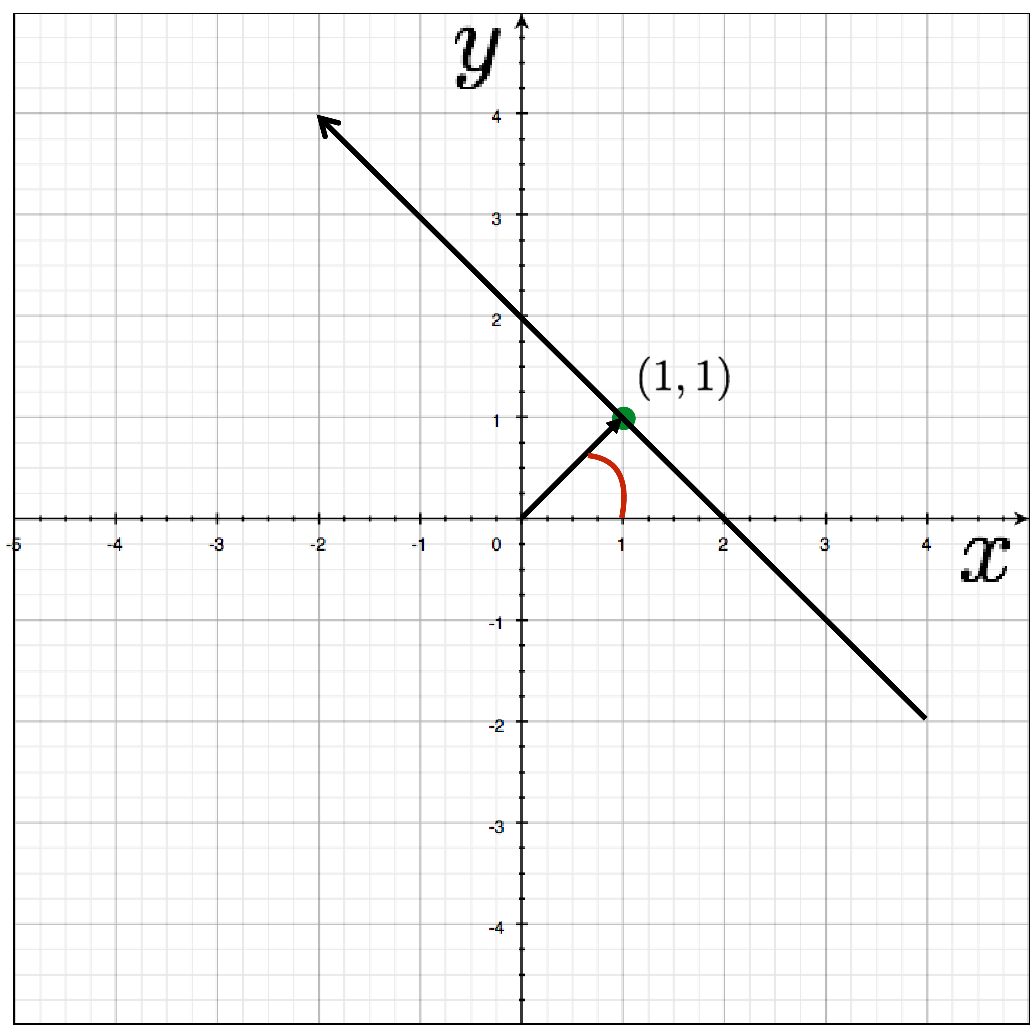
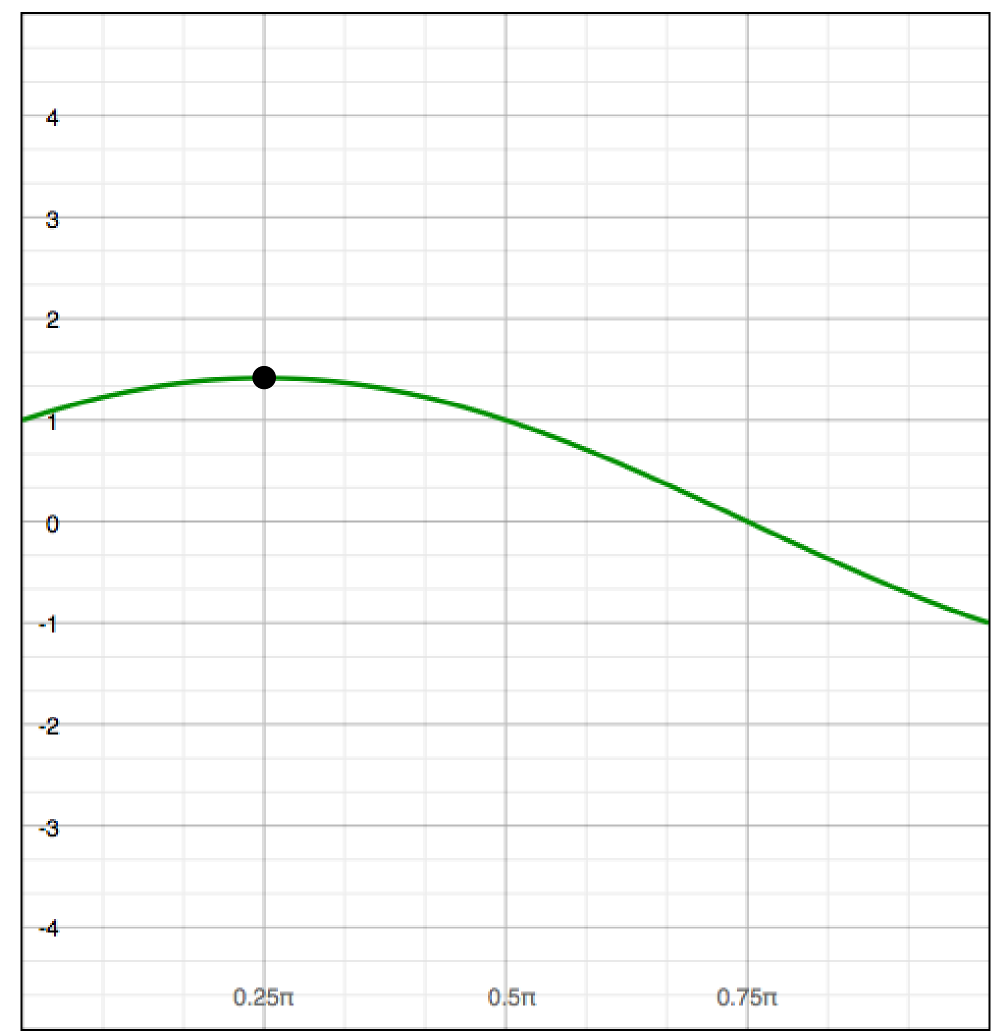


Image space

a line becomes a point



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

$$x\cos\theta + y\sin\theta = \rho$$

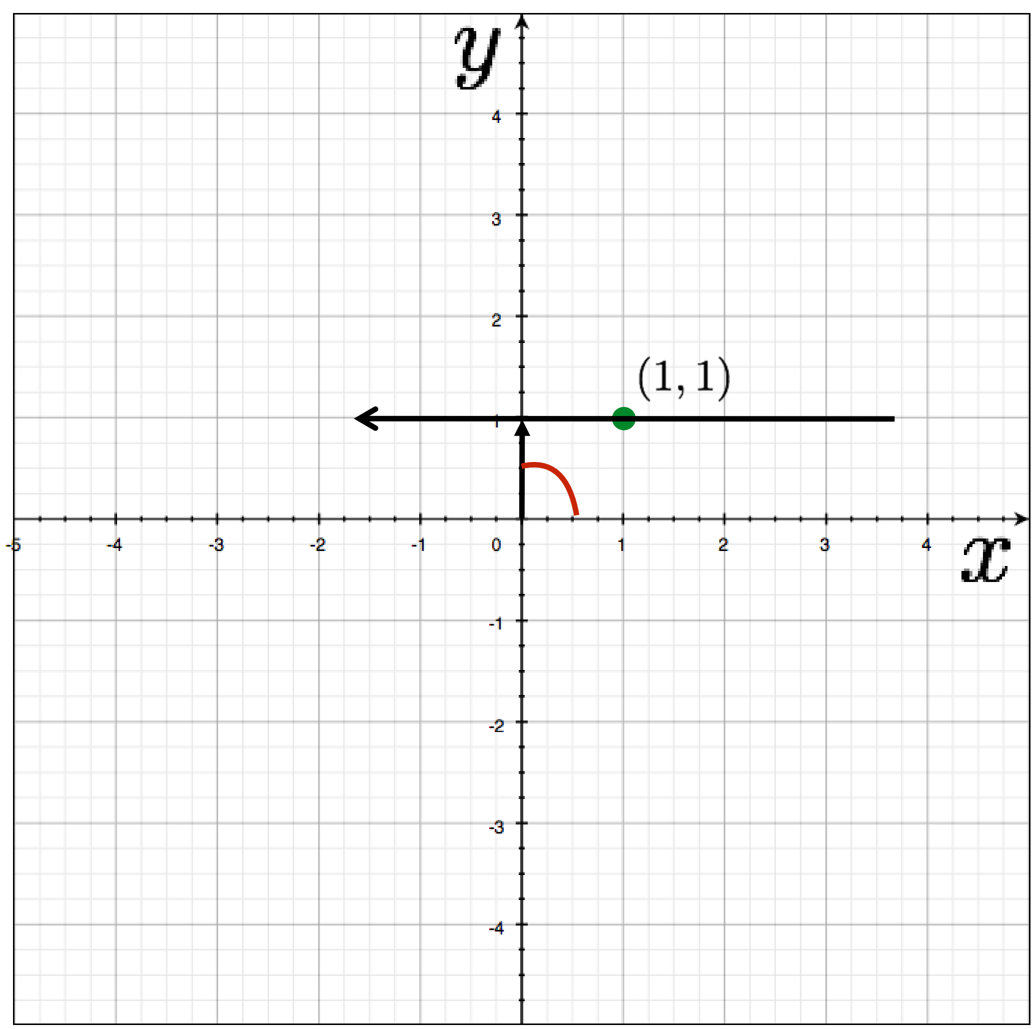
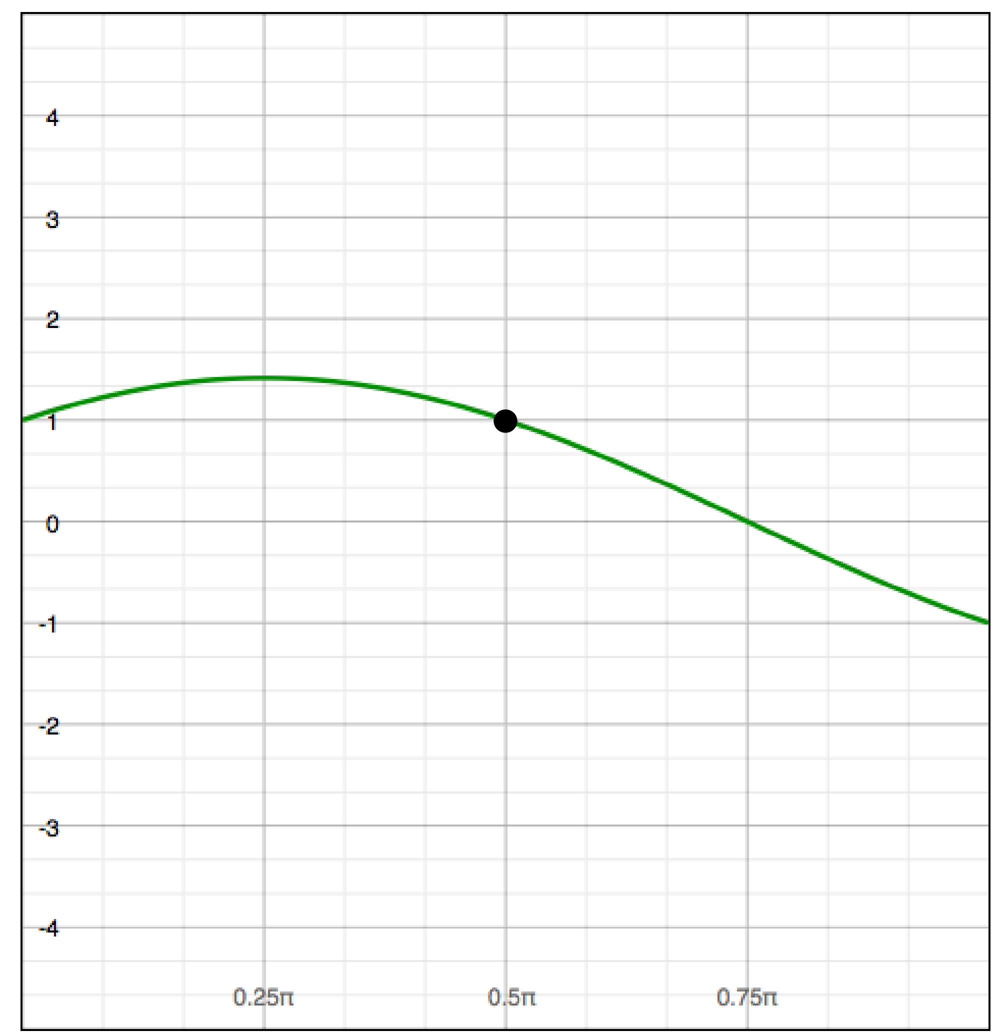


Image space

a line becomes a point



Parameter space

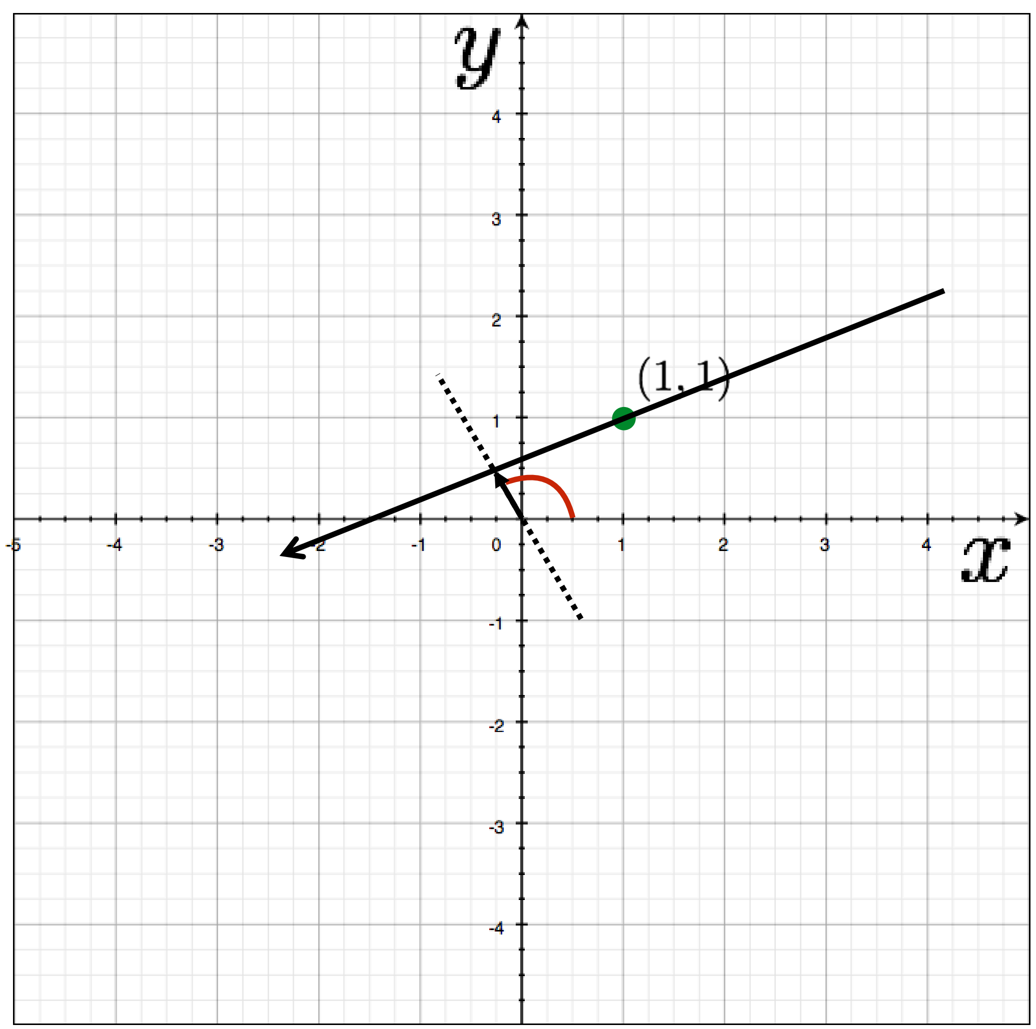
Image and parameter space

variables

$$y = mx + b$$

parameters

$$x \cos\theta + y \sin\theta = \rho$$



a line becomes a point

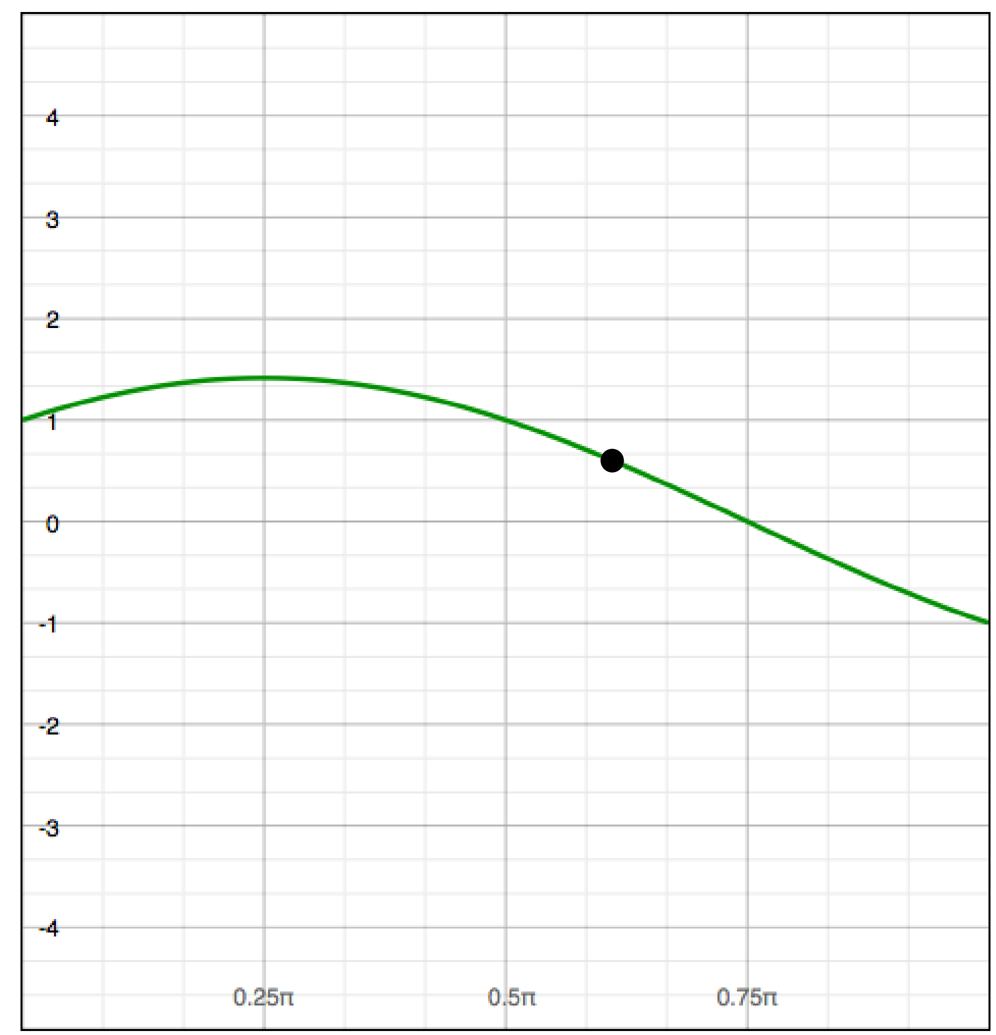


Image space

Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

$$x \cos\theta + y \sin\theta = \rho$$

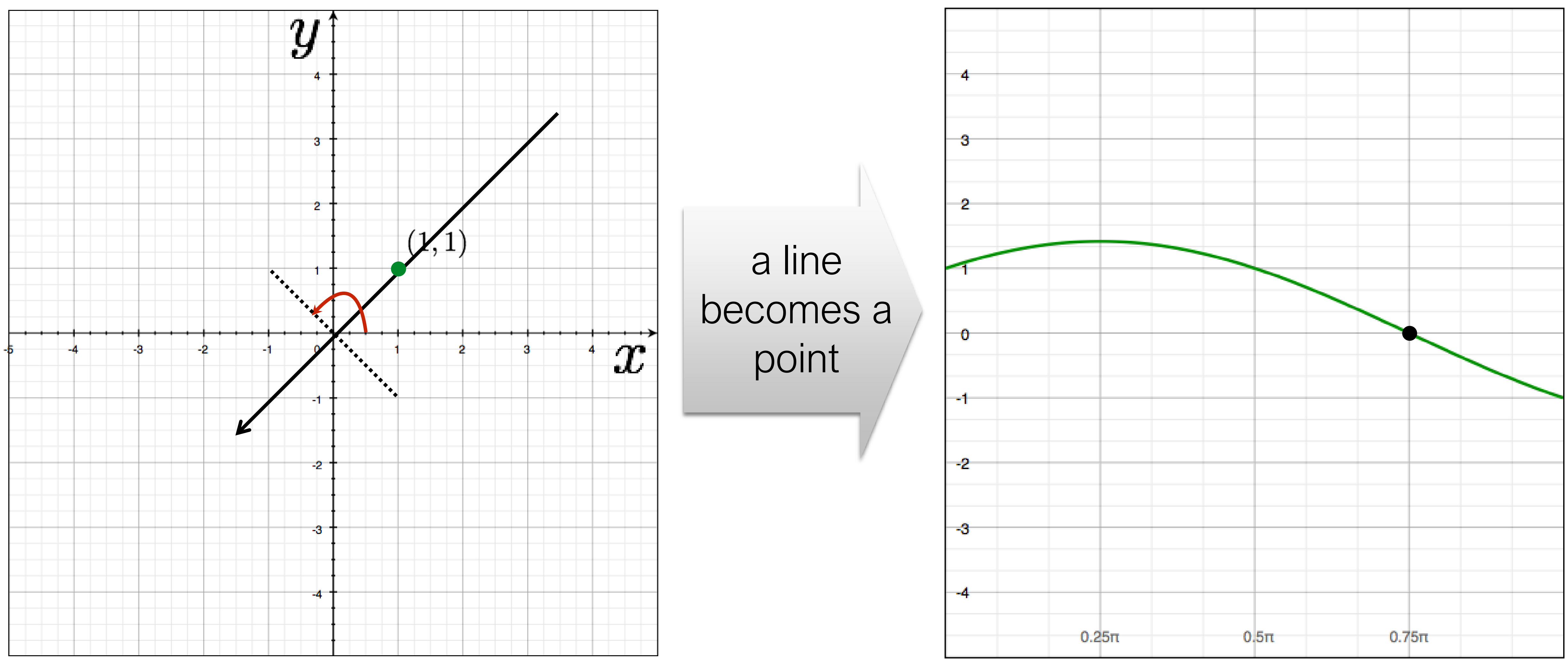


Image space

Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

$$x \cos\theta + y \sin\theta = \rho$$

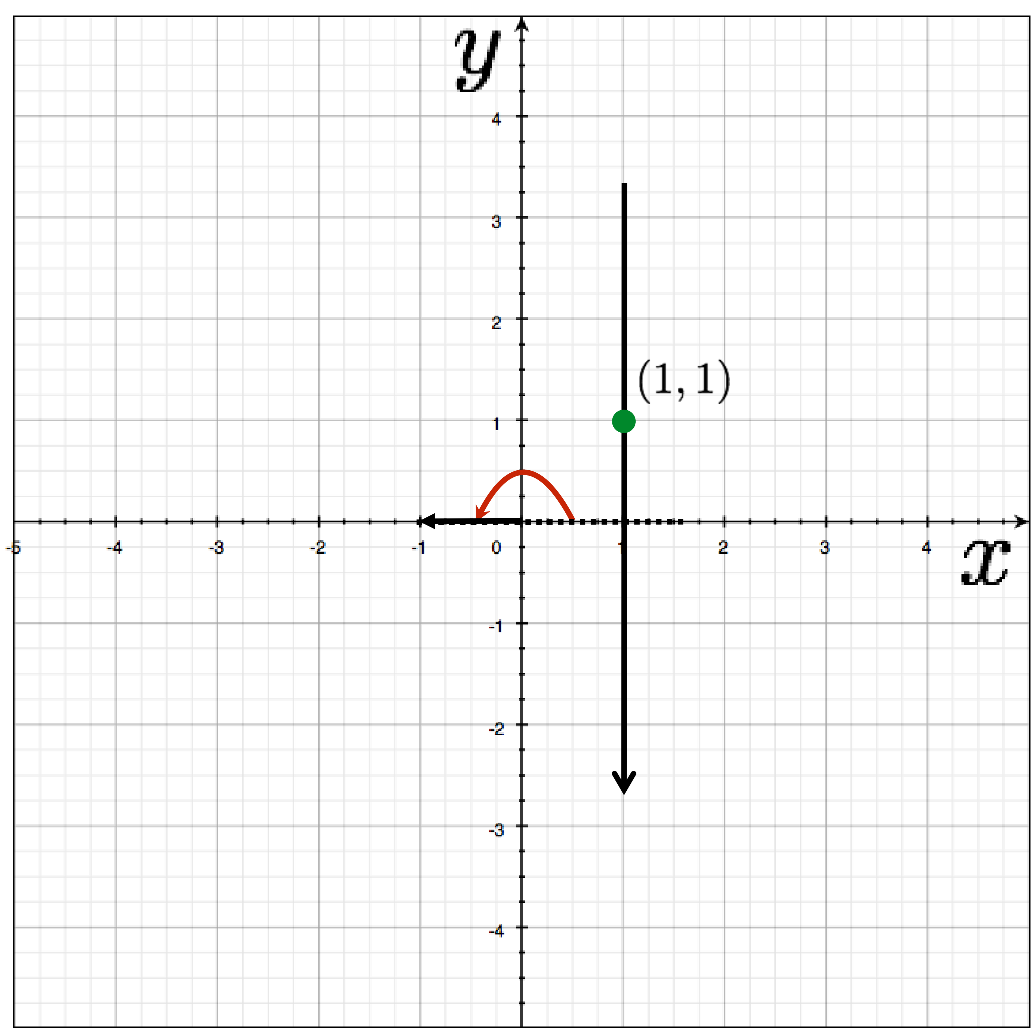
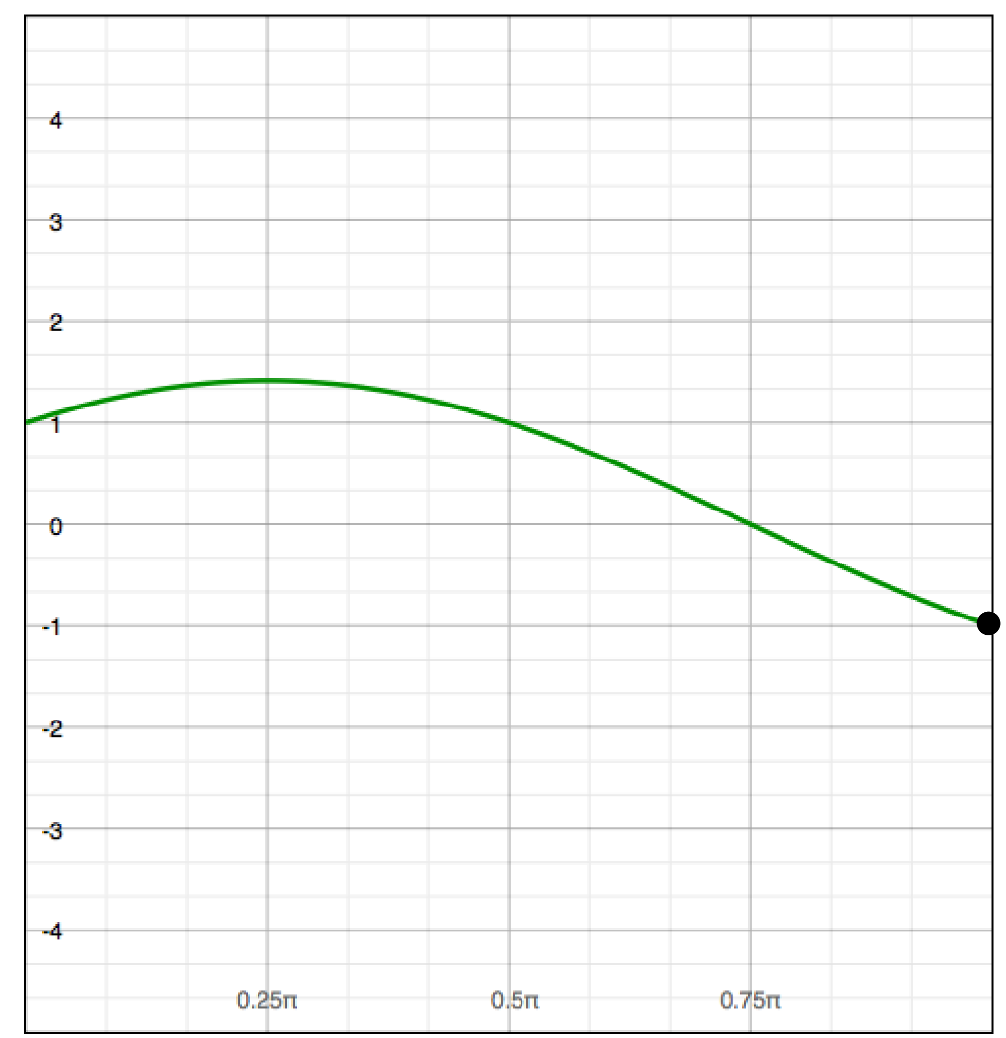


Image space

a line becomes a point



Parameter space

Image and parameter space

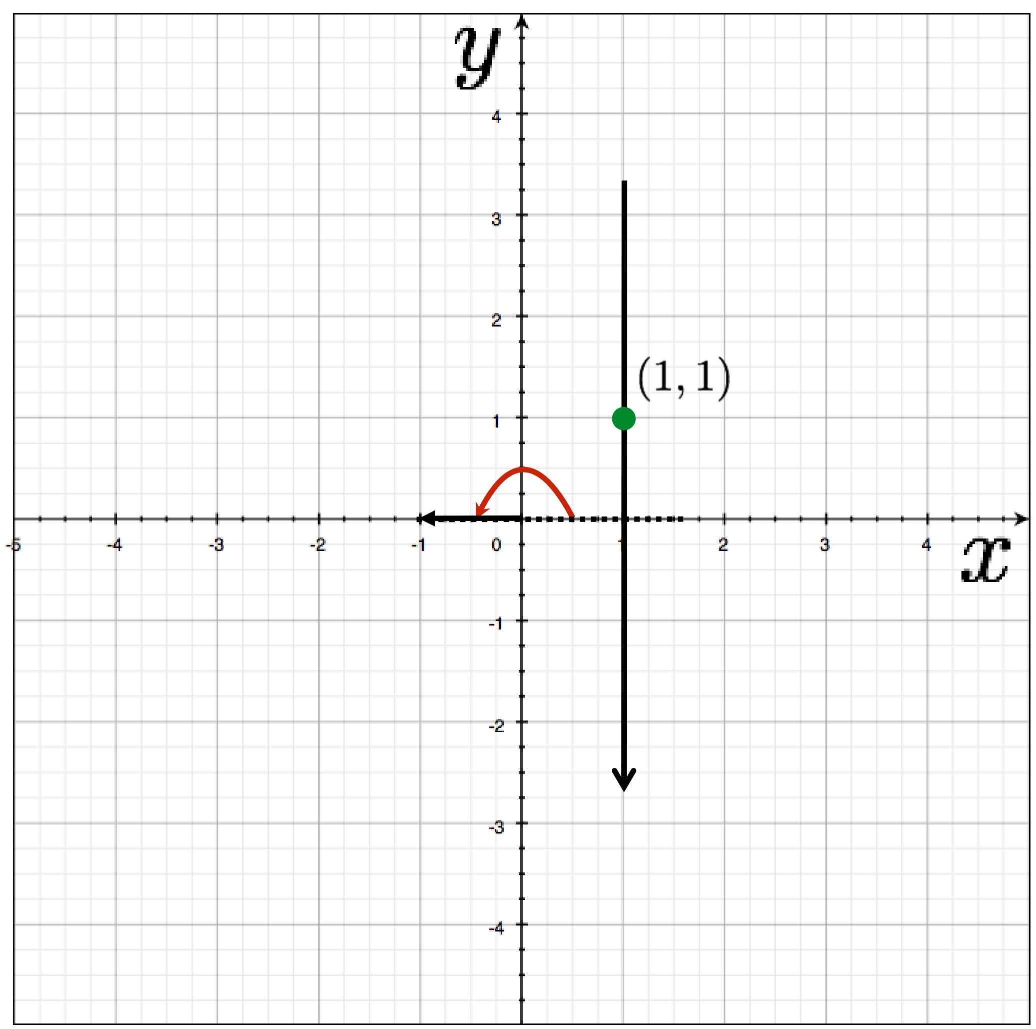
variables

$$y = mx + b$$

parameters

$$x \cos\theta + y \sin\theta = \rho$$

Wait …why is rho negative?



a line becomes a point

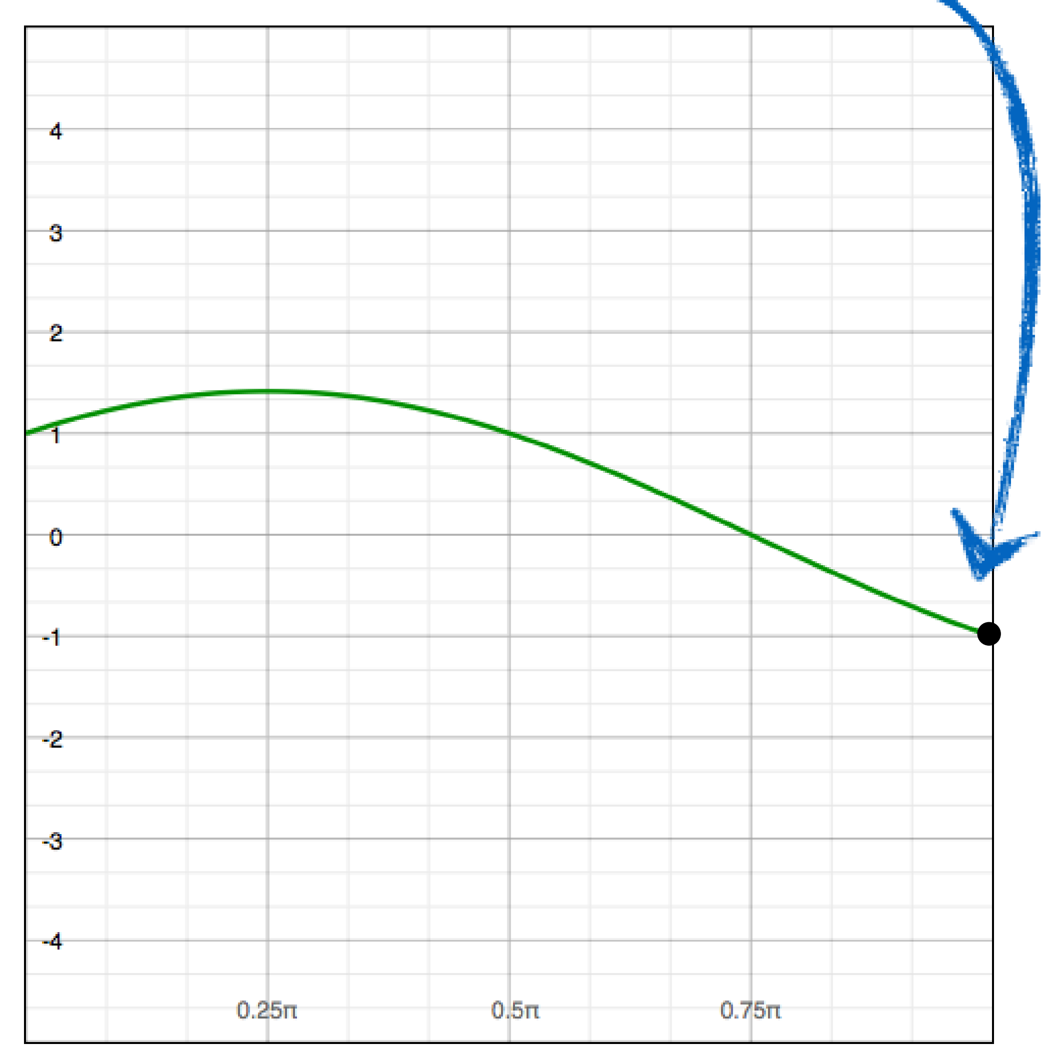


Image space

Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

$$x \cos\theta + y \sin\theta = \rho$$

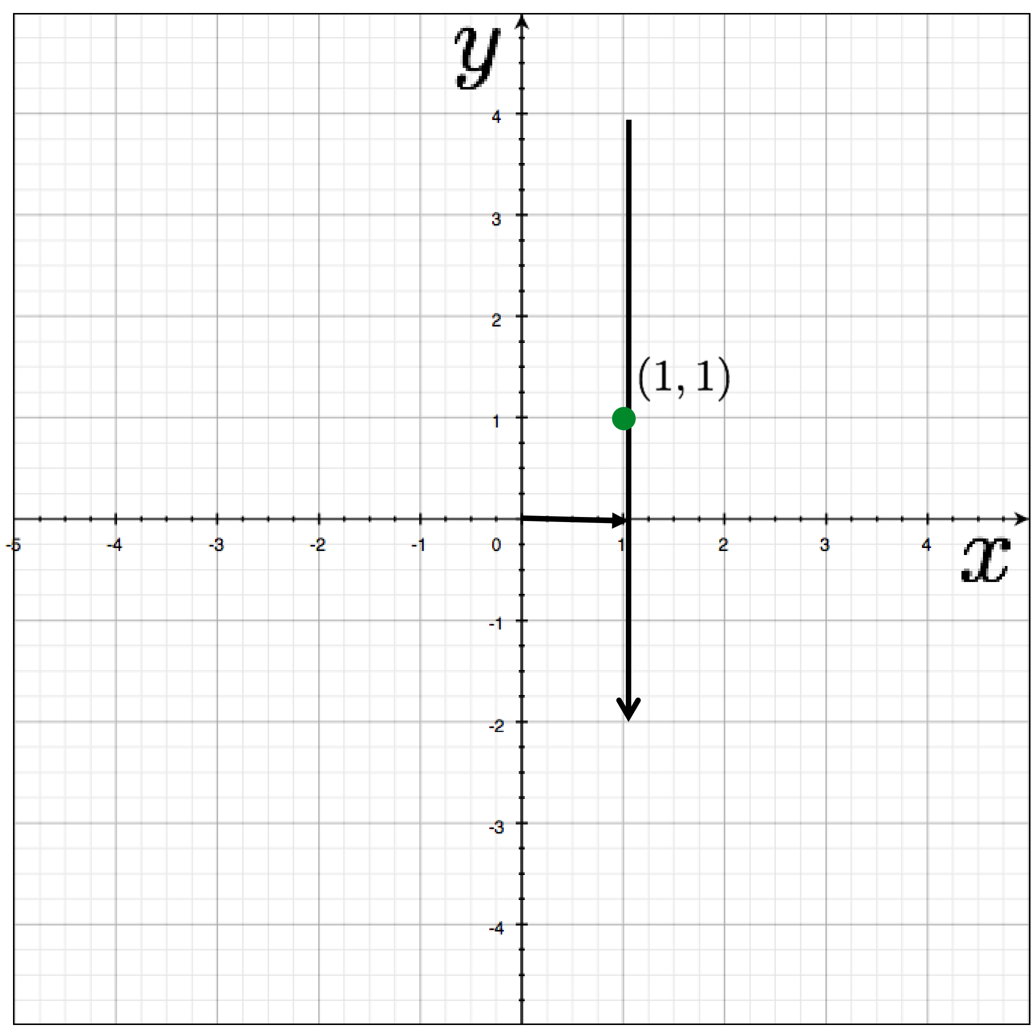
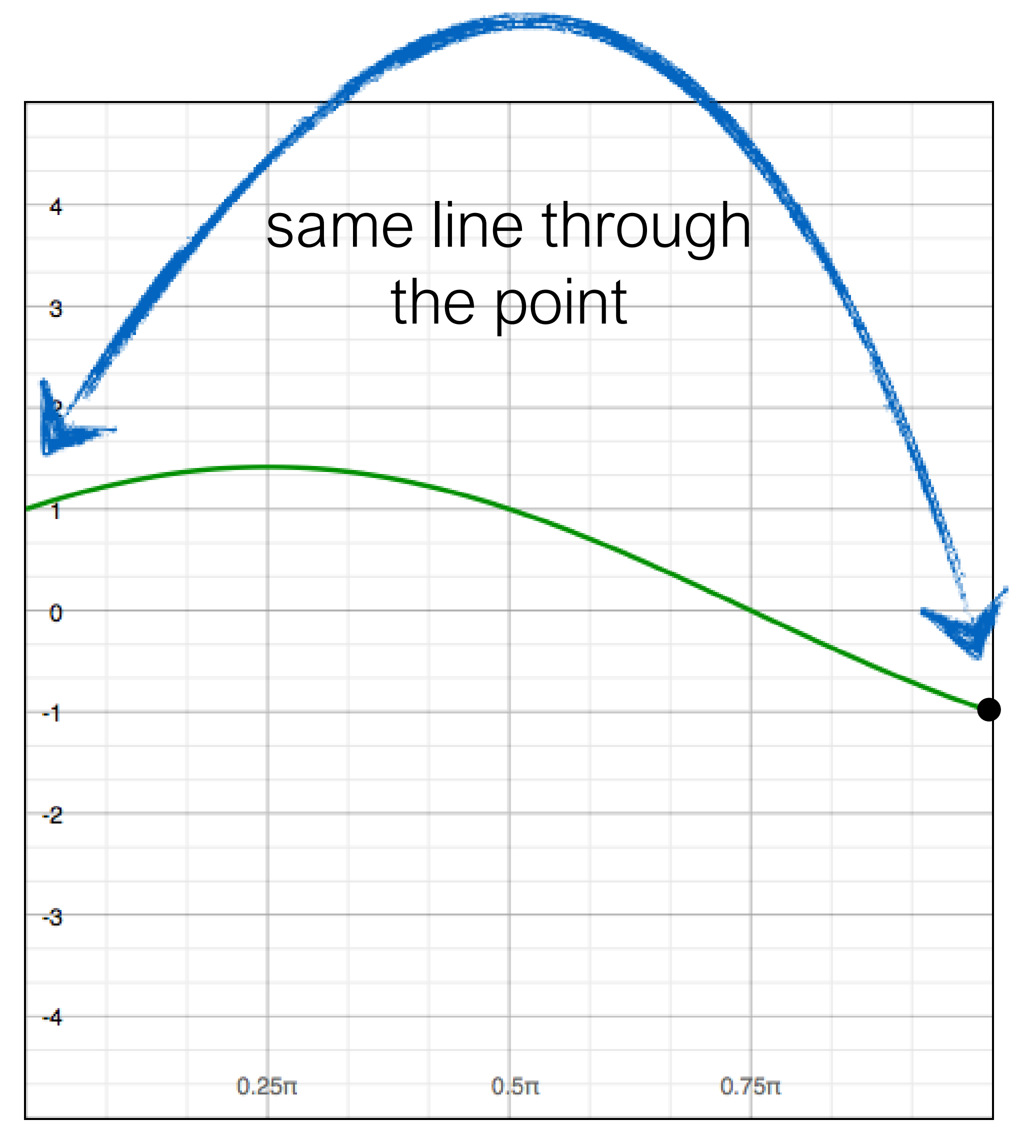


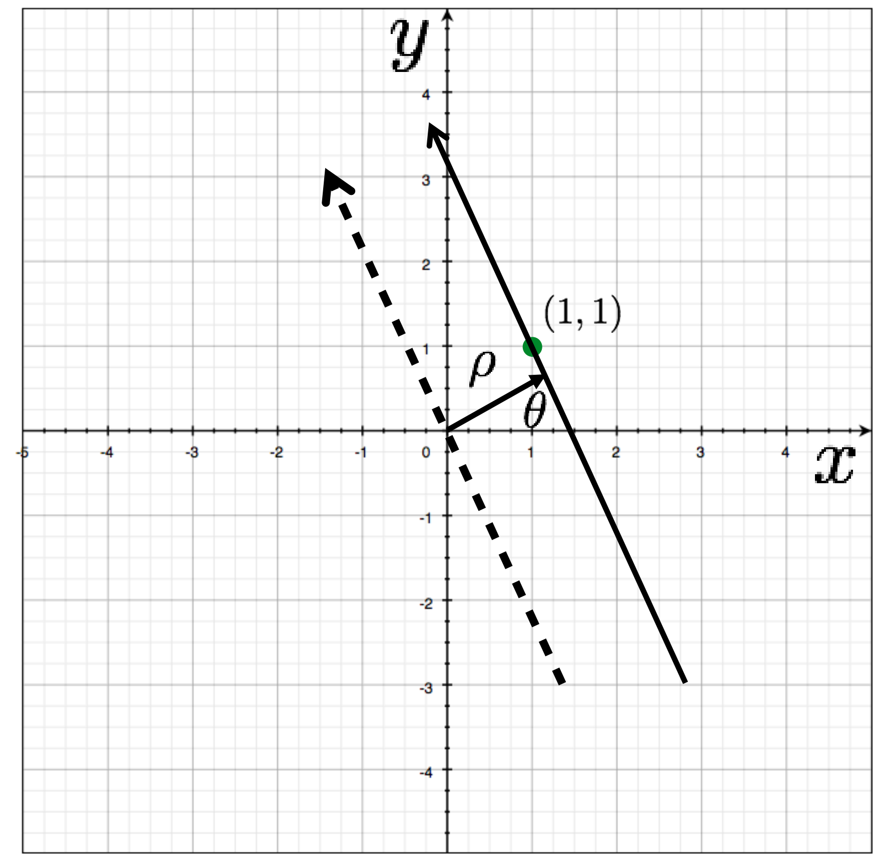
Image space

a line becomes a point



Parameter space

There are two ways to write the same line:

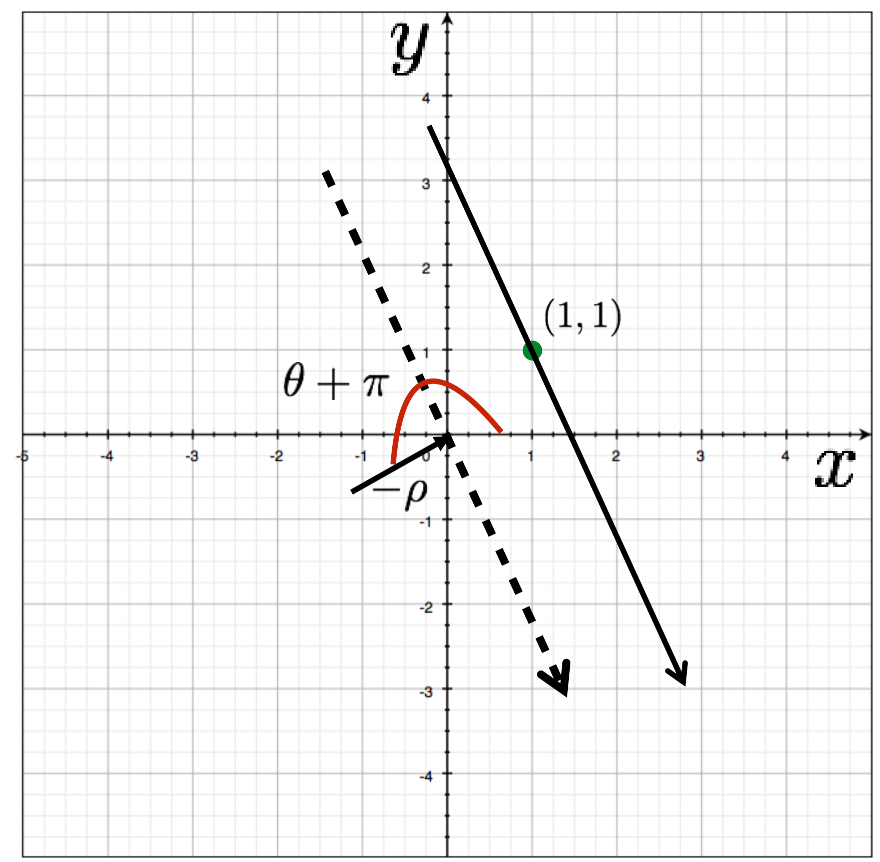


Positive rho version:

$$x \cos\theta + y \sin\theta = \rho$$

Negative rho version:

$$x \cos(\theta + \pi) + y \sin(\theta + \pi) = -\rho$$



Recall:

$$\sin(\theta) = -\sin(\theta + \pi)$$
$$\cos(\theta) = -\cos(\theta + \pi)$$

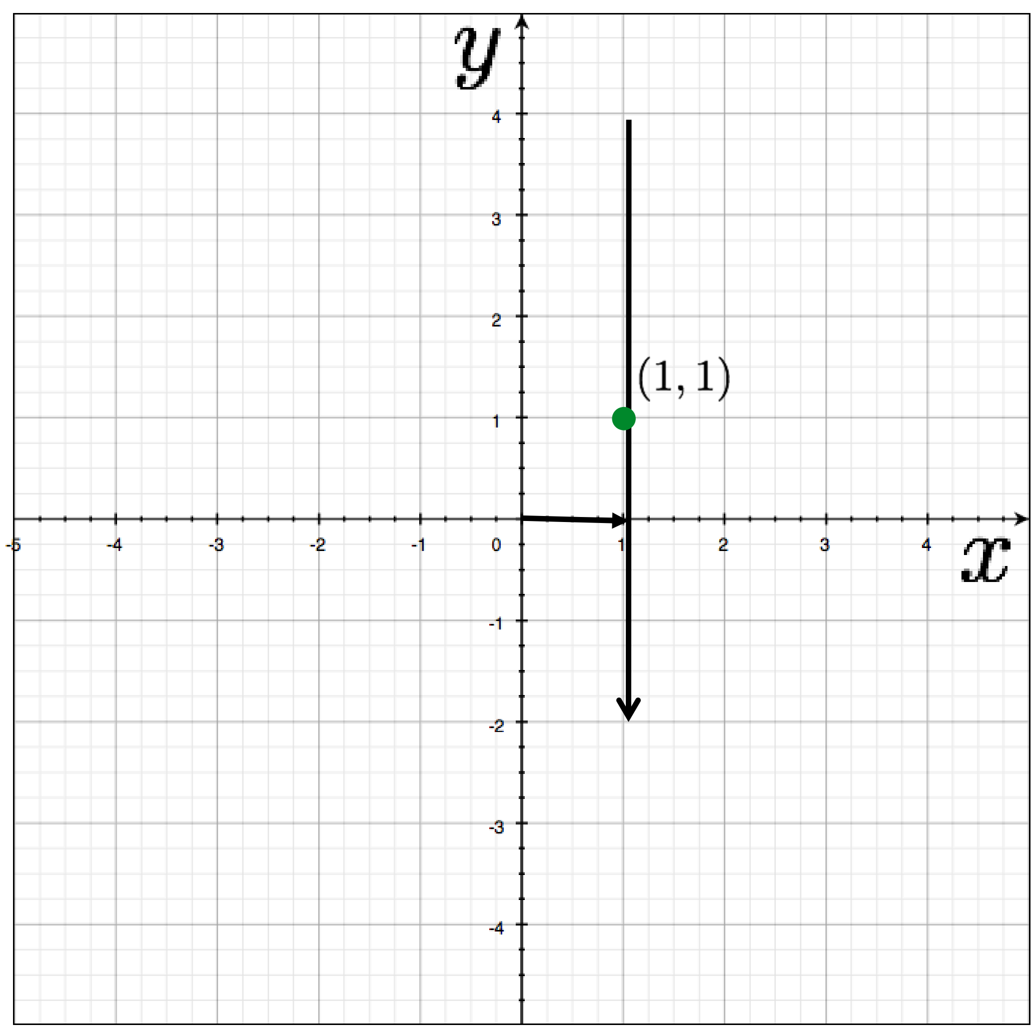
Image and parameter space

variables

$$y = mx + b$$

parameters

$$x\cos\theta + y\sin\theta = \rho$$



a line becomes a point

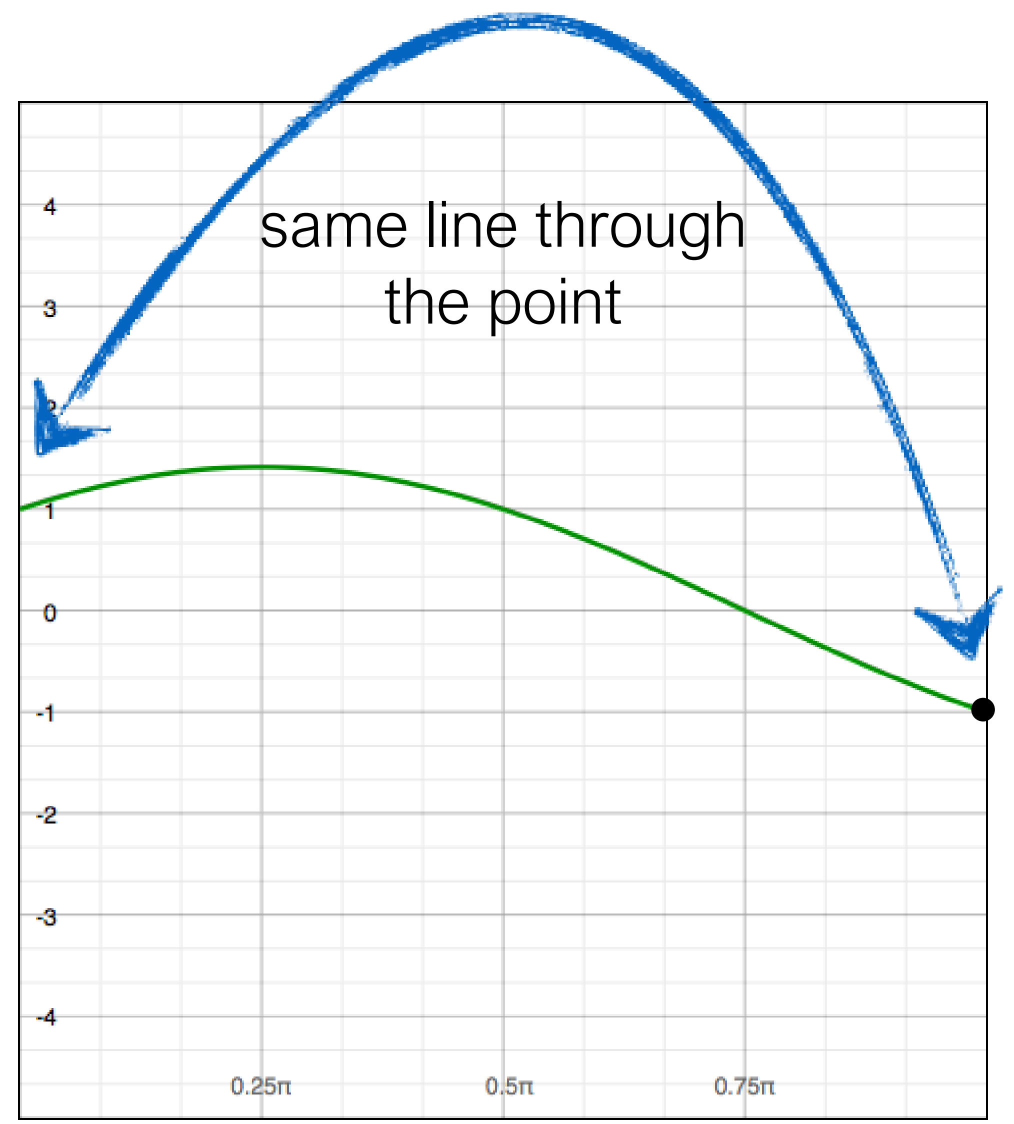


Image space

Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

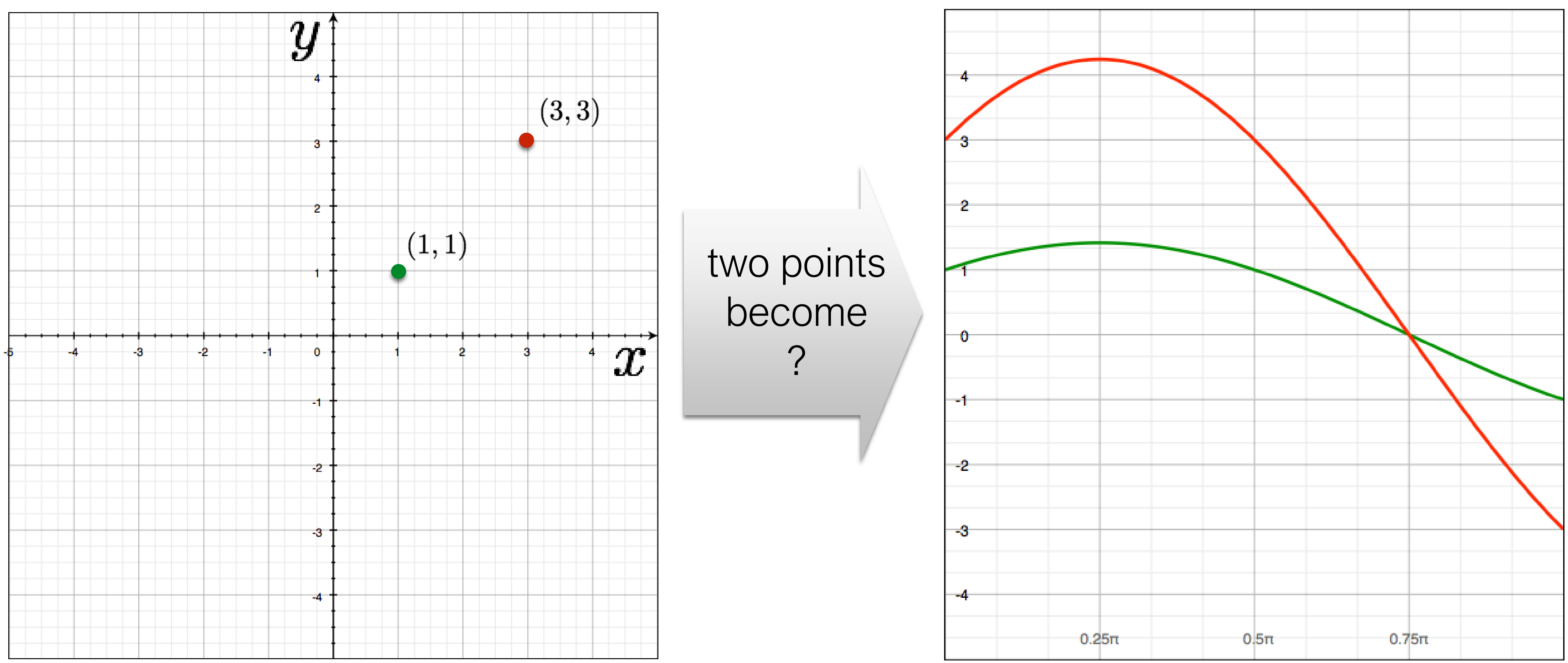


Image space

Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

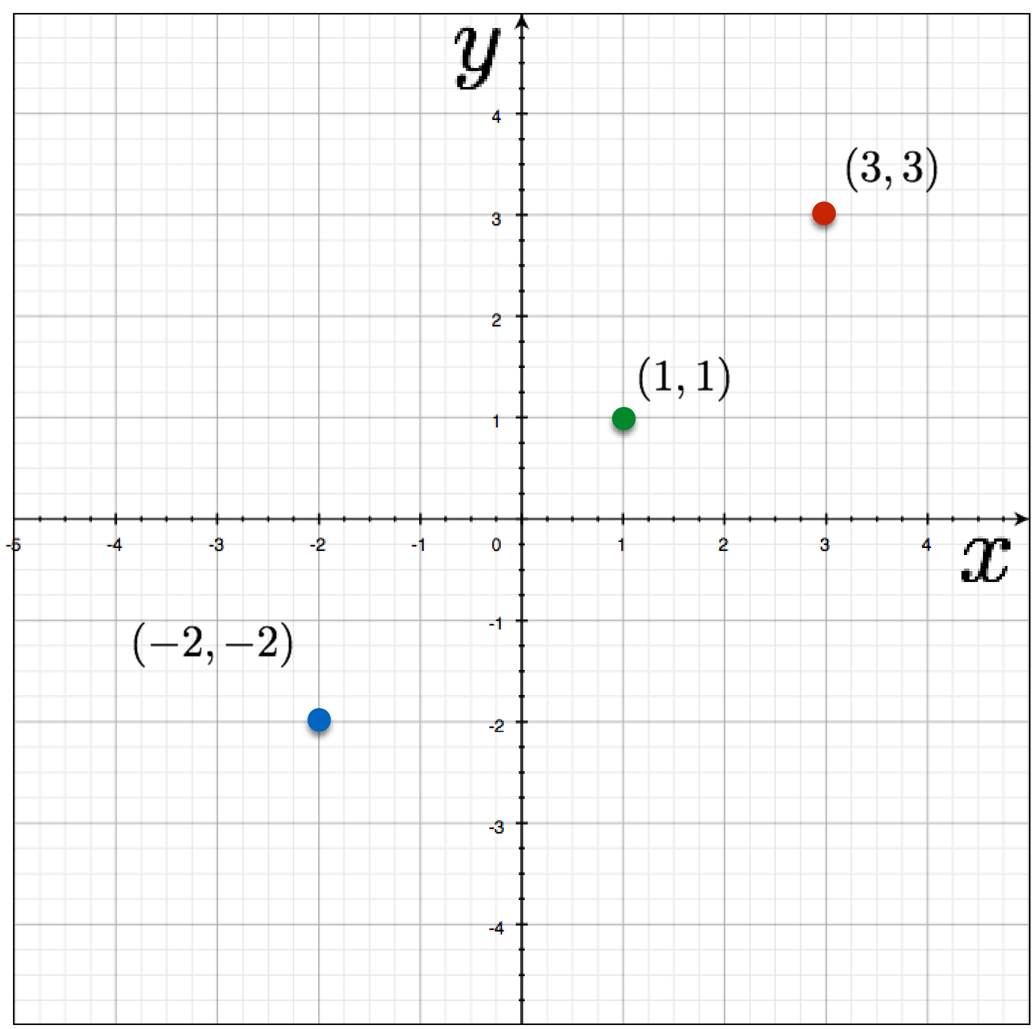
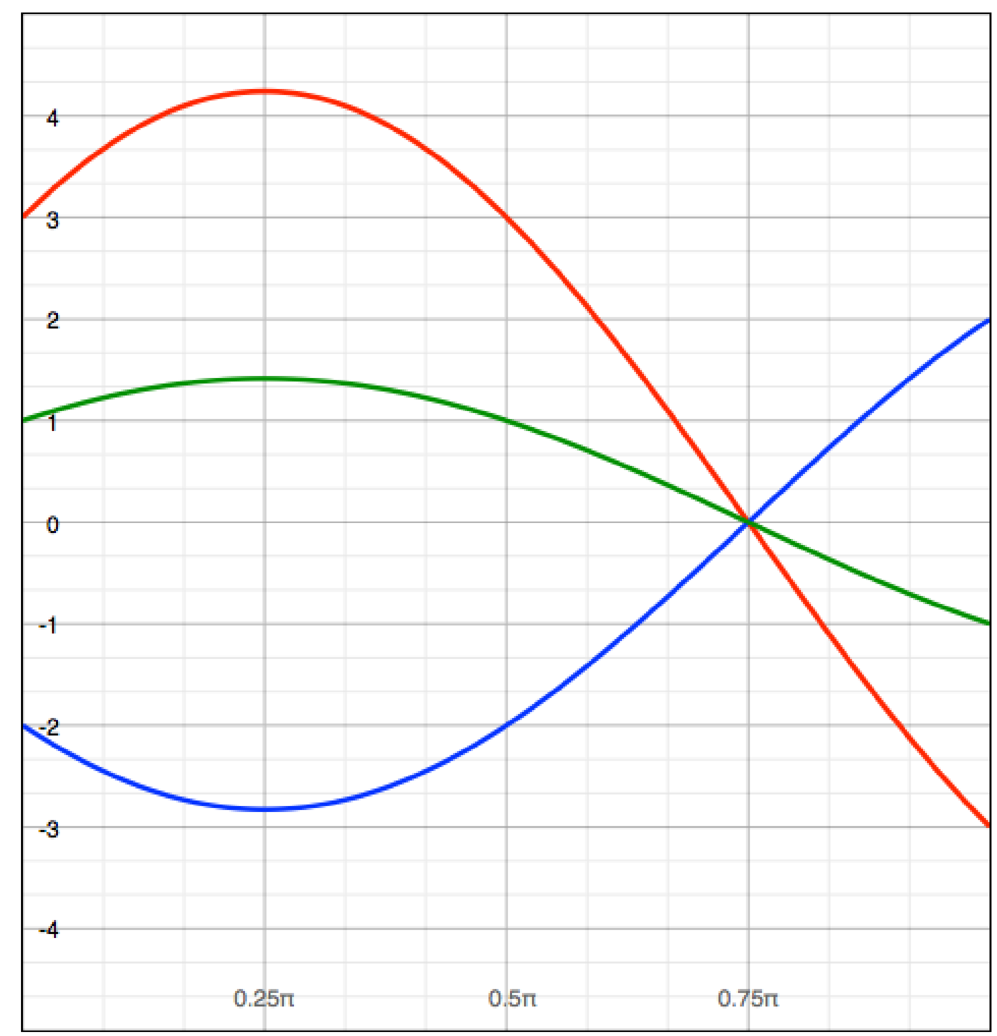


Image space

three points become ?



Parameter space

Image and parameter space

variables

$$y = mx + b$$

parameters

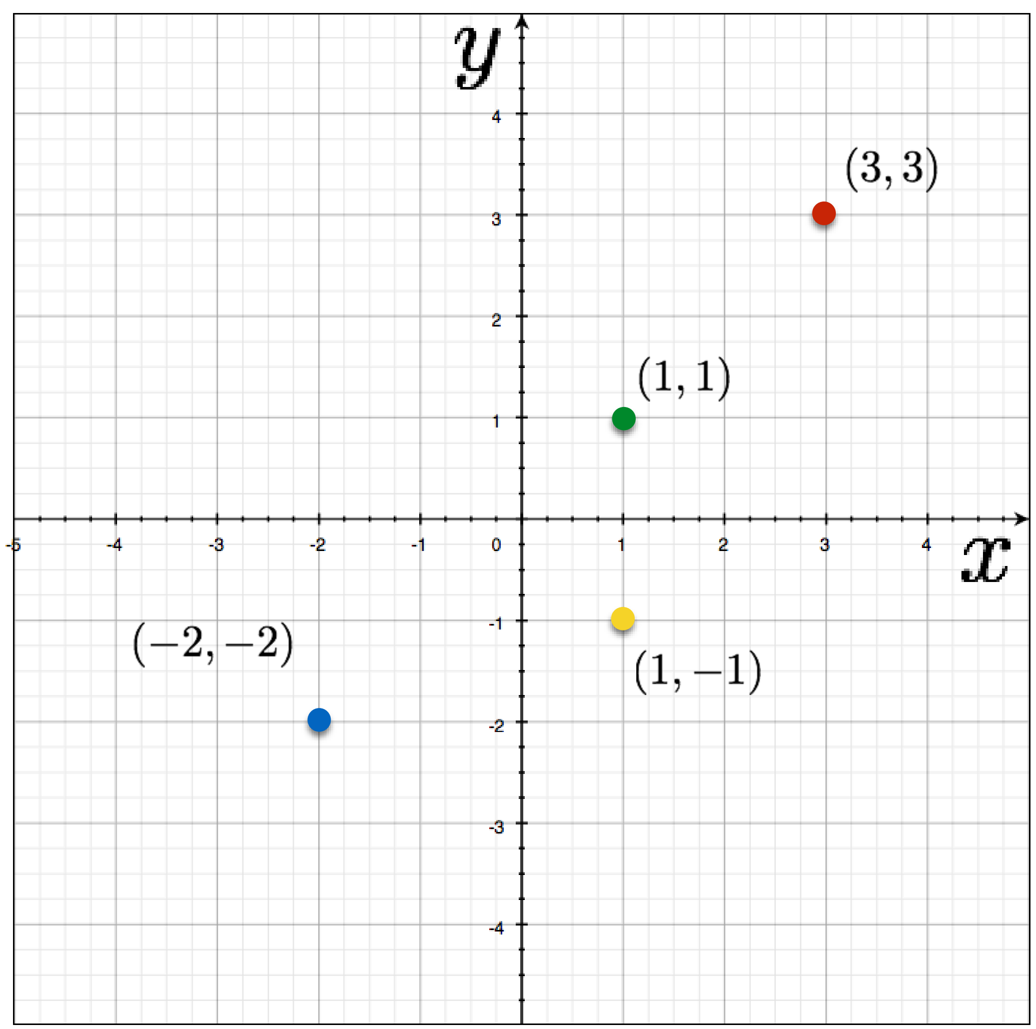
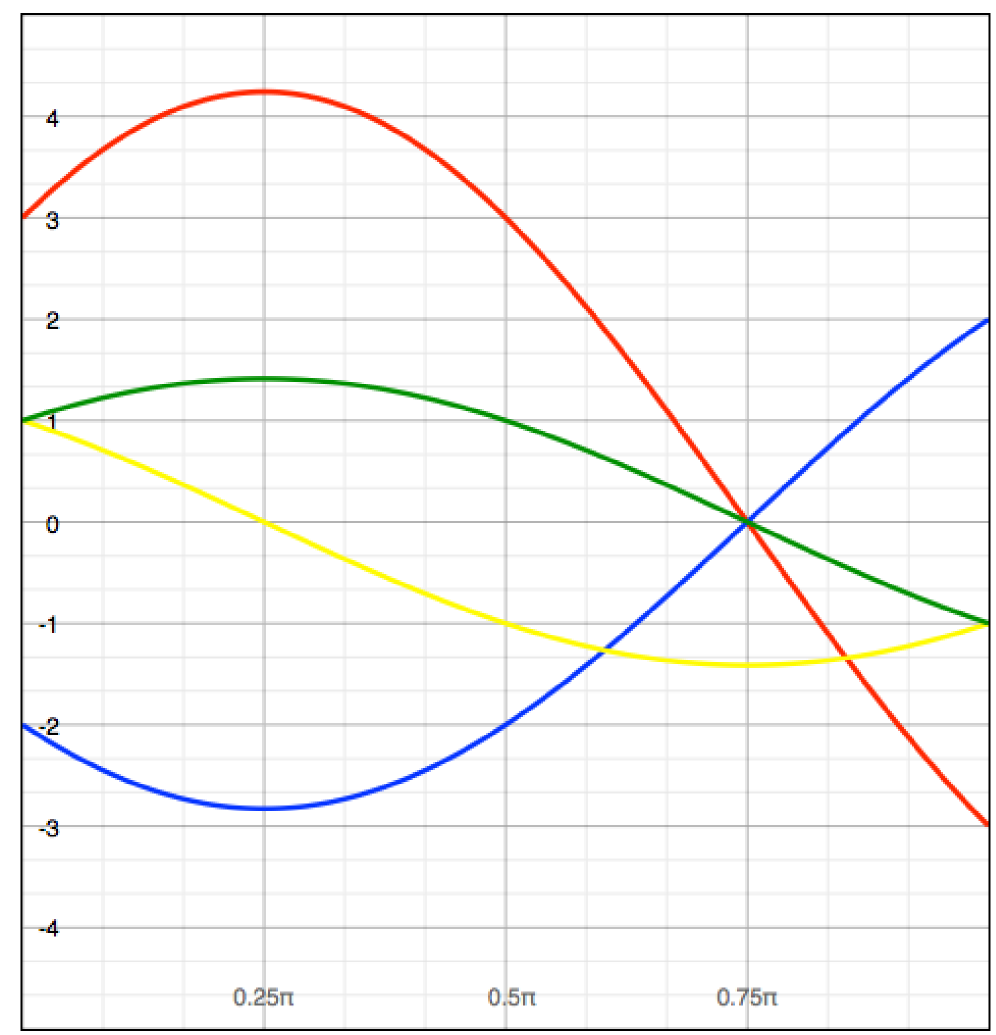


Image space

four points become ?



Parameter space

Implementation

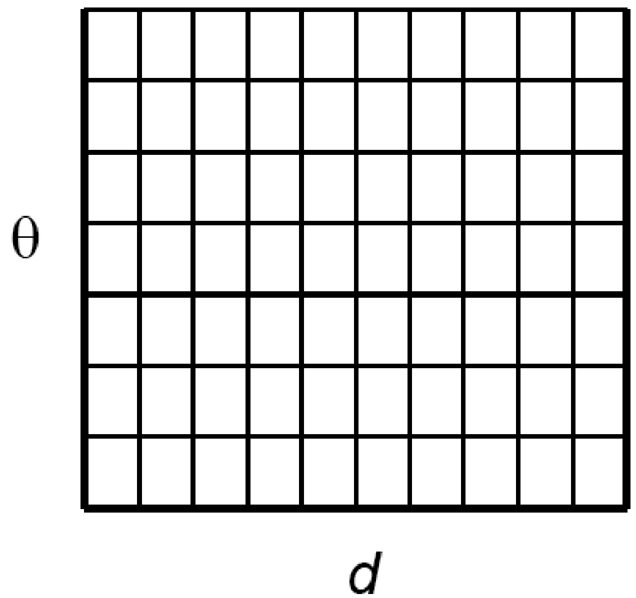
```
1. Initialize accumulator H to all zeros

2. For each edge point (x,y) in the image
       For θ = 0 to 180
           ρ = x cos θ + y sin θ
           H(θ, ρ) = H(θ, ρ) + 1
       end
   end

3. Find the value(s) of (θ, ρ) where H(θ, ρ)
   is a local maximum

4. The detected line in the image is given by
    ρ = x cos θ + y sin θ
```

H: accumulator array (votes)



NOTE: Watch your coordinates. Image origin is top left!

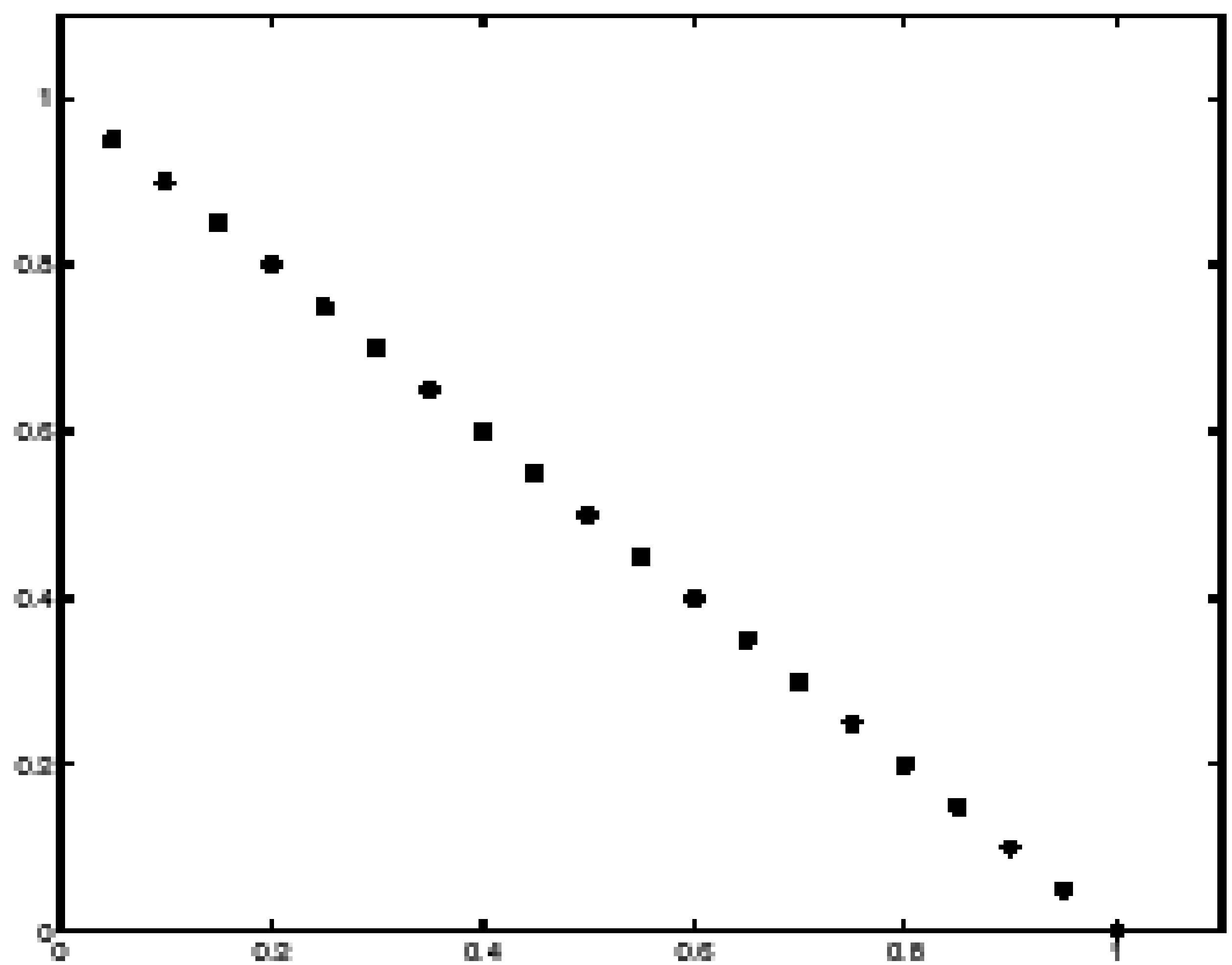
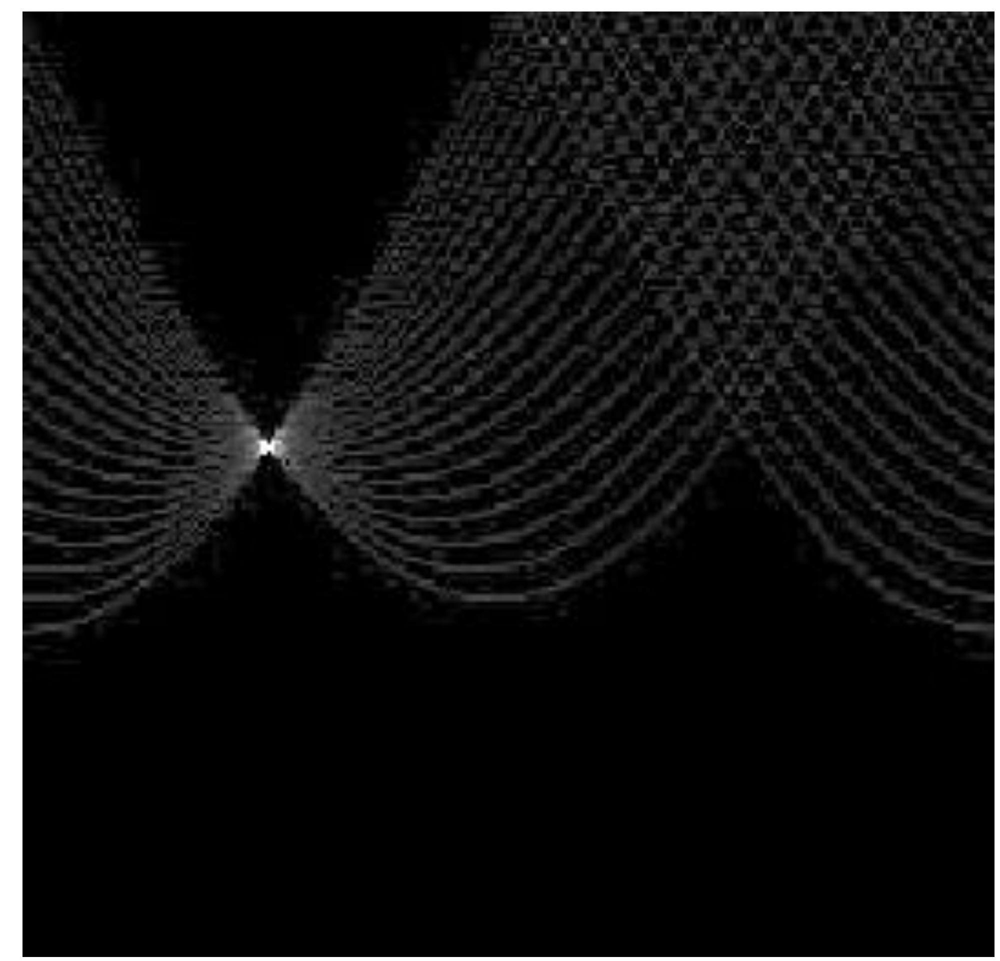


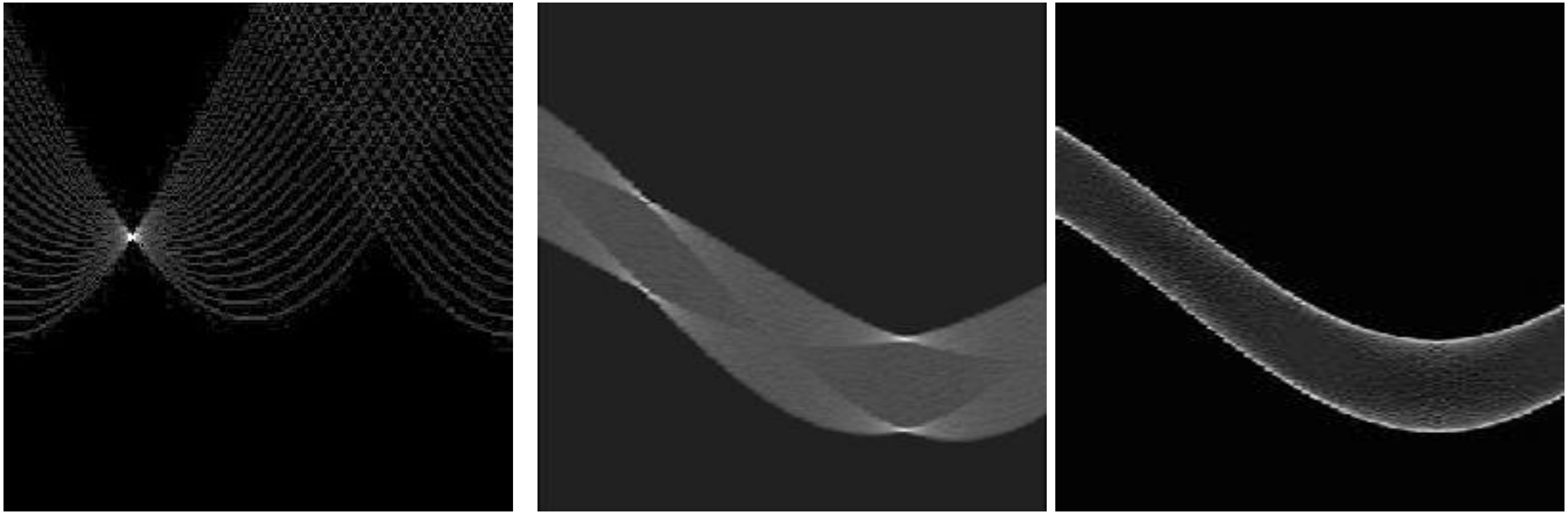
Image space



Votes

Basic shapes

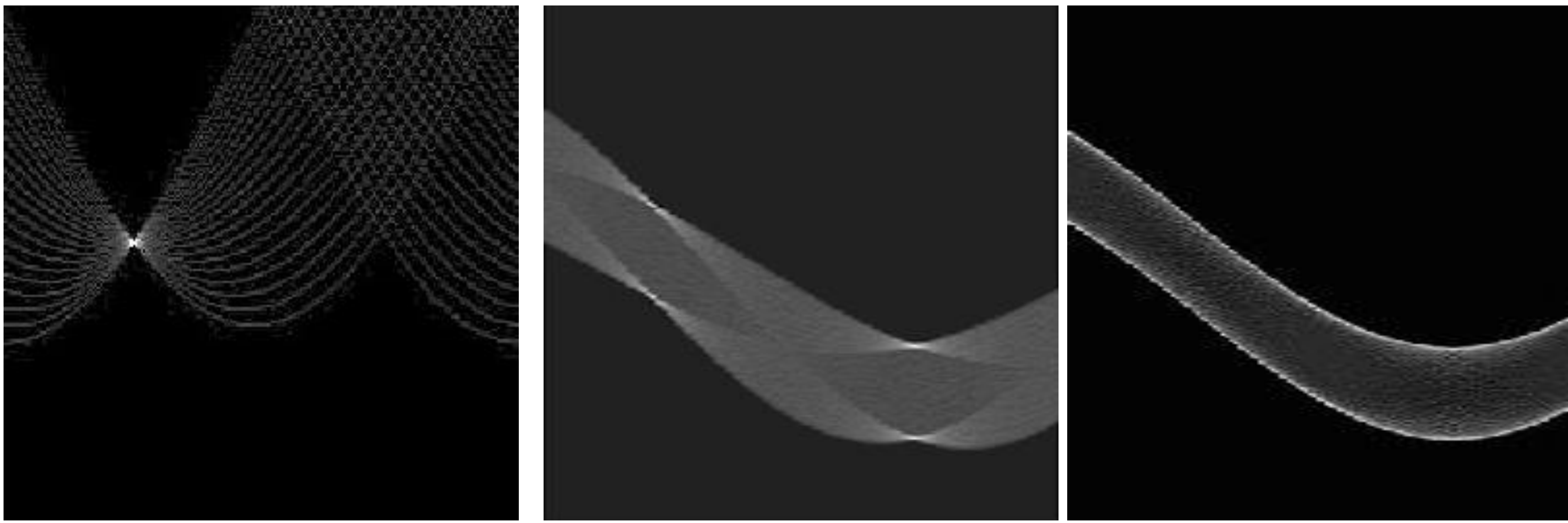
(in parameter space)



can you guess the shape?

Basic shapes

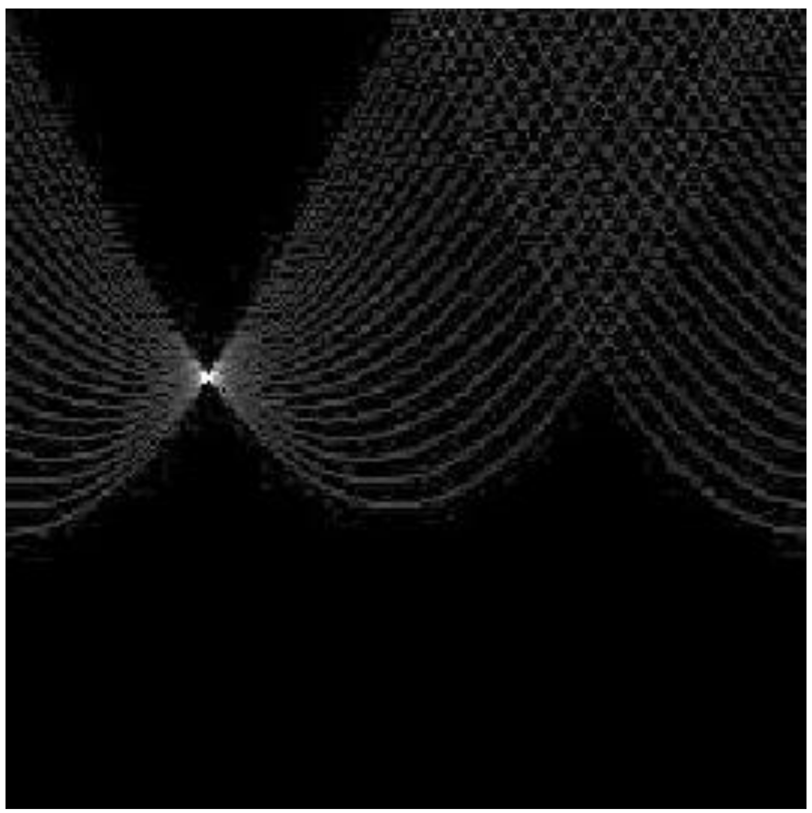
(in parameter space)



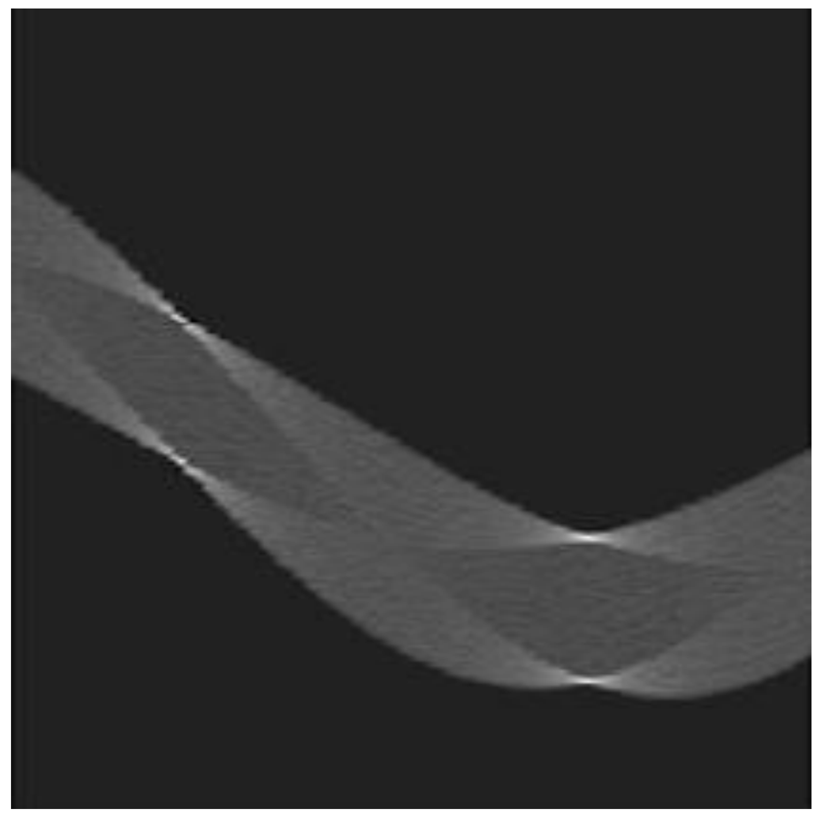
line

Basic shapes

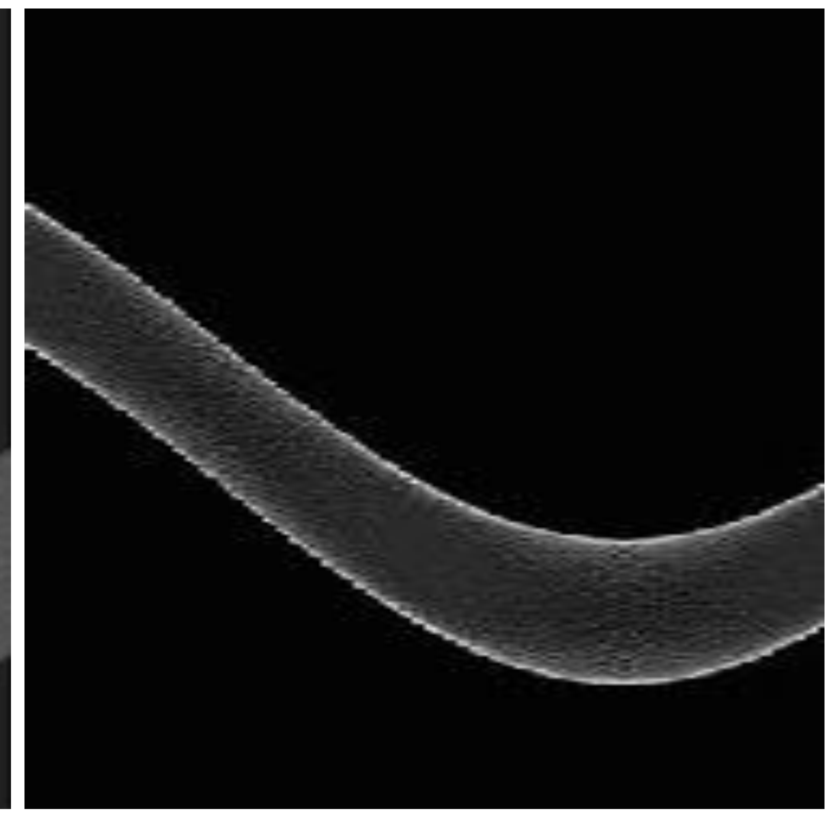
(in parameter space)



line

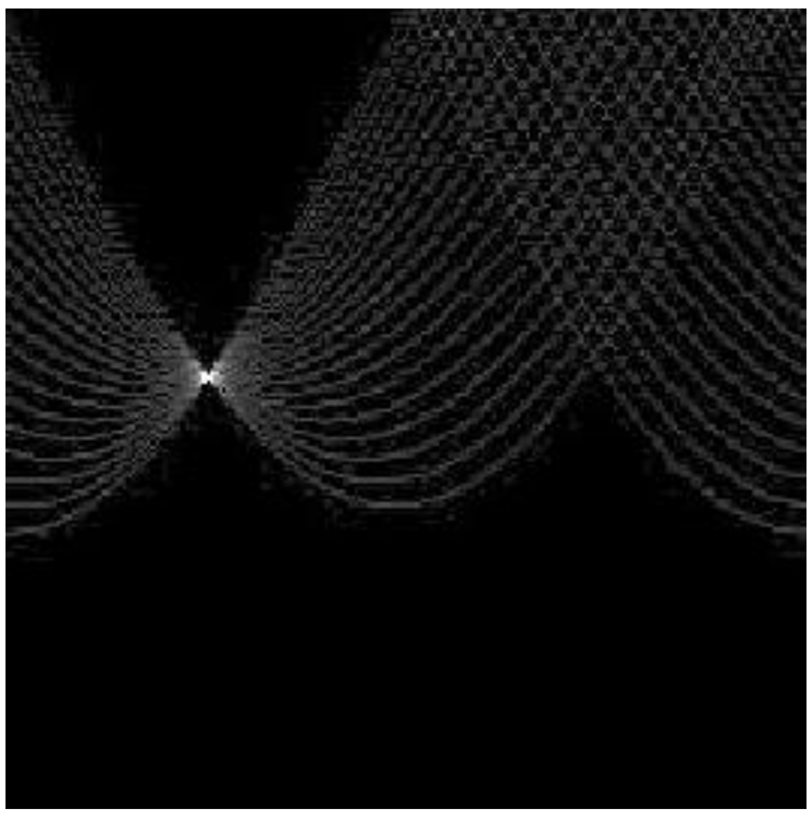


rectangle

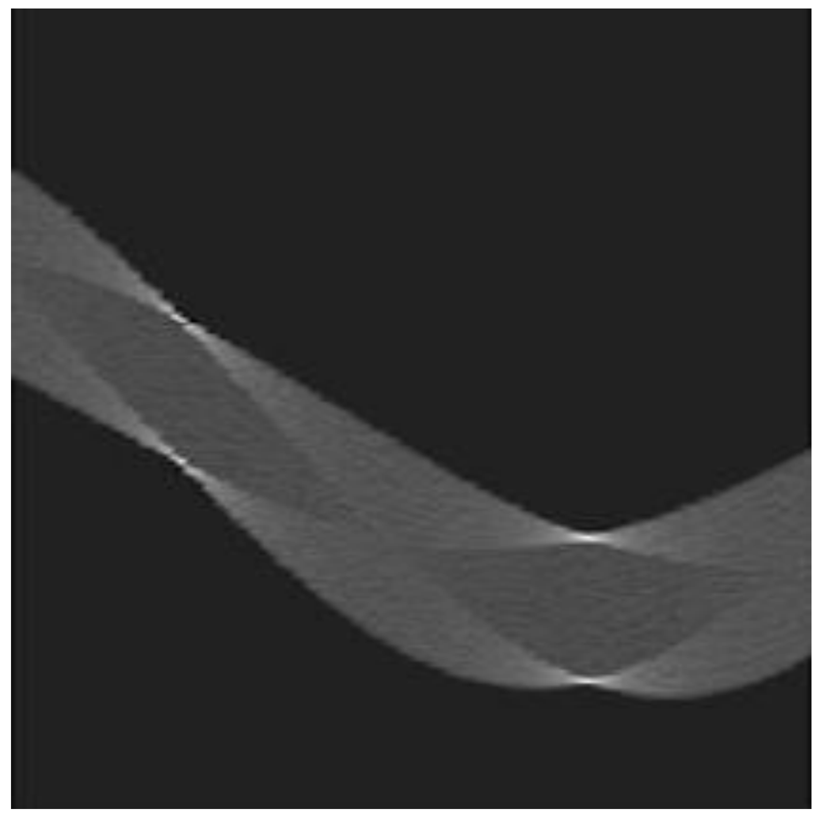


Basic shapes

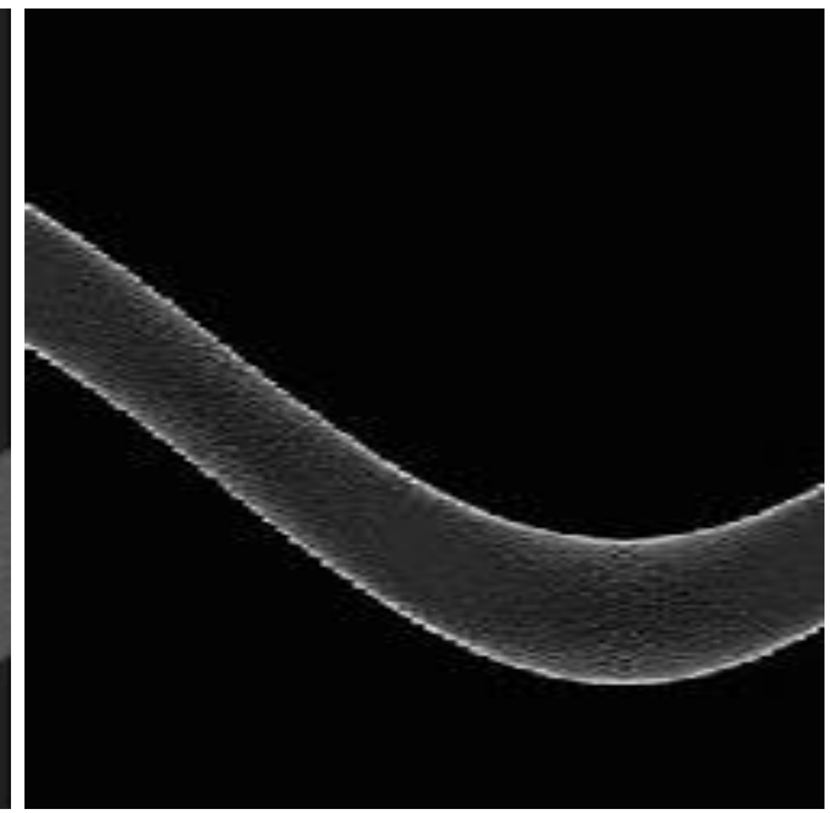
(in parameter space)



line

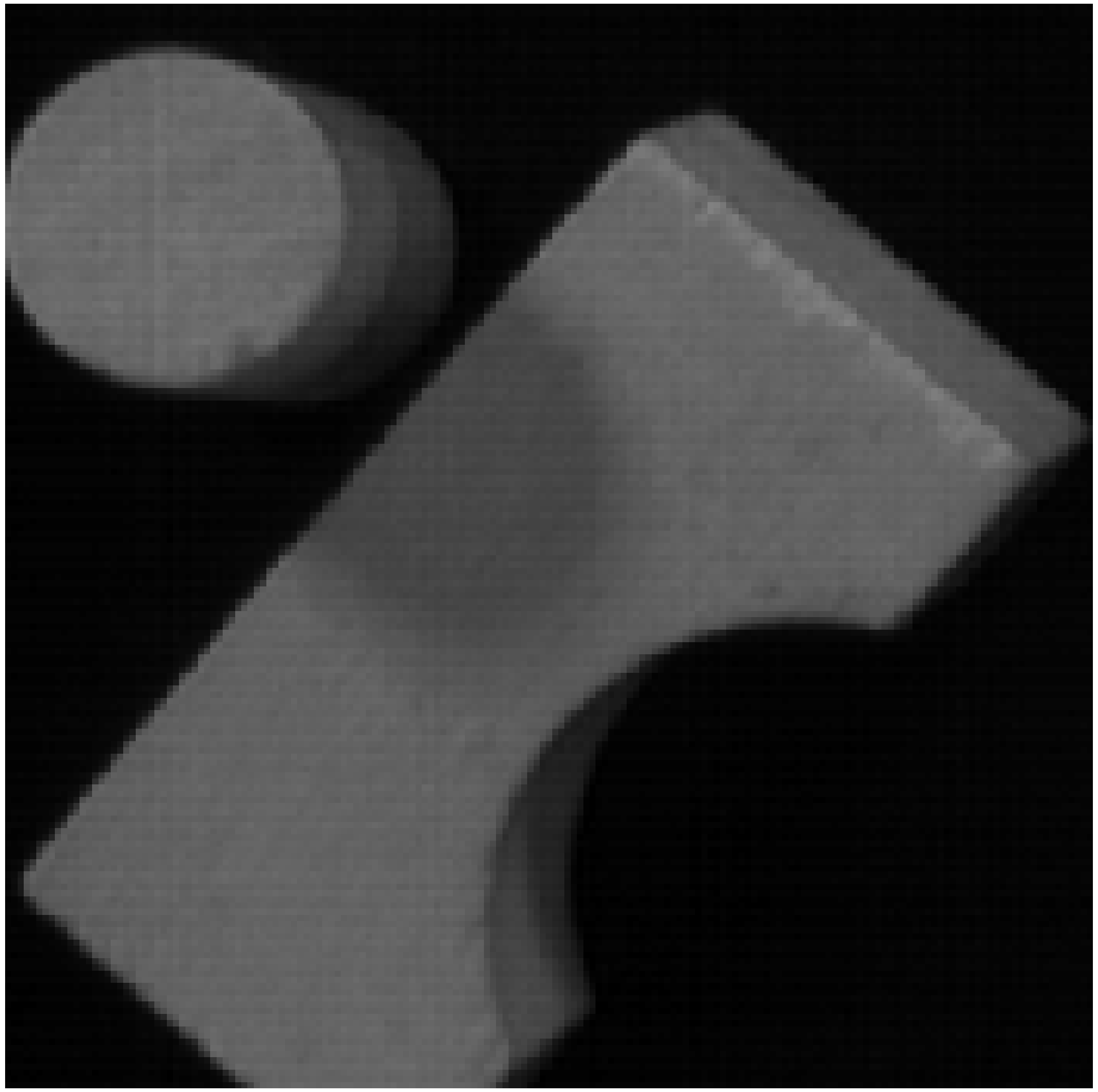


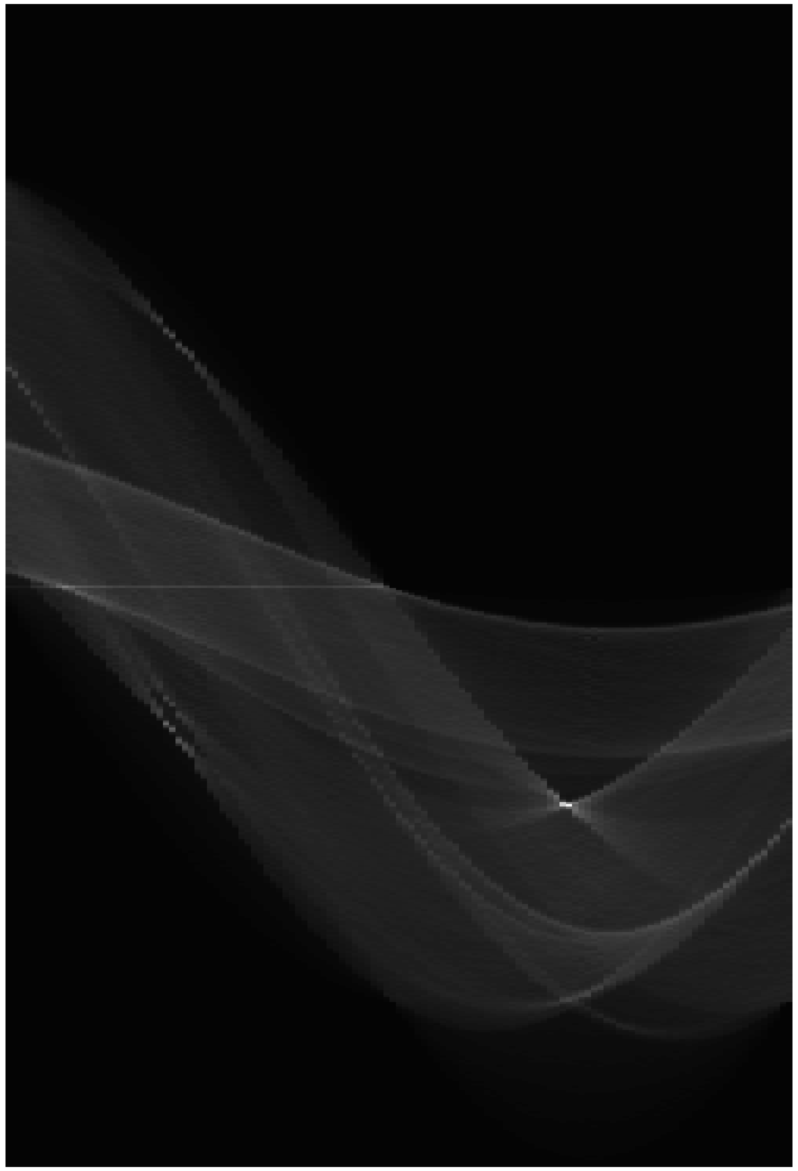
rectangle



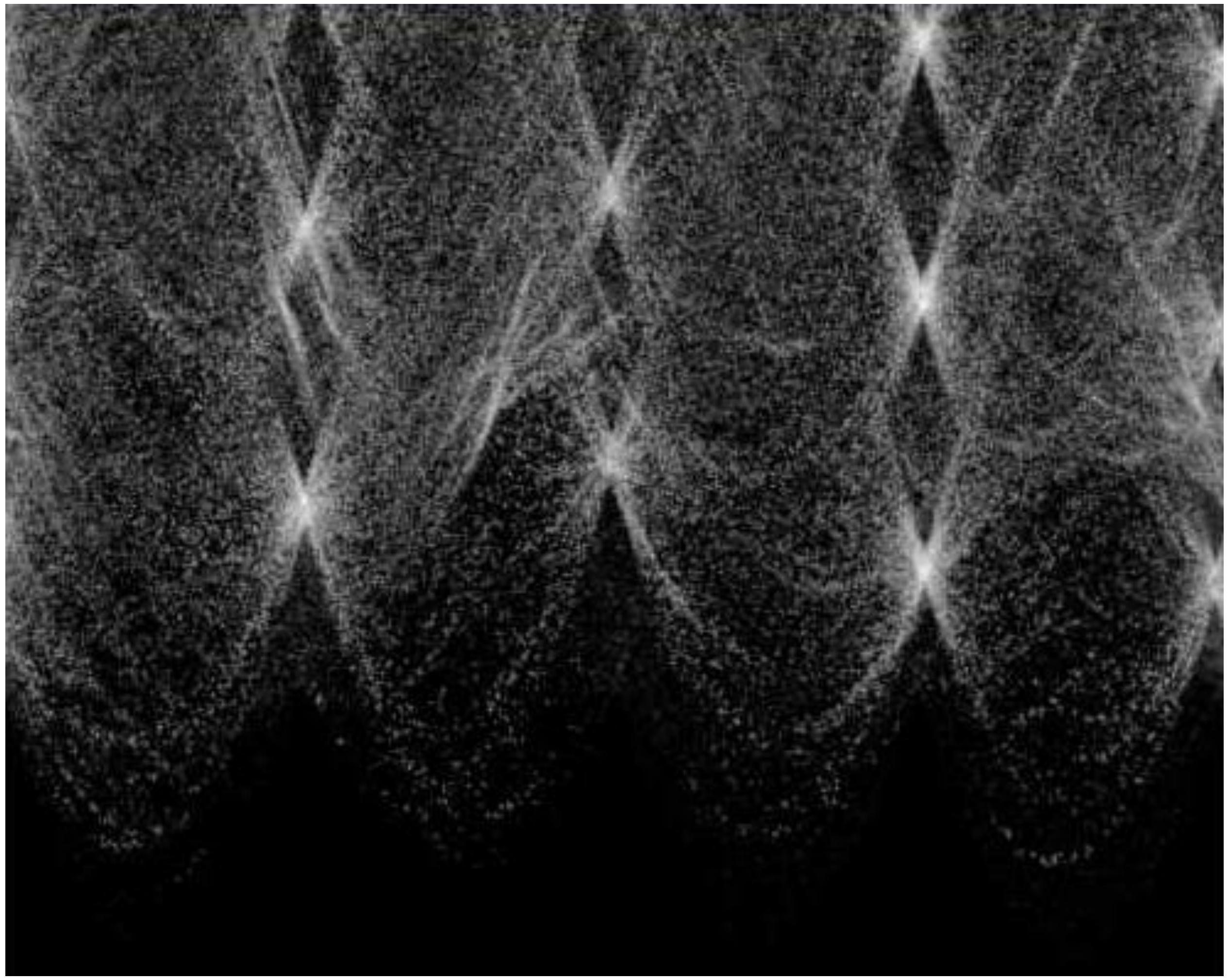
circle

Basic Shapes





More complex image



In practice, measurements are noisy…

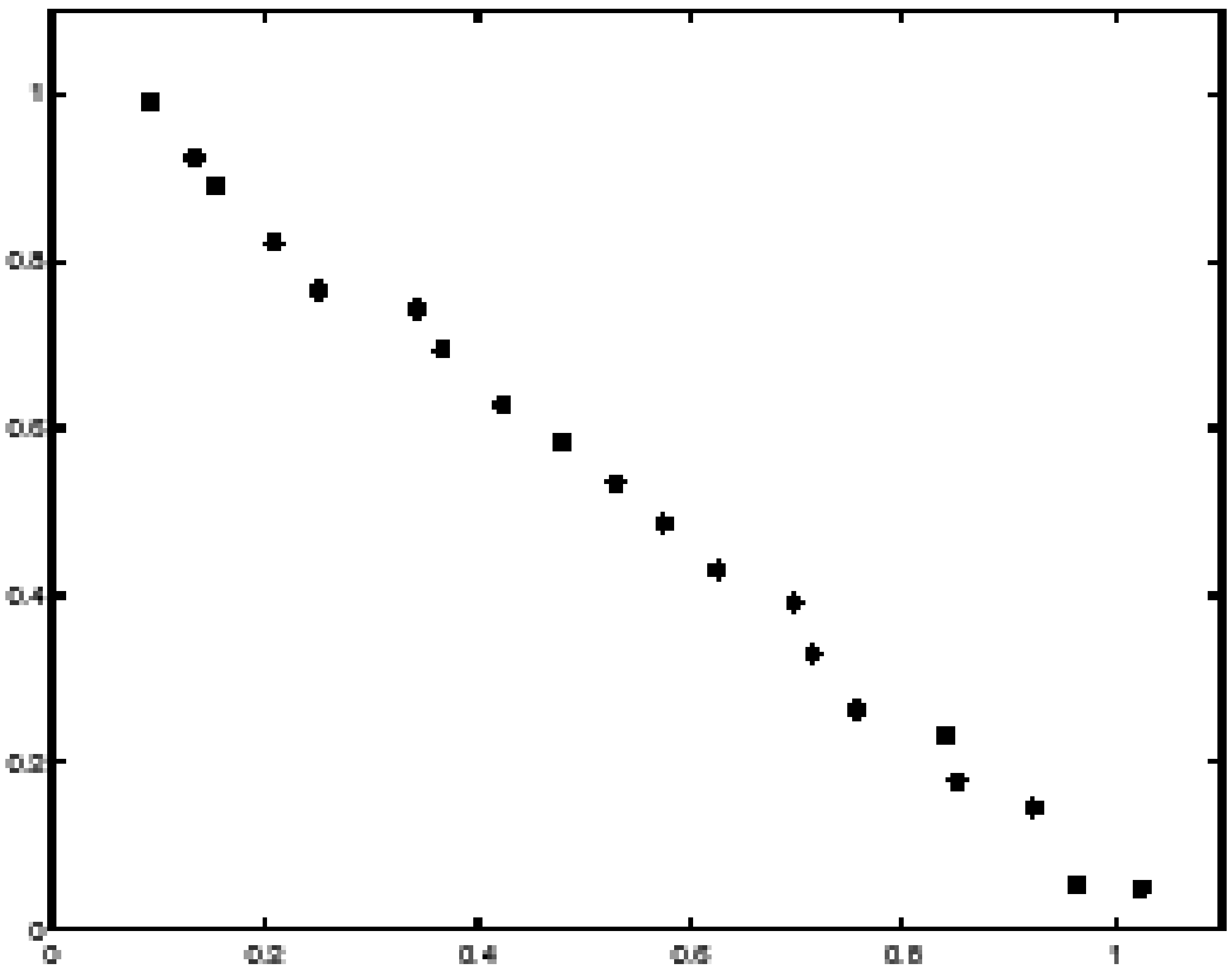
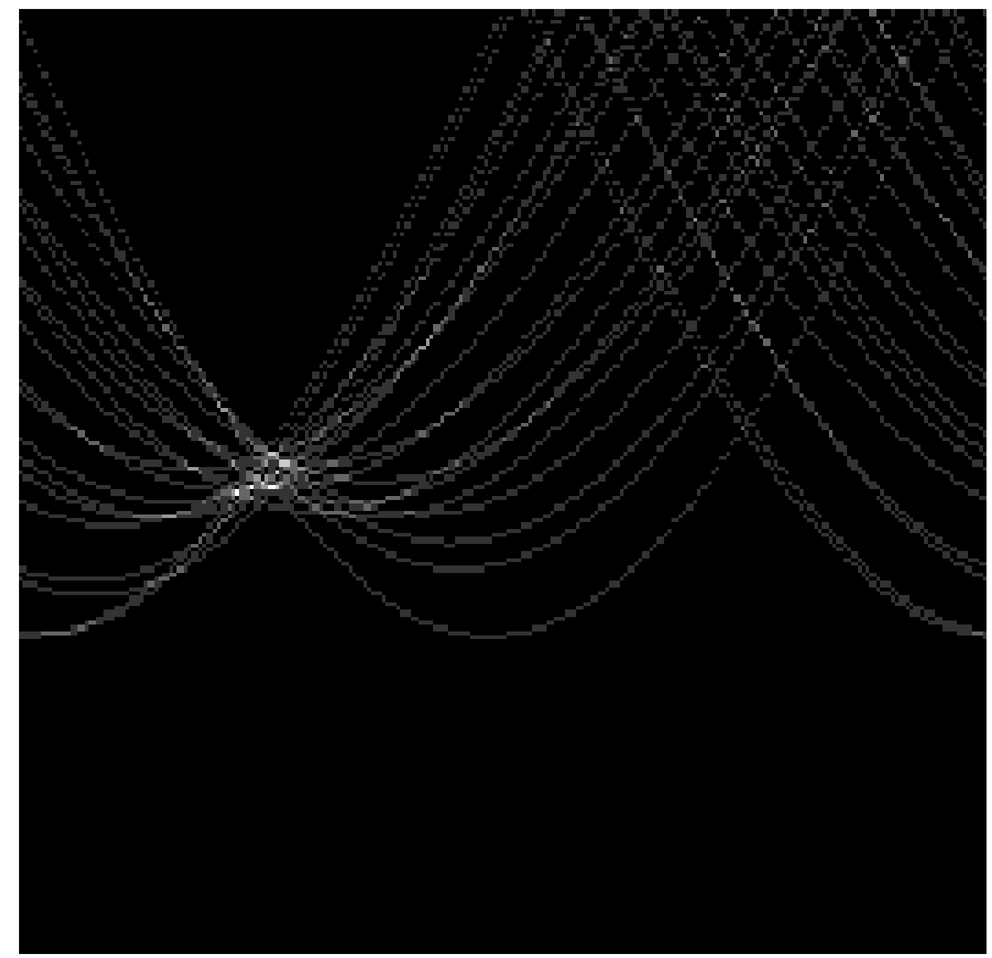


Image space



Votes

Too much noise …

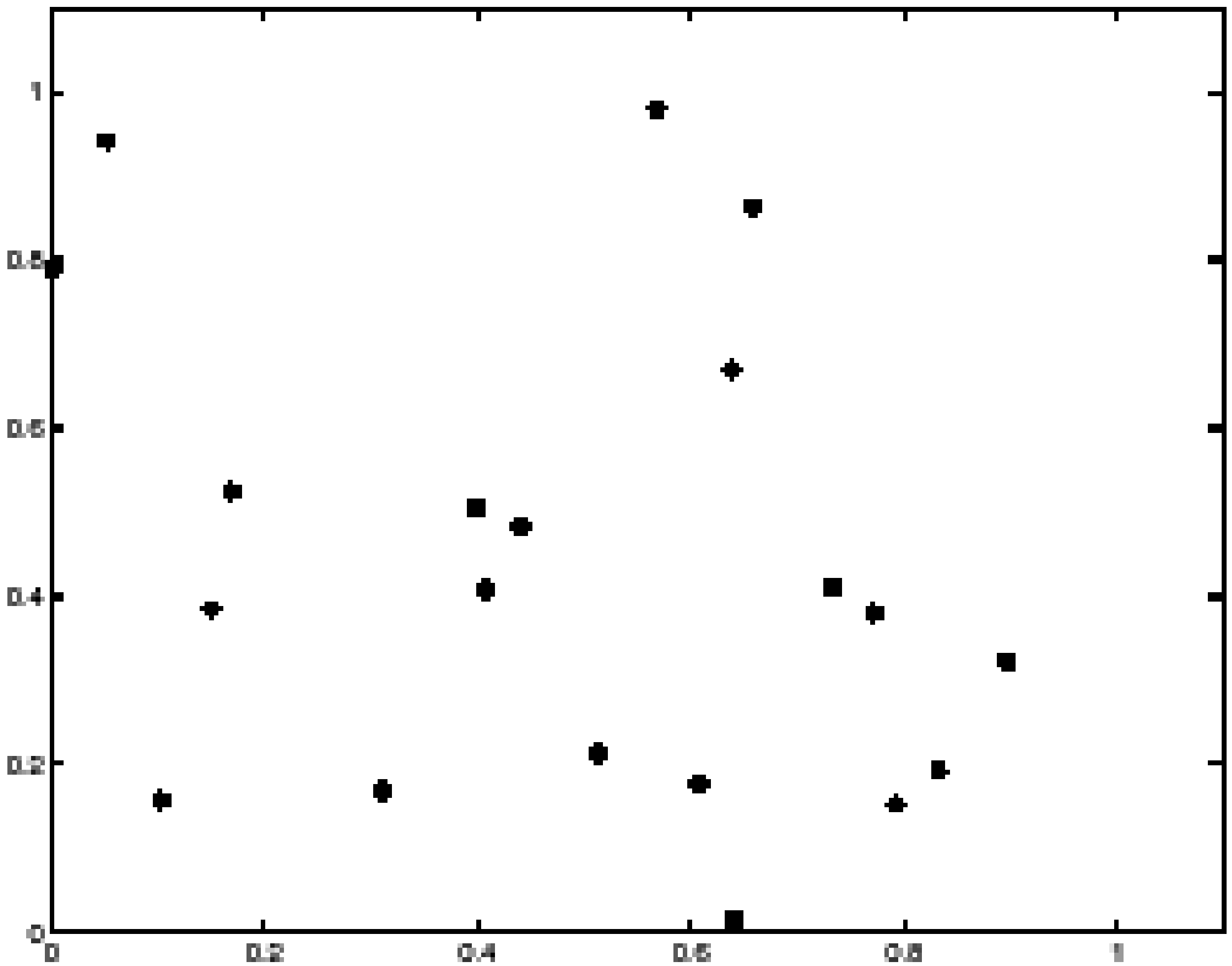
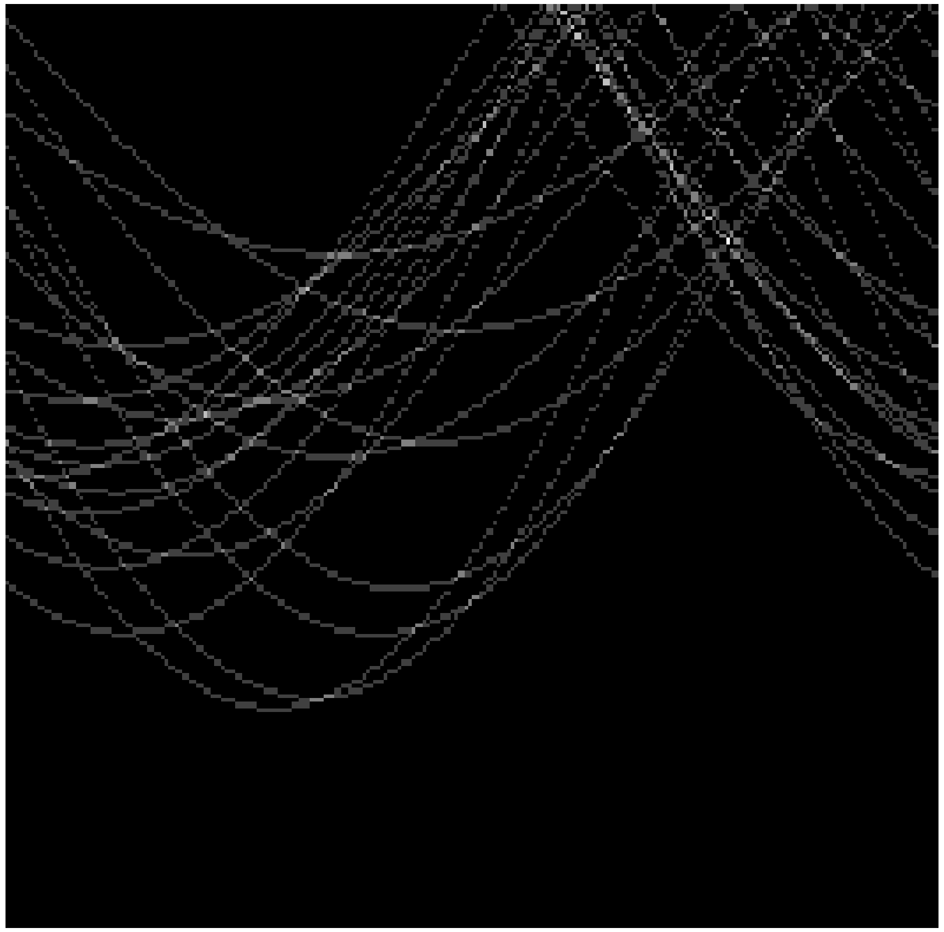
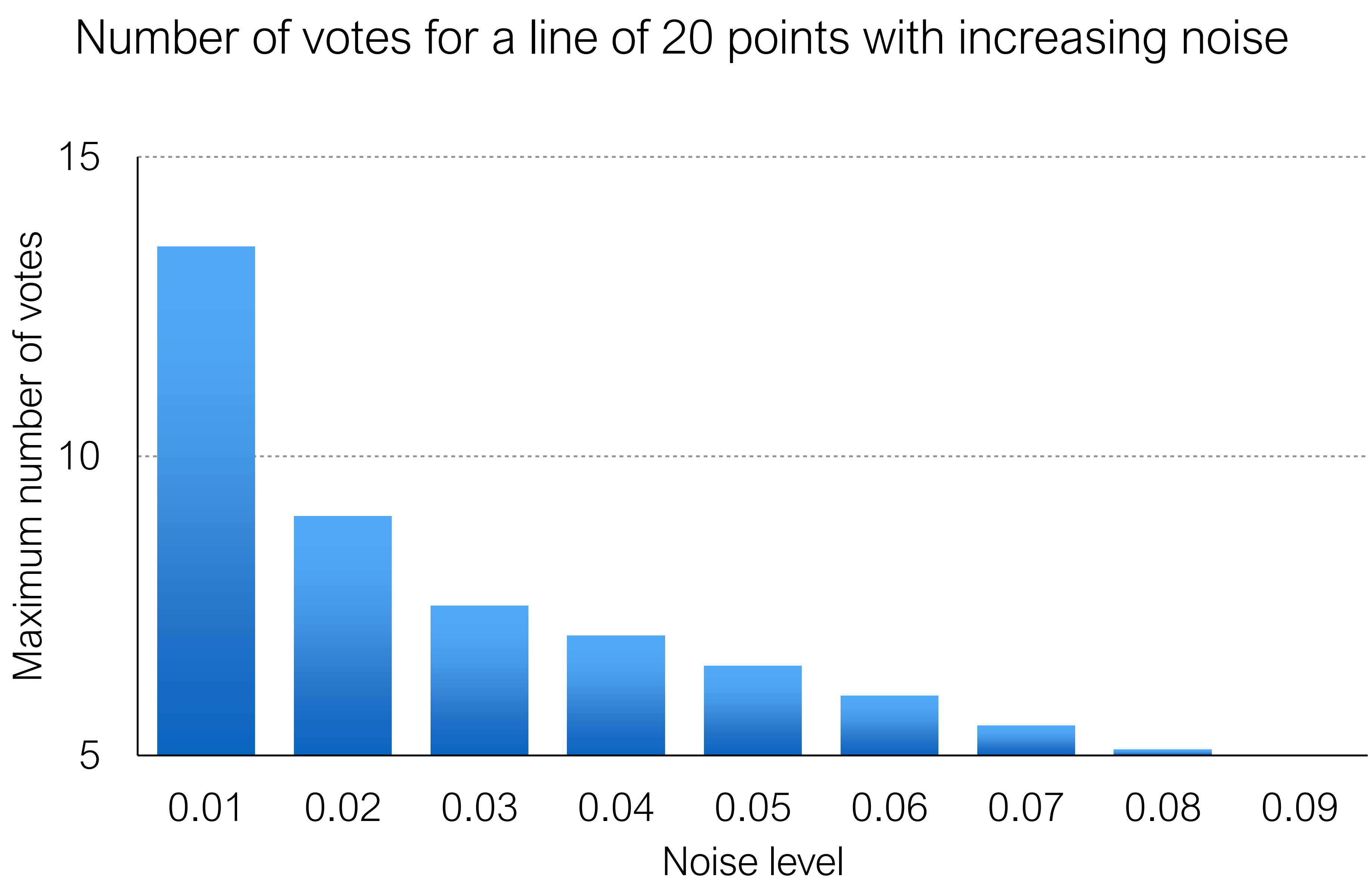


Image space



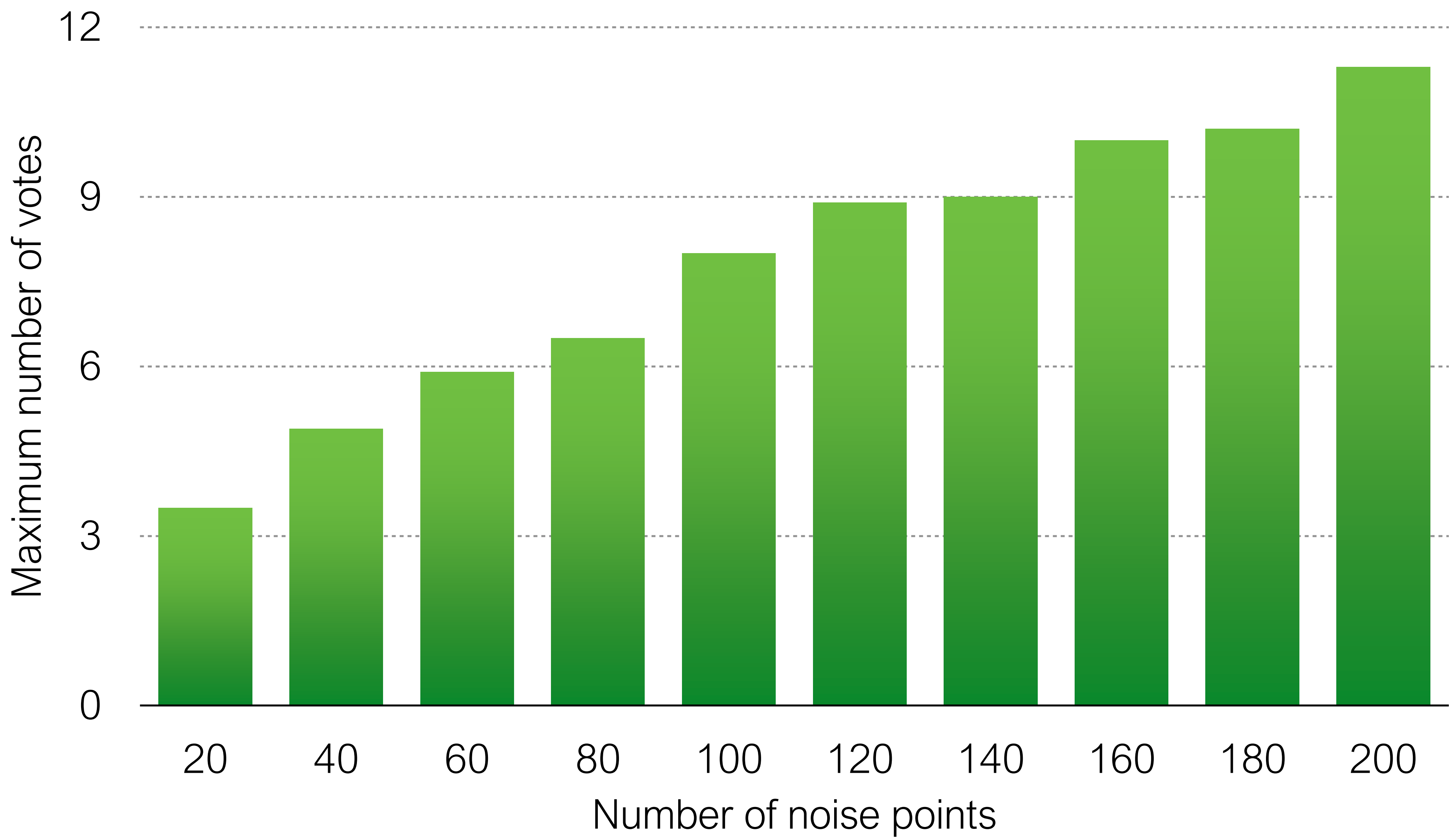
Votes

Effects of noise level



More noise, less votes (in the right bin)

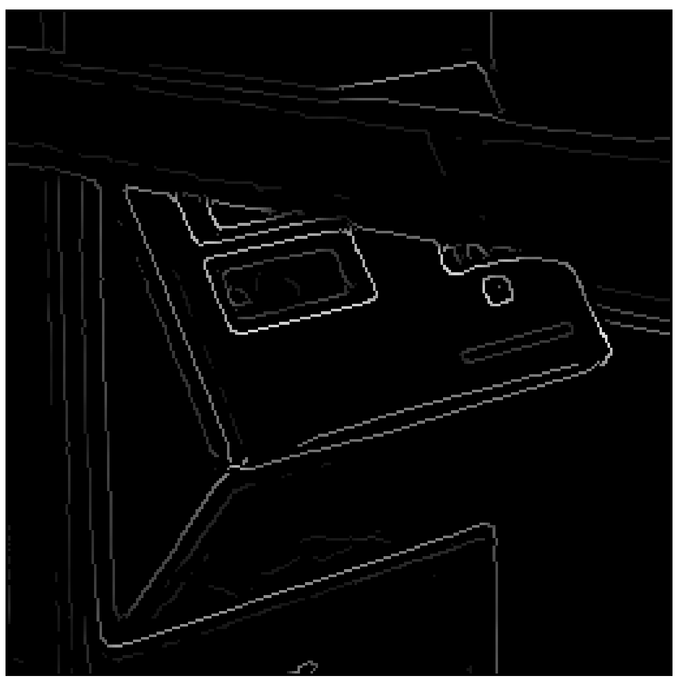
Effect of noise points



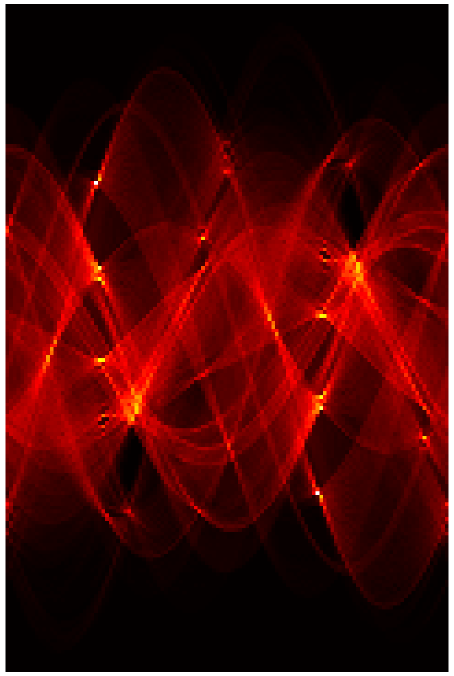
More noise, more votes (in the wrong bin)

Real-world example

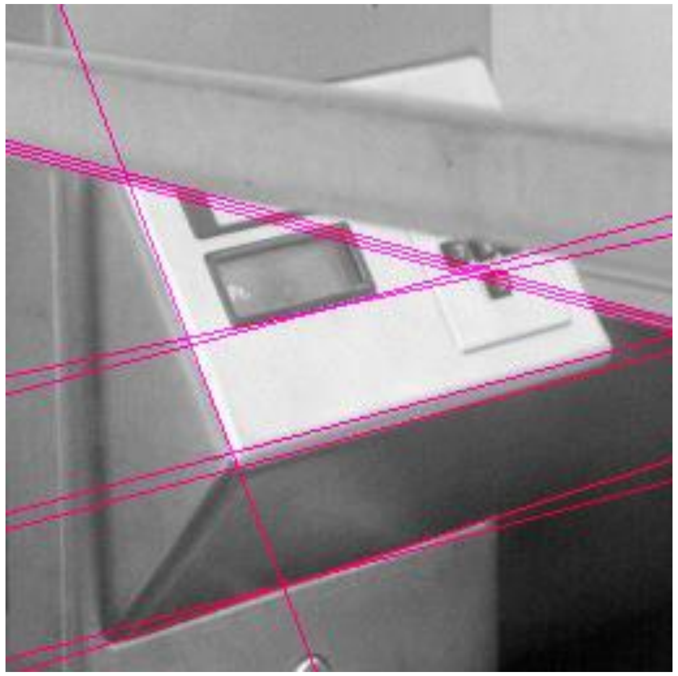
Original



Edges



parameter space



Hough Lines

Hough Circles

Let’s assume radius known

parameters

$$(x-a)^2+(y-b)^2=r^2$$

variables

parameters

$$(x-a)^2+(y-b)^2=r^2$$

variables

What is the dimension of the parameter space?

parameters

$$(x - a)^2 + (y - b)^2 = r^2$$

variables

parameters

$$(x - a)^2 + (y - b)^2 = r^2$$

variables

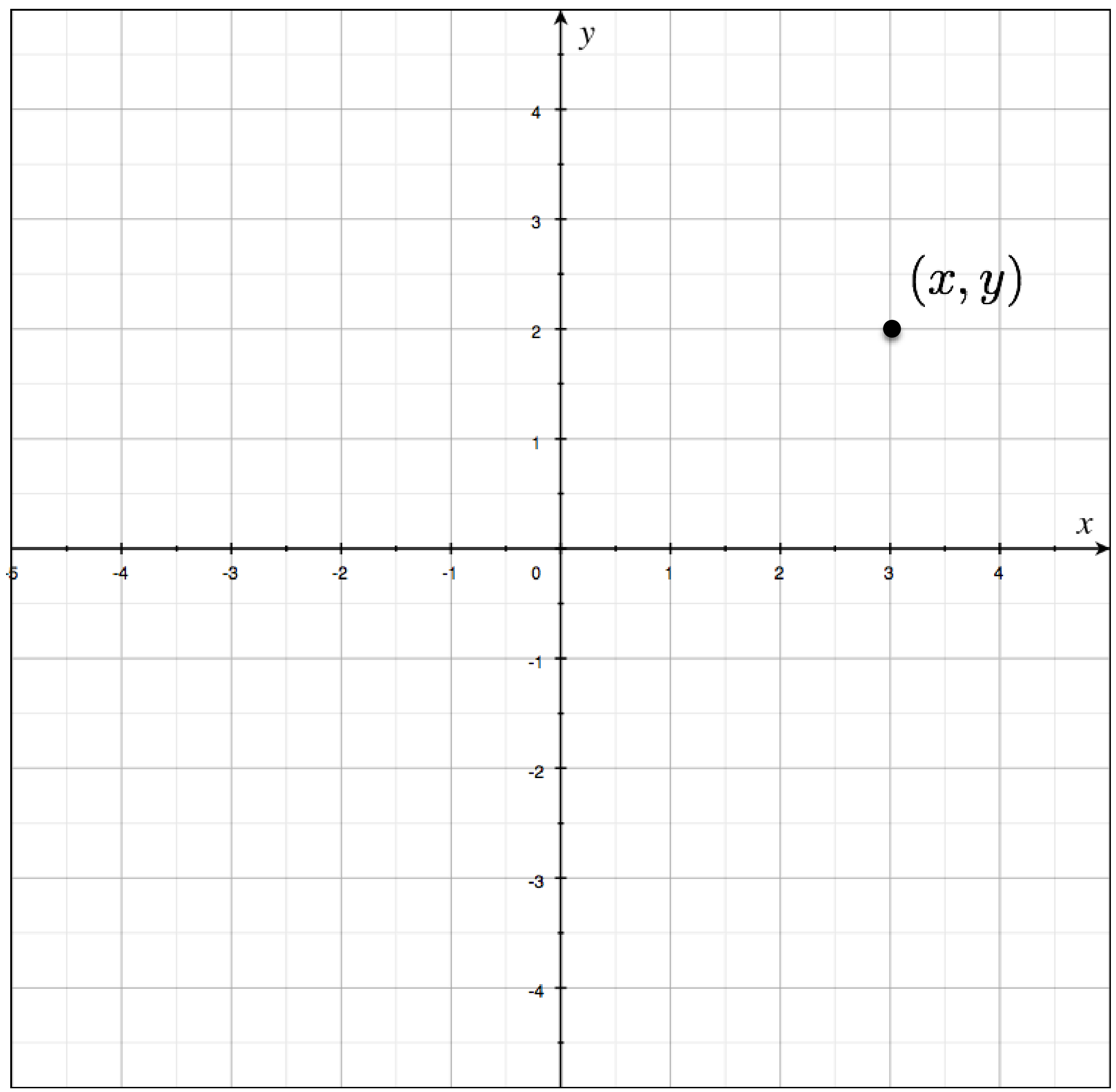
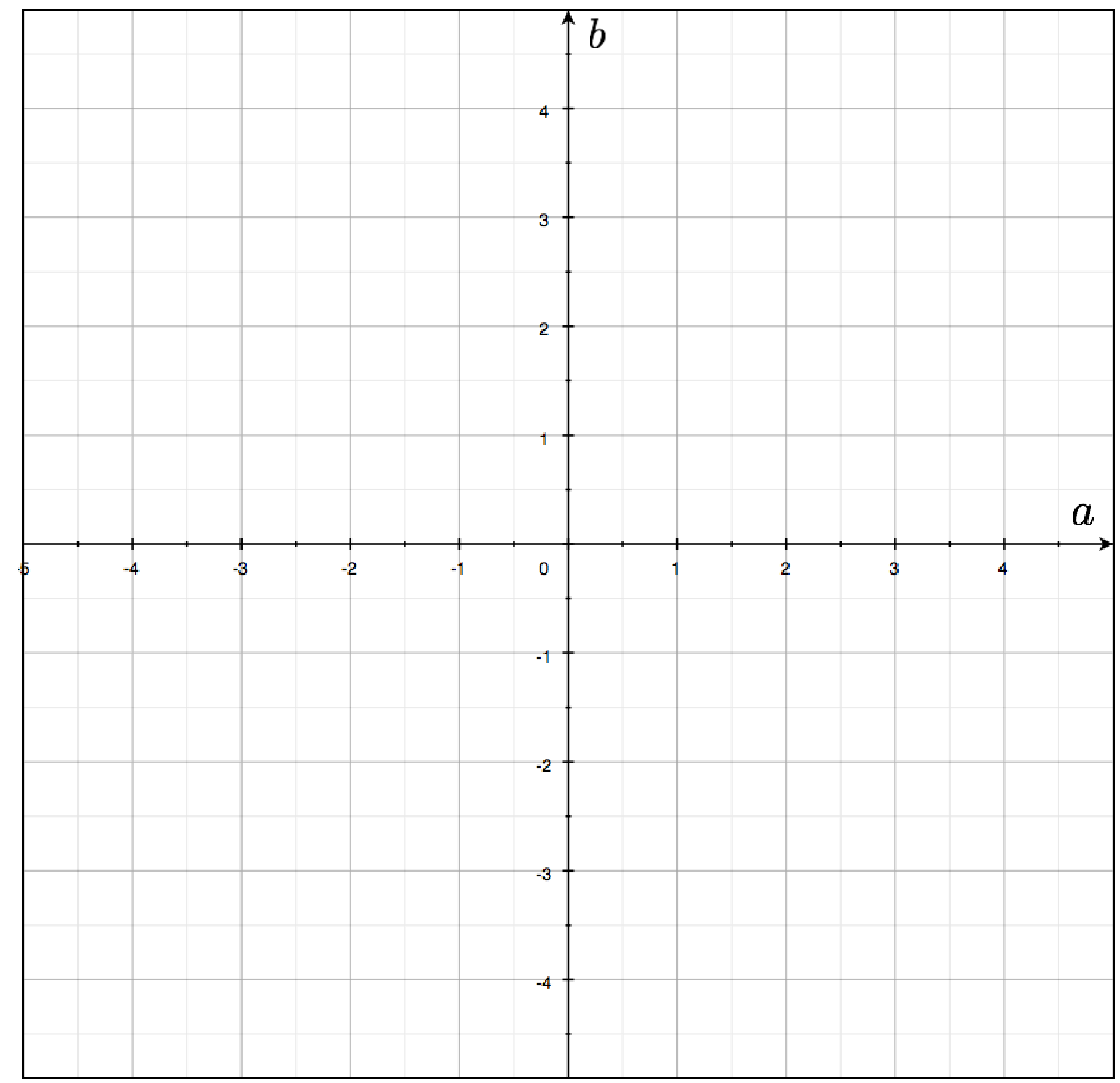


Image space



Parameter space

What does a point in image space correspond to in parameter space?

parameters

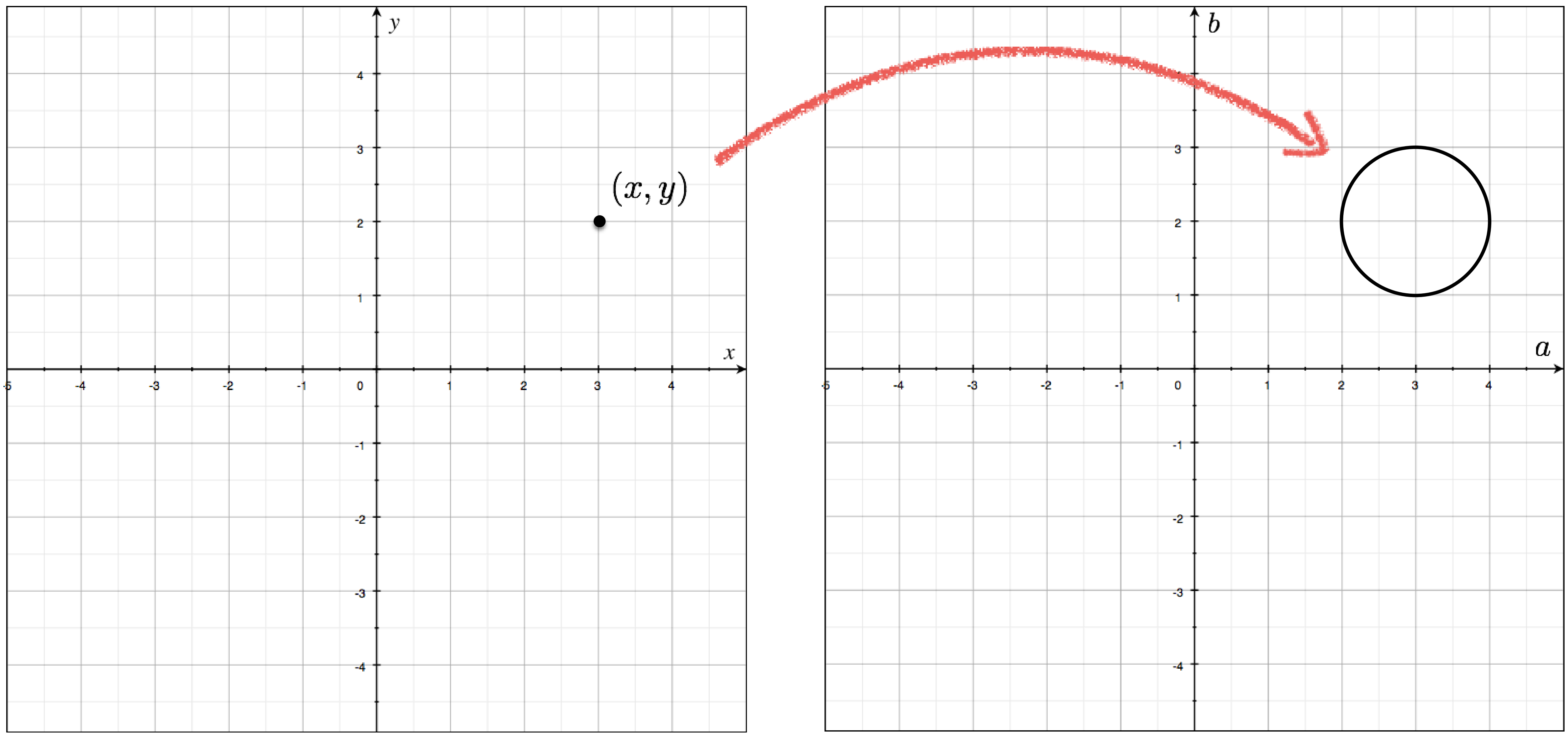
$$(x - a)^2 + (y - b)^2 = r^2$$

variables

parameters

$$(x - a)^2 + (y - b)^2 = r^2$$

variables



parameters

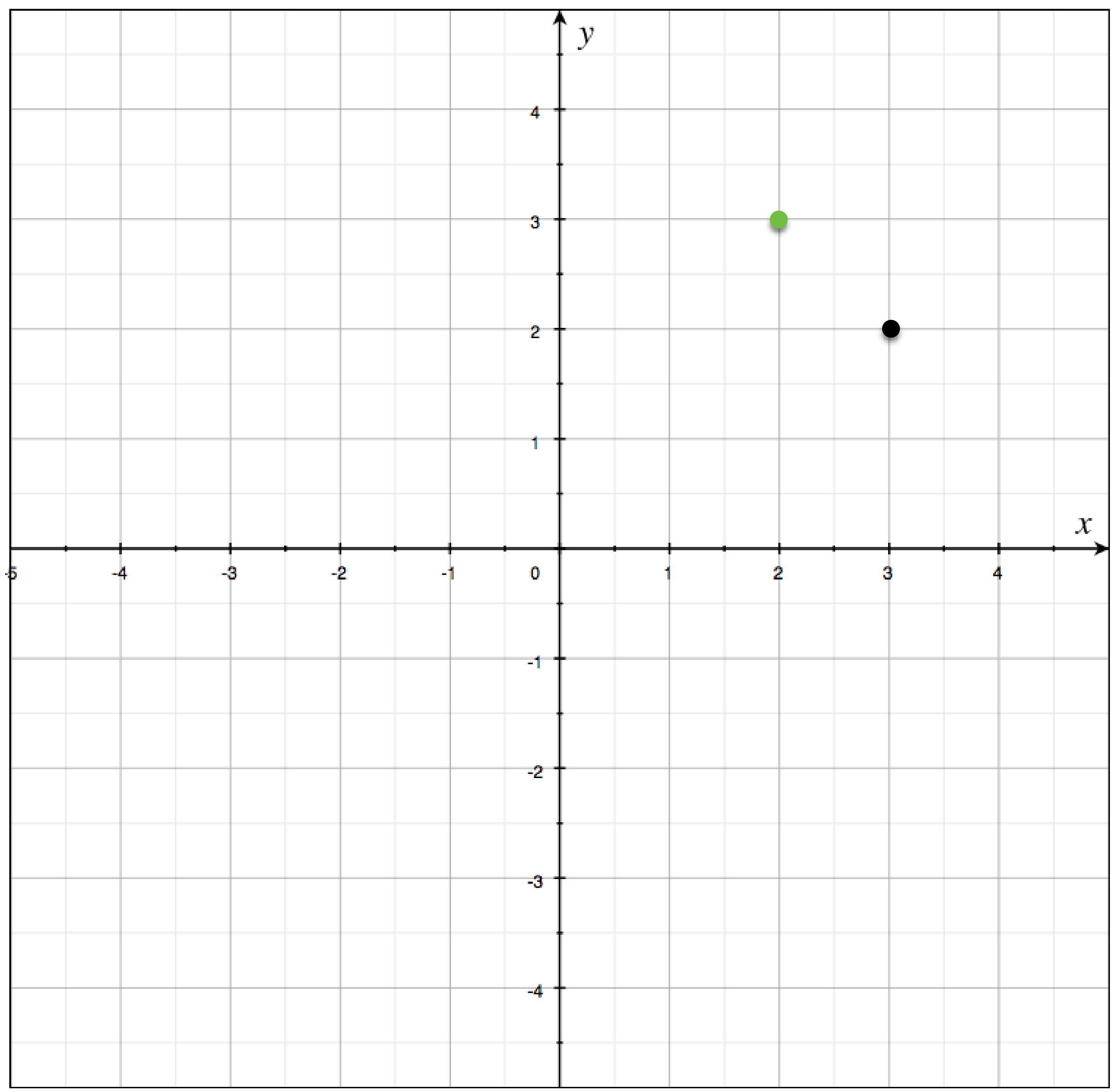
$$(x - a)^2 + (y - b)^2 = r^2$$

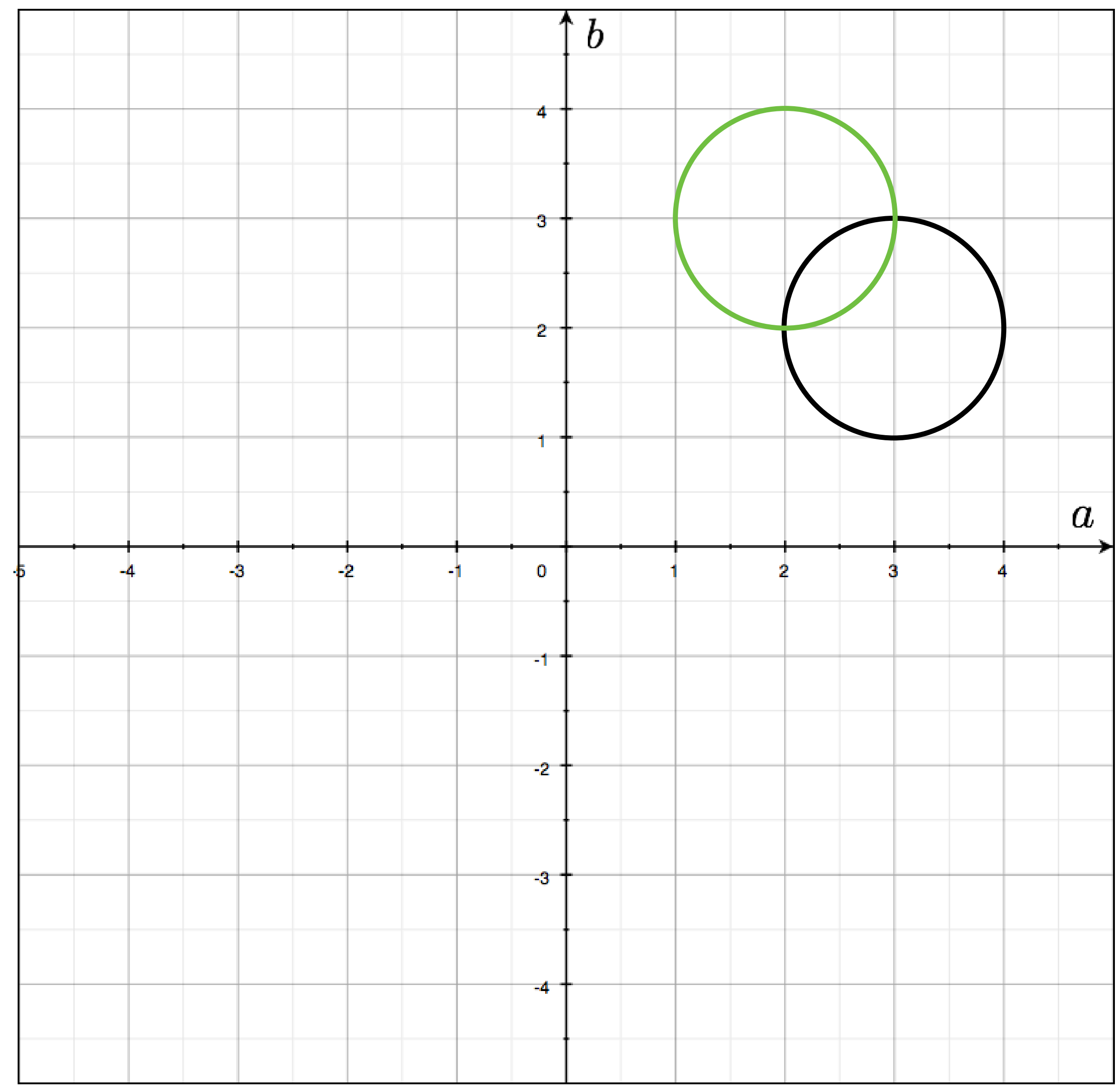
variables

parameters

$$(x - a)^2 + (y - b)^2 = r^2$$

variables





parameters

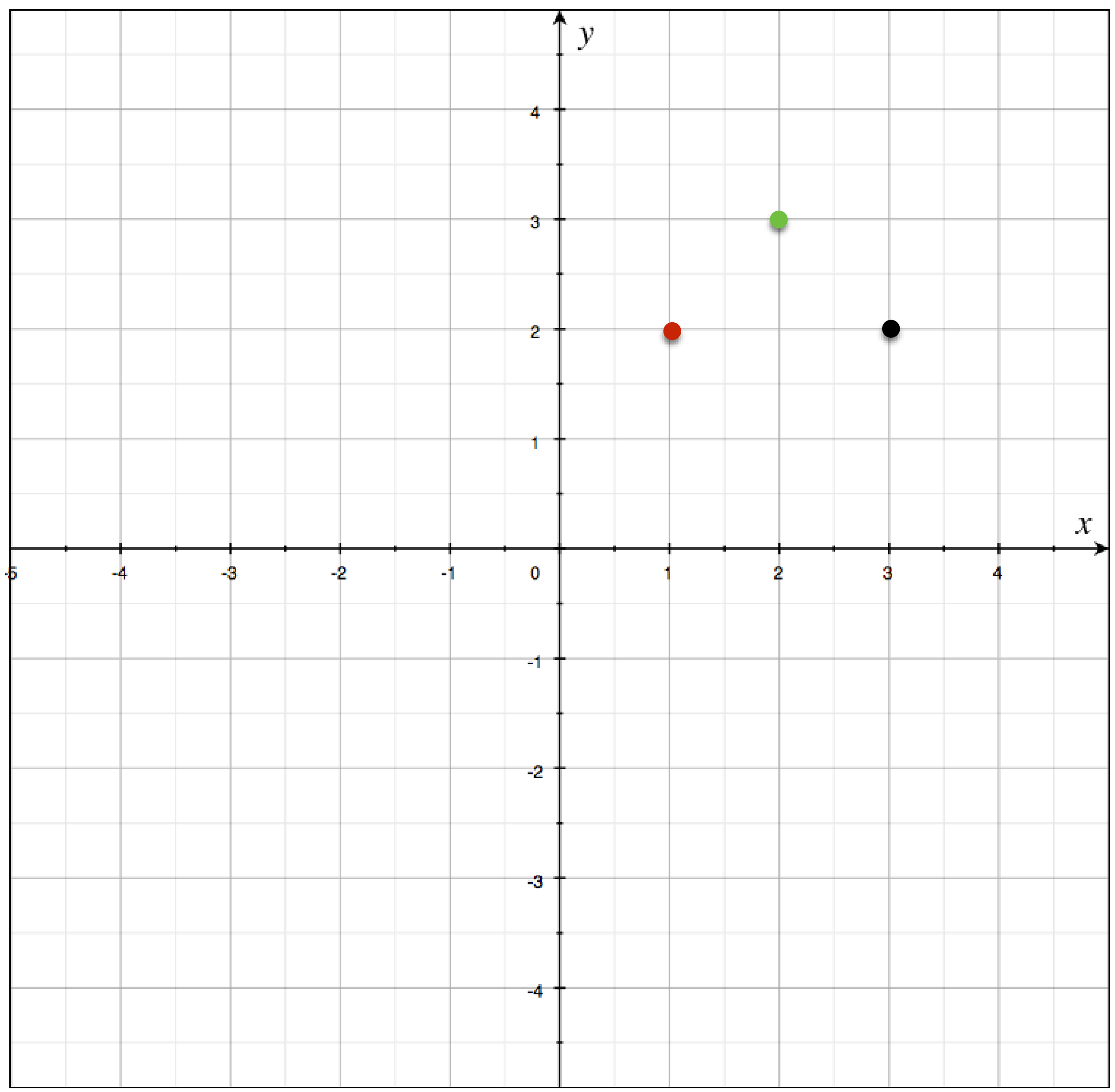
$$(x - a)^2 + (y - b)^2 = r^2$$

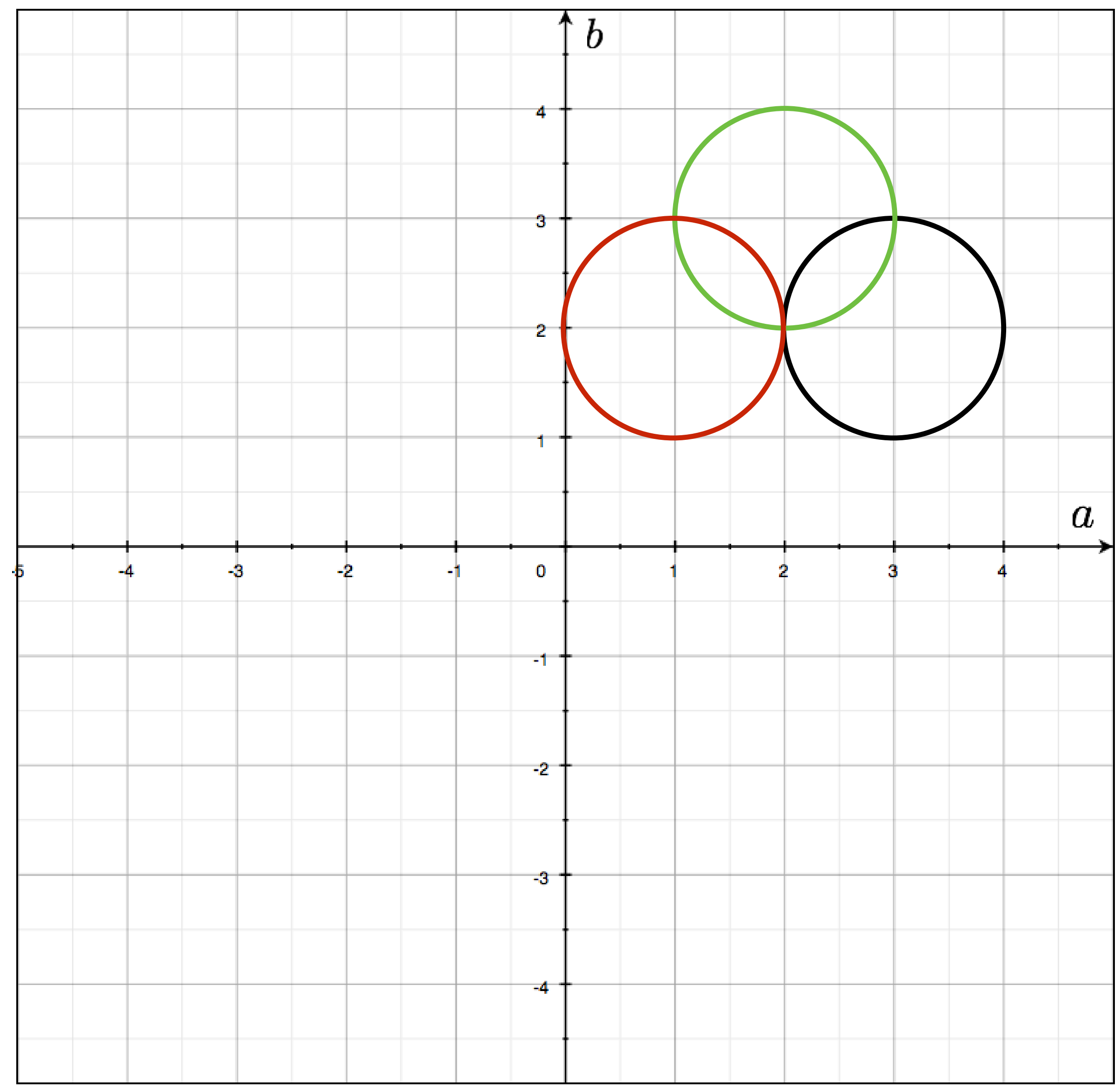
variables

parameters

$$(x - a)^2 + (y - b)^2 = r^2$$

variables





parameters

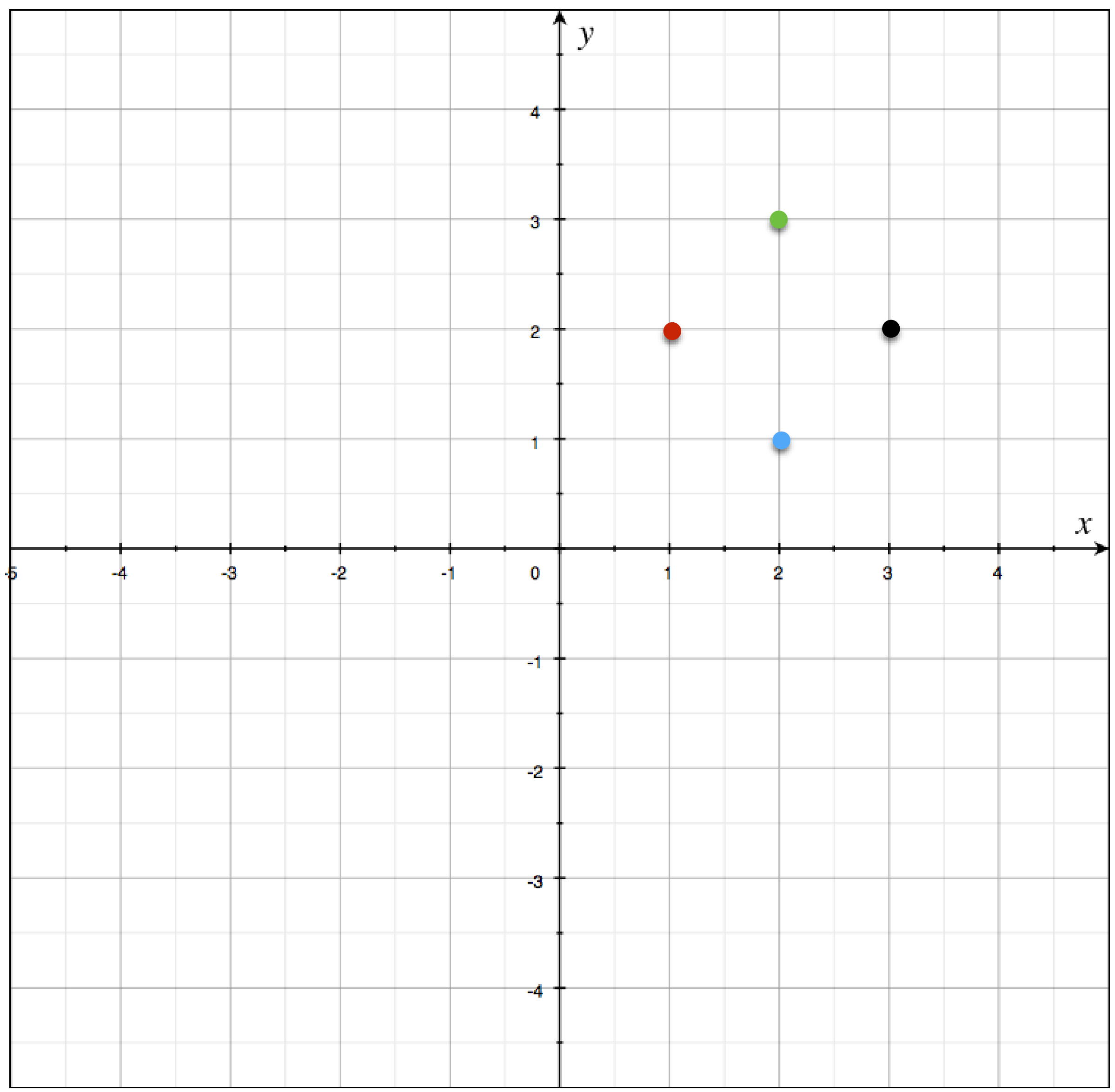
$$(x - a)^2 + (y - b)^2 = r^2$$

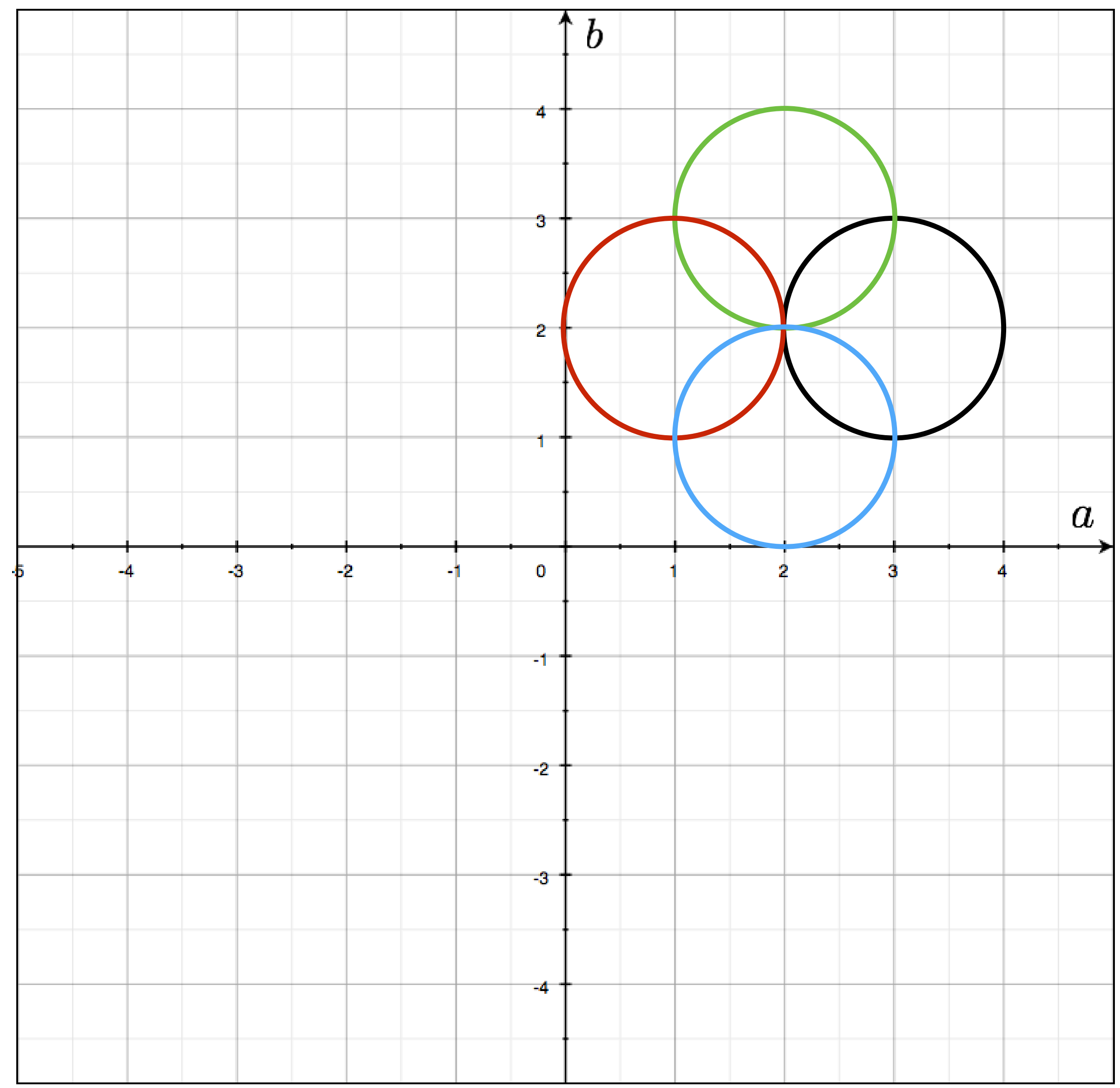
variables

parameters

$$(x - a)^2 + (y - b)^2 = r^2$$

variables





What if radius is unknown?

parameters

$$(x-a)^2+(y-b)^2=r^2$$

variables

parameters

$$(x-a)^2+(y-b)^2=r^2$$

variables

What if radius is unknown?

parameters

$$(x-a)^2+(y-b)^2=r^2$$

variables

parameters

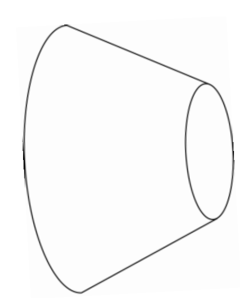
$$(x-a)^2+(y-b)^2=r^2$$

variables

If radius is not known: 3D Hough Space!

Use Accumulator array $A(a,b,r)$

Surface shape in Hough space is complicated

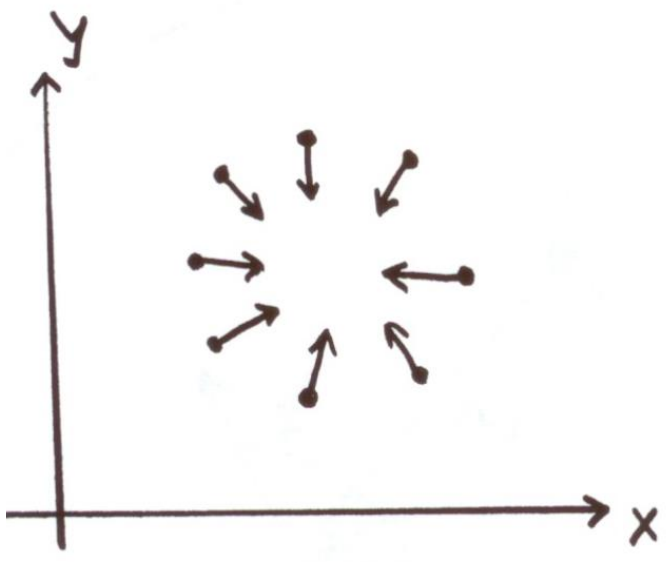


Using Gradient Information

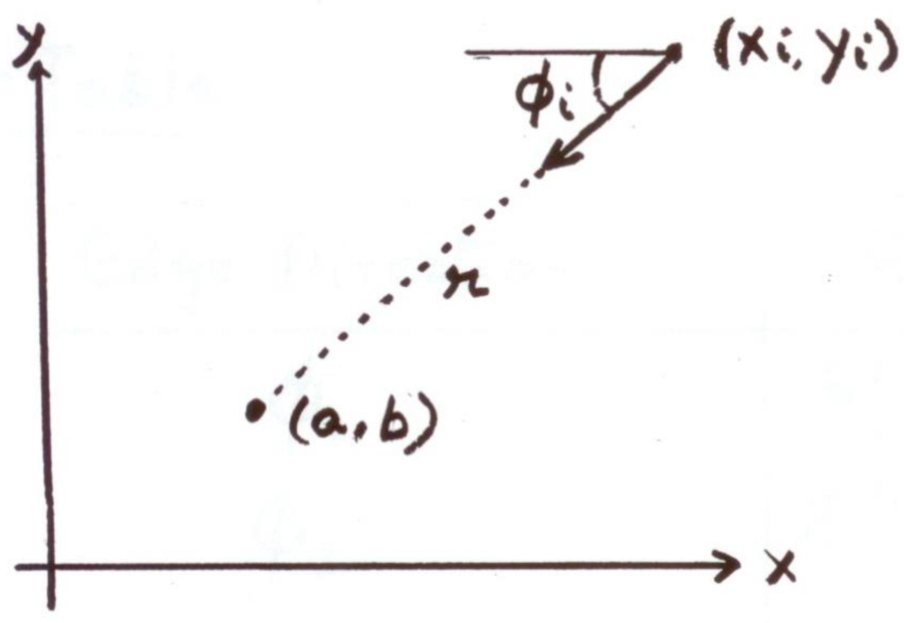
Gradient information can save lot of computation:

Edge Location (x_i, y_i)

Edge Direction ϕ_i



Assume radius is known:



$$a = x - r\cos\phi$$

$$b = y - r\sin\phi$$

Need to increment only one point in accumulator!

parameters

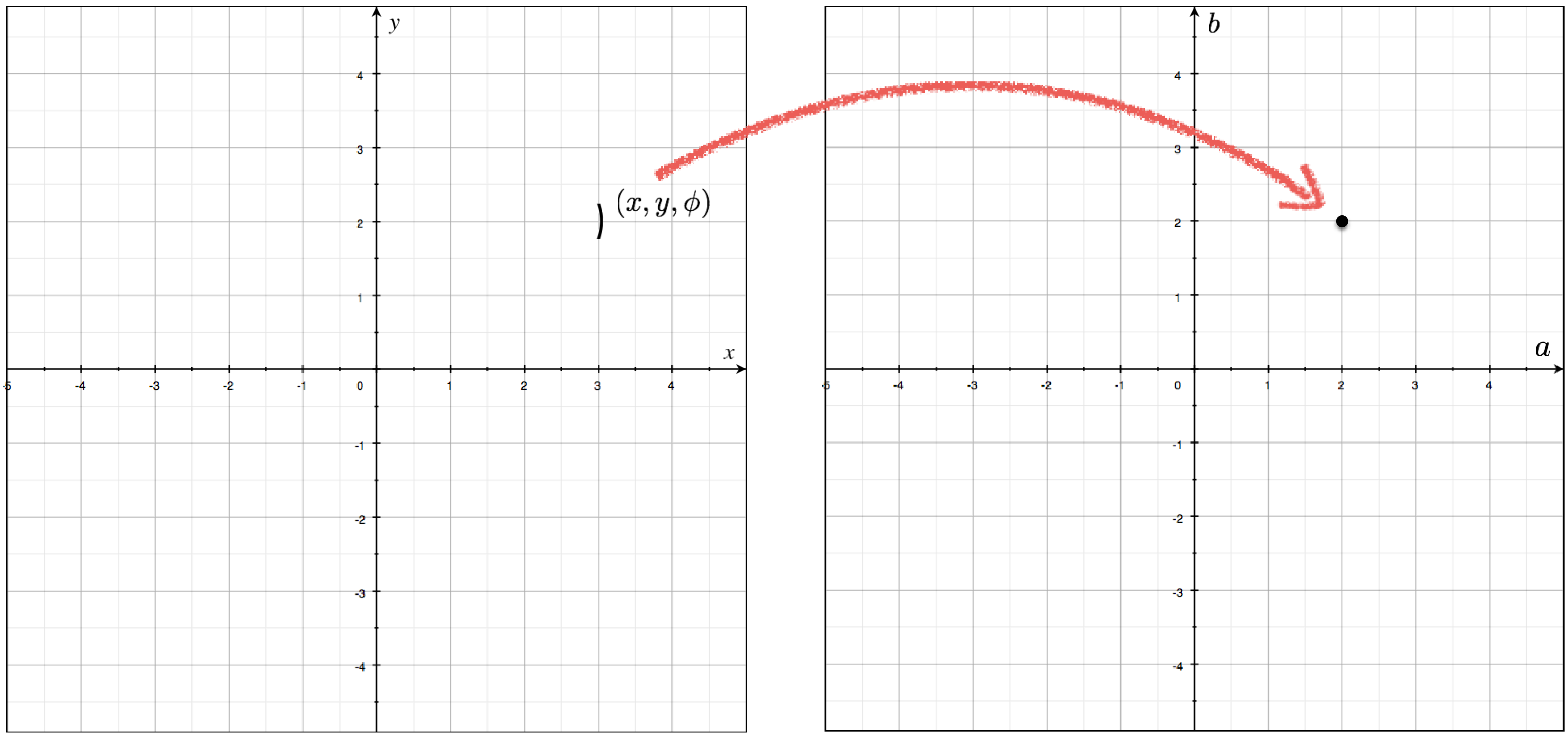
$$(x - a)^2 + (y - b)^2 = r^2$$

variables

parameters

$$(x - a)^2 + (y - b)^2 = r^2$$

variables



parameters

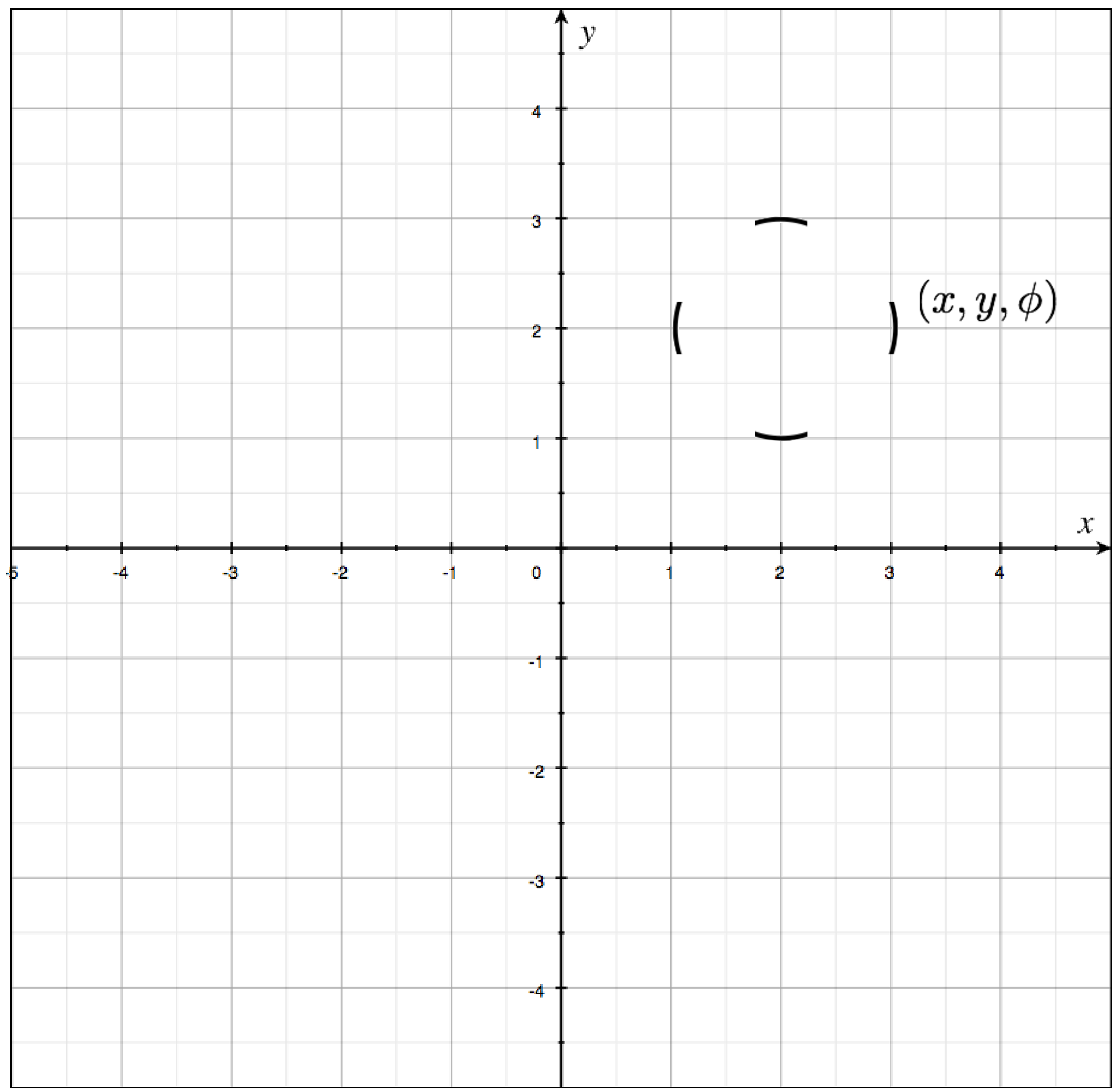
$$(x - a)^2 + (y - b)^2 = r^2$$

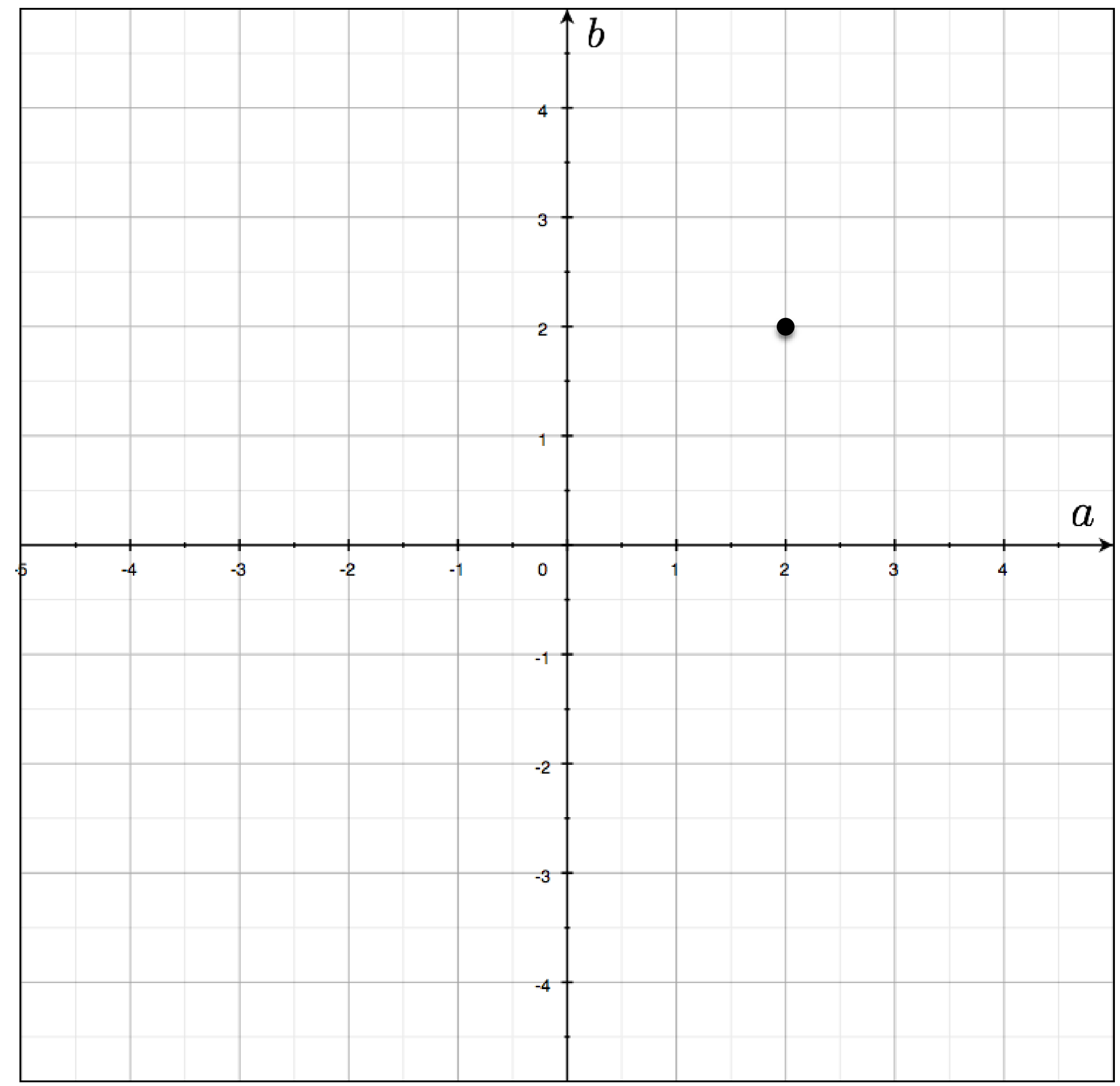
variables

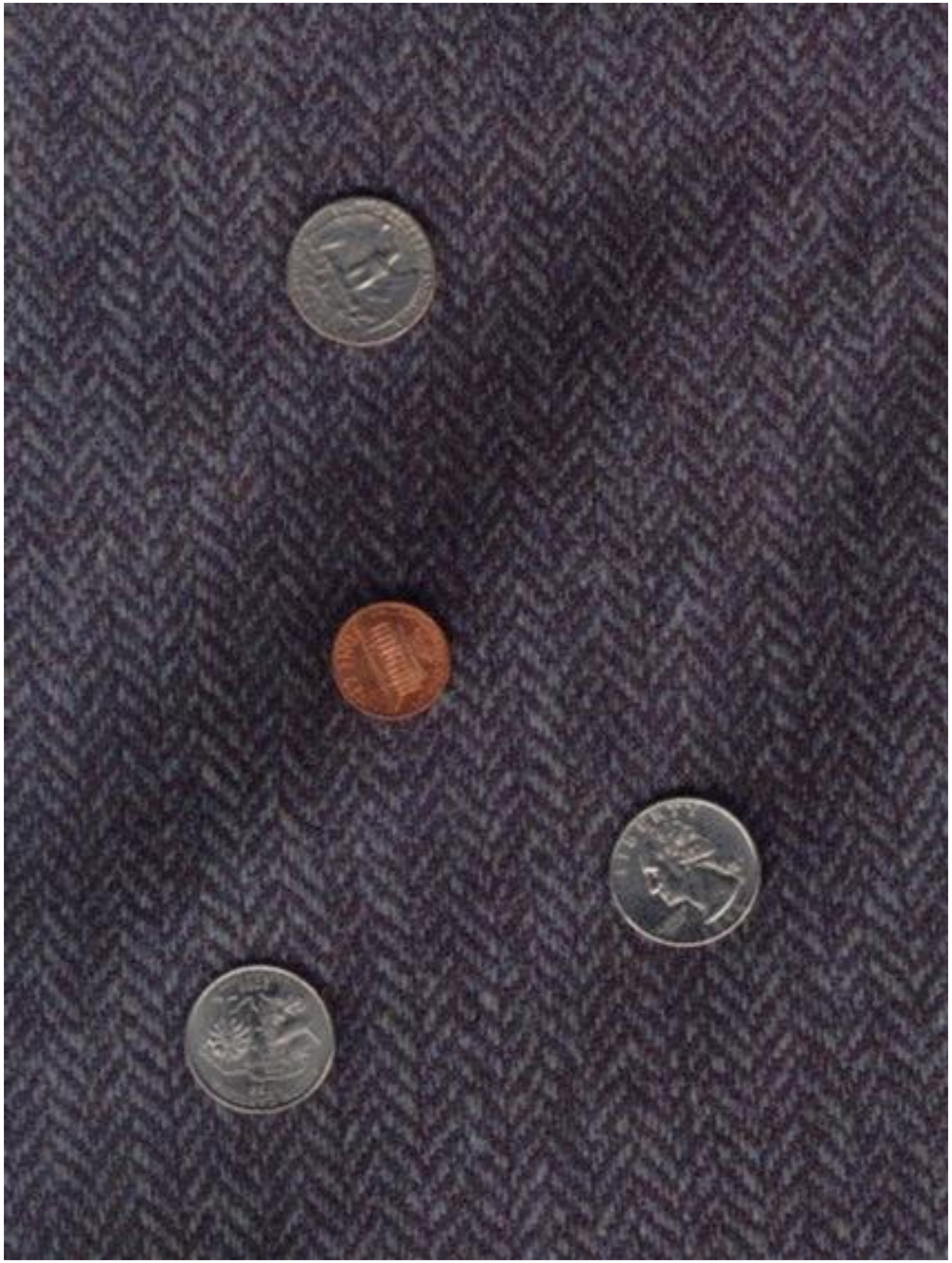
parameters

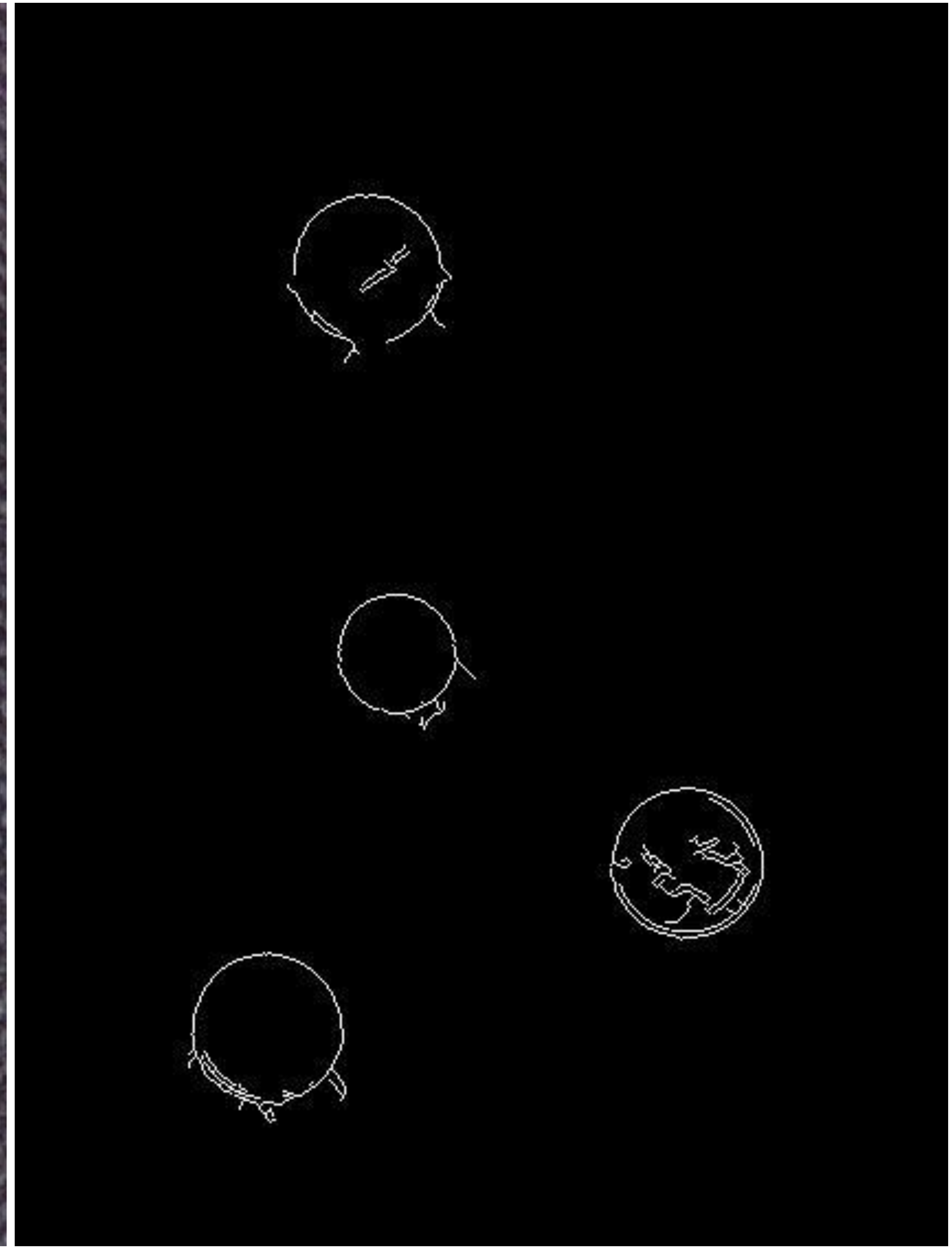
$$(x - a)^2 + (y - b)^2 = r^2$$

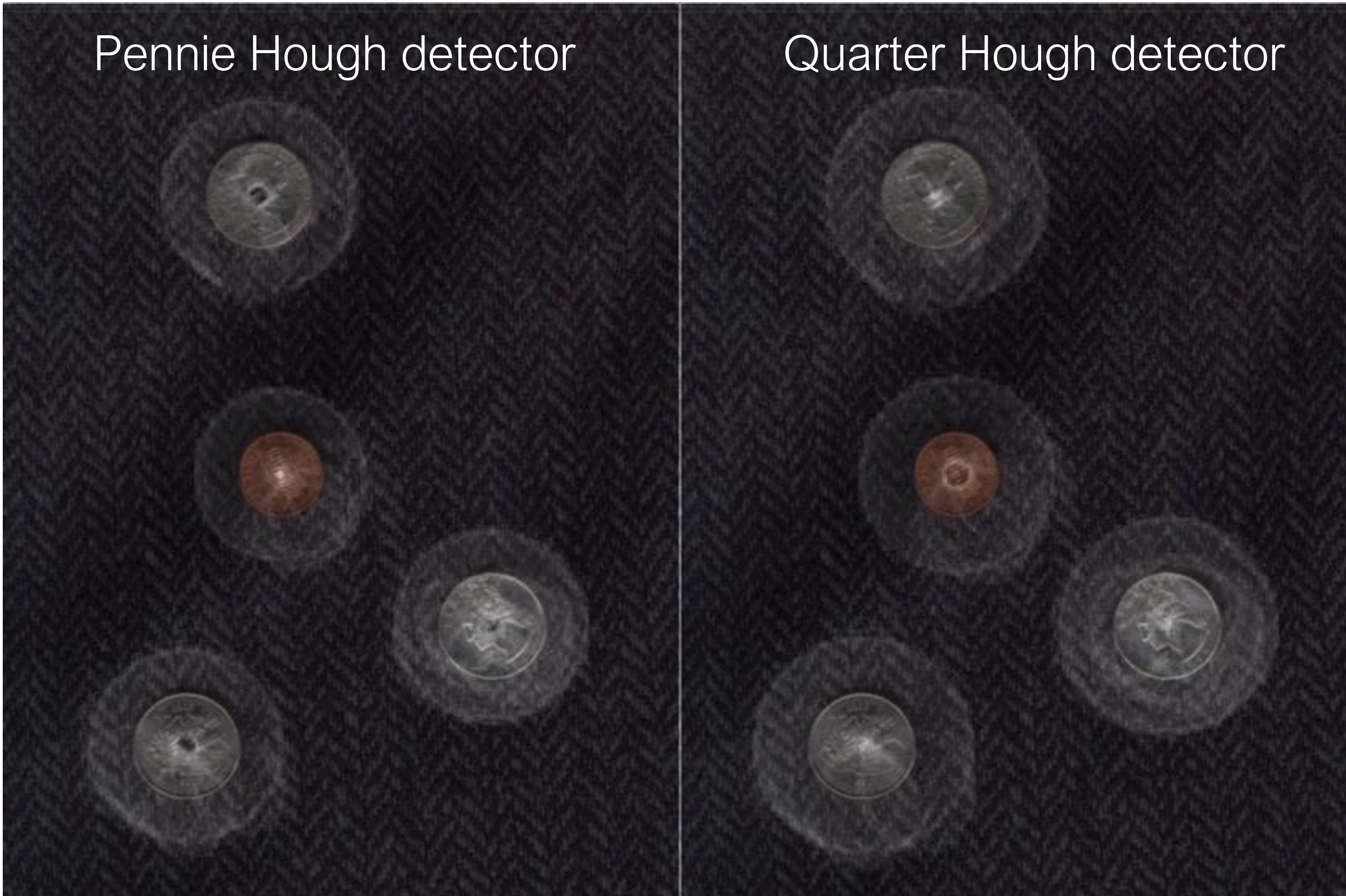
variables

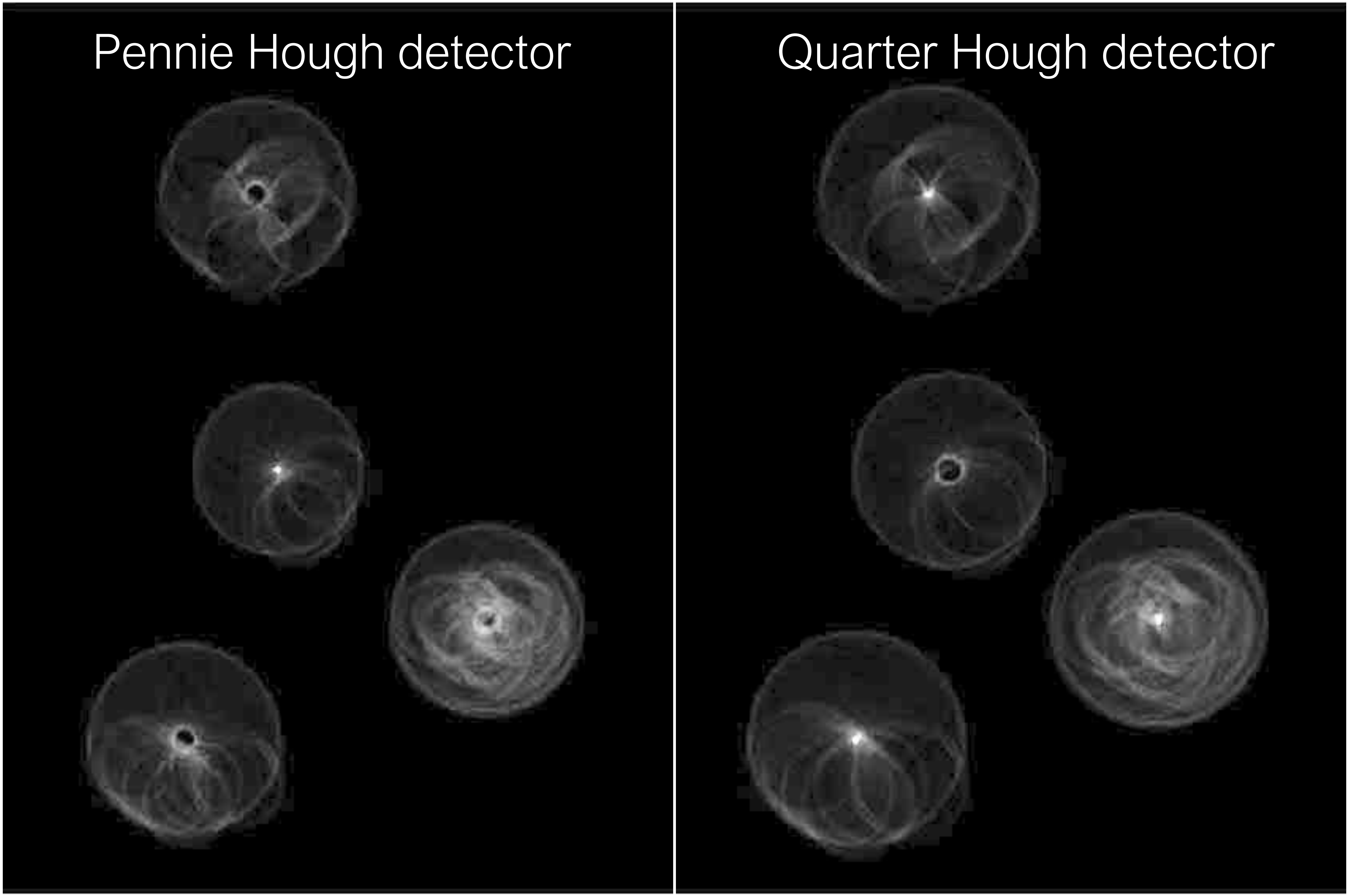









Pennie Hough detector
Quarter Hough detector


Pennie Hough detector
Quarter Hough detector

Can you use Hough Transforms for other objects, beyond lines and circles?

Do you have to use edge detectors to vote in Hough Space?

The Hough transform …

Deals with occlusion well? 

Detects multiple instances? 

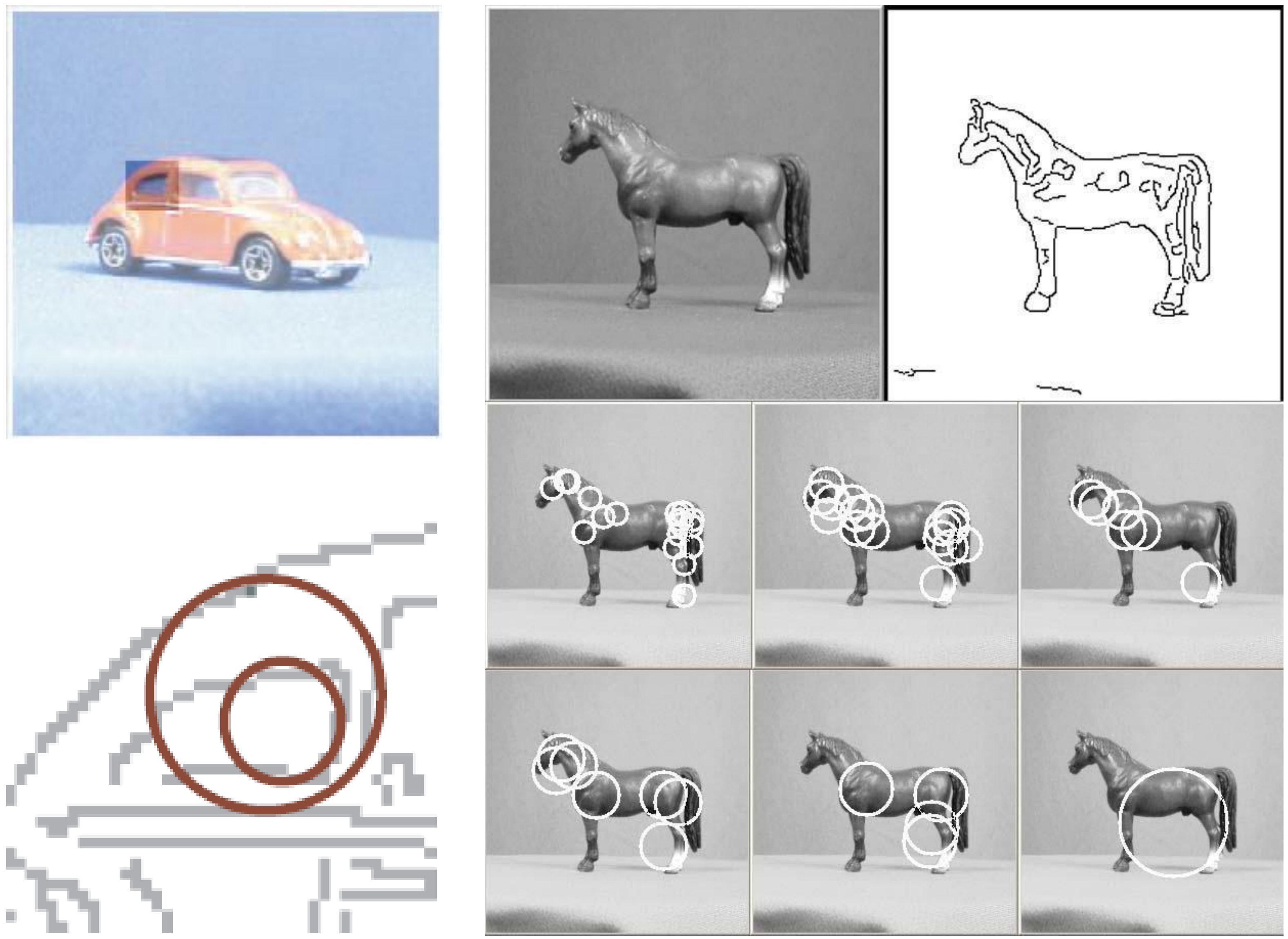
Robust to noise? 

Good computational complexity? 

Easy to set parameters? 

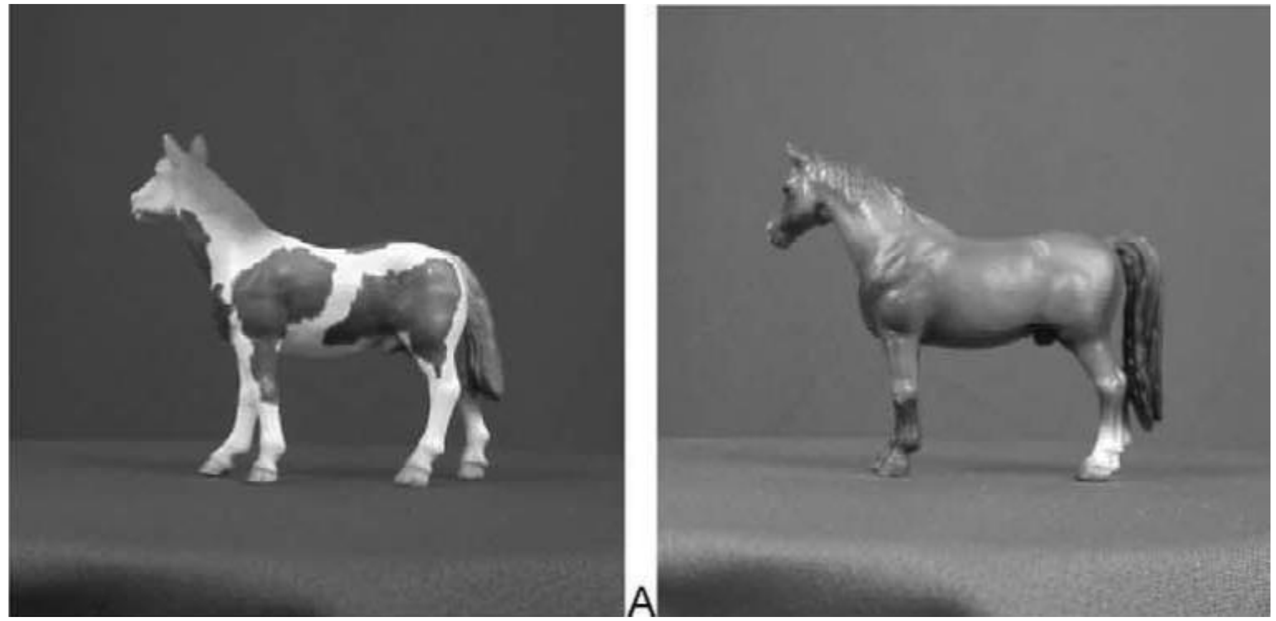
Application of Hough transforms

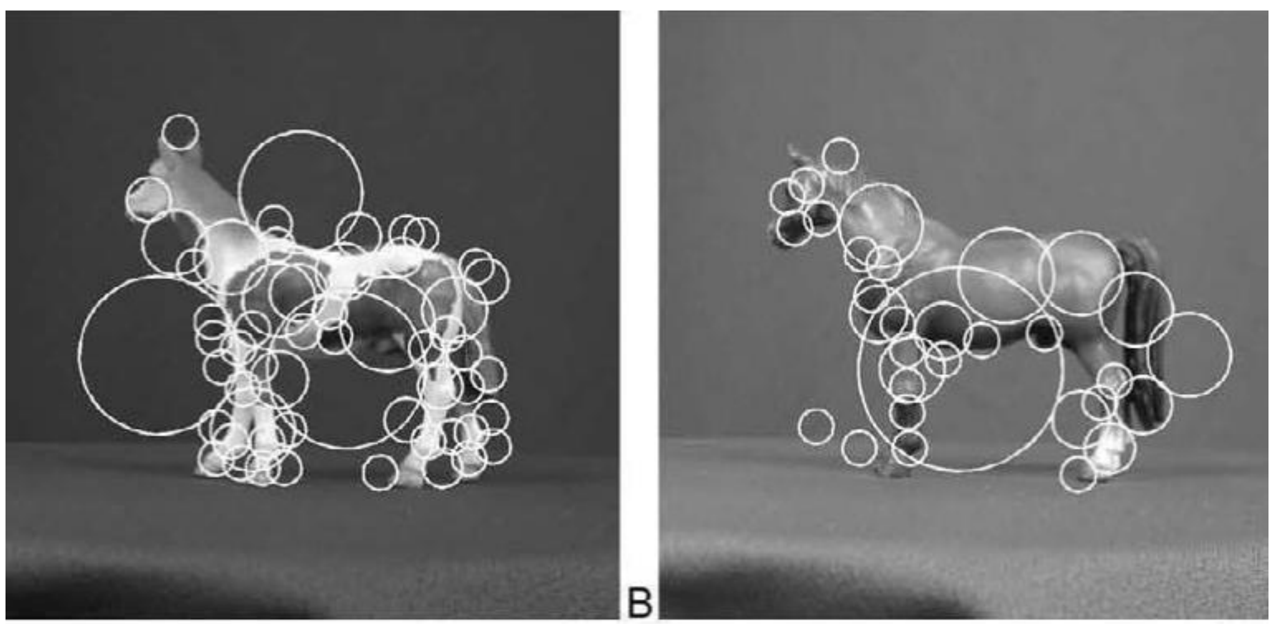
Detecting shape features



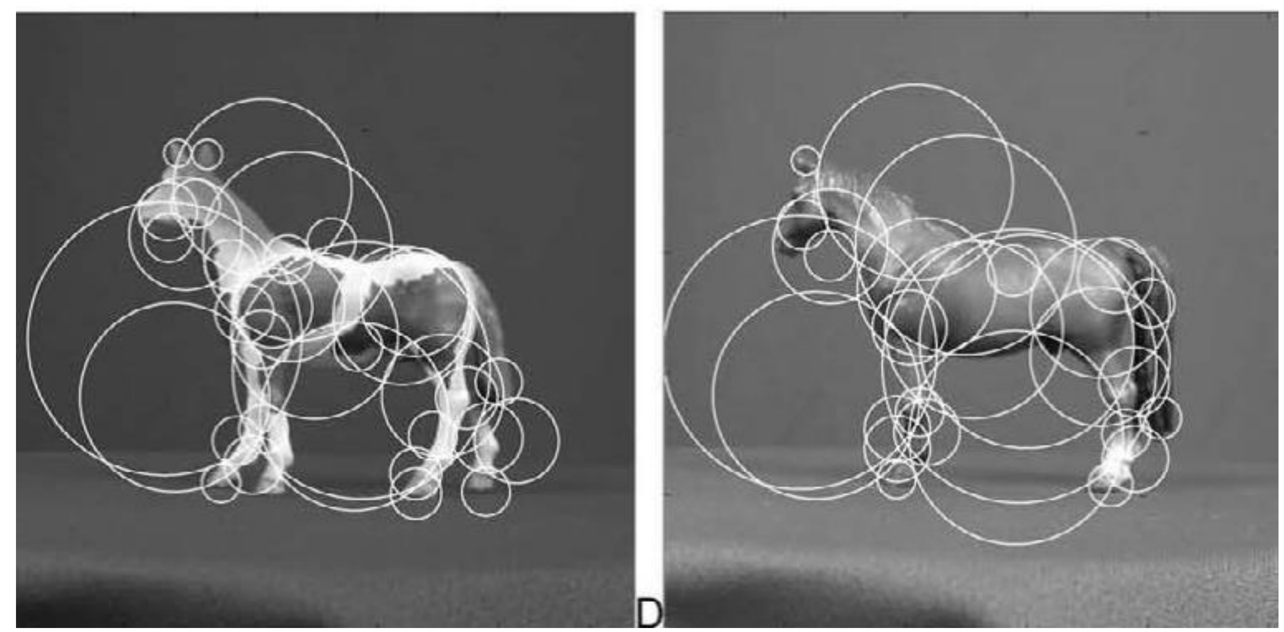
hmid, Scale-invariant shape features for recognition of object categ

Original images



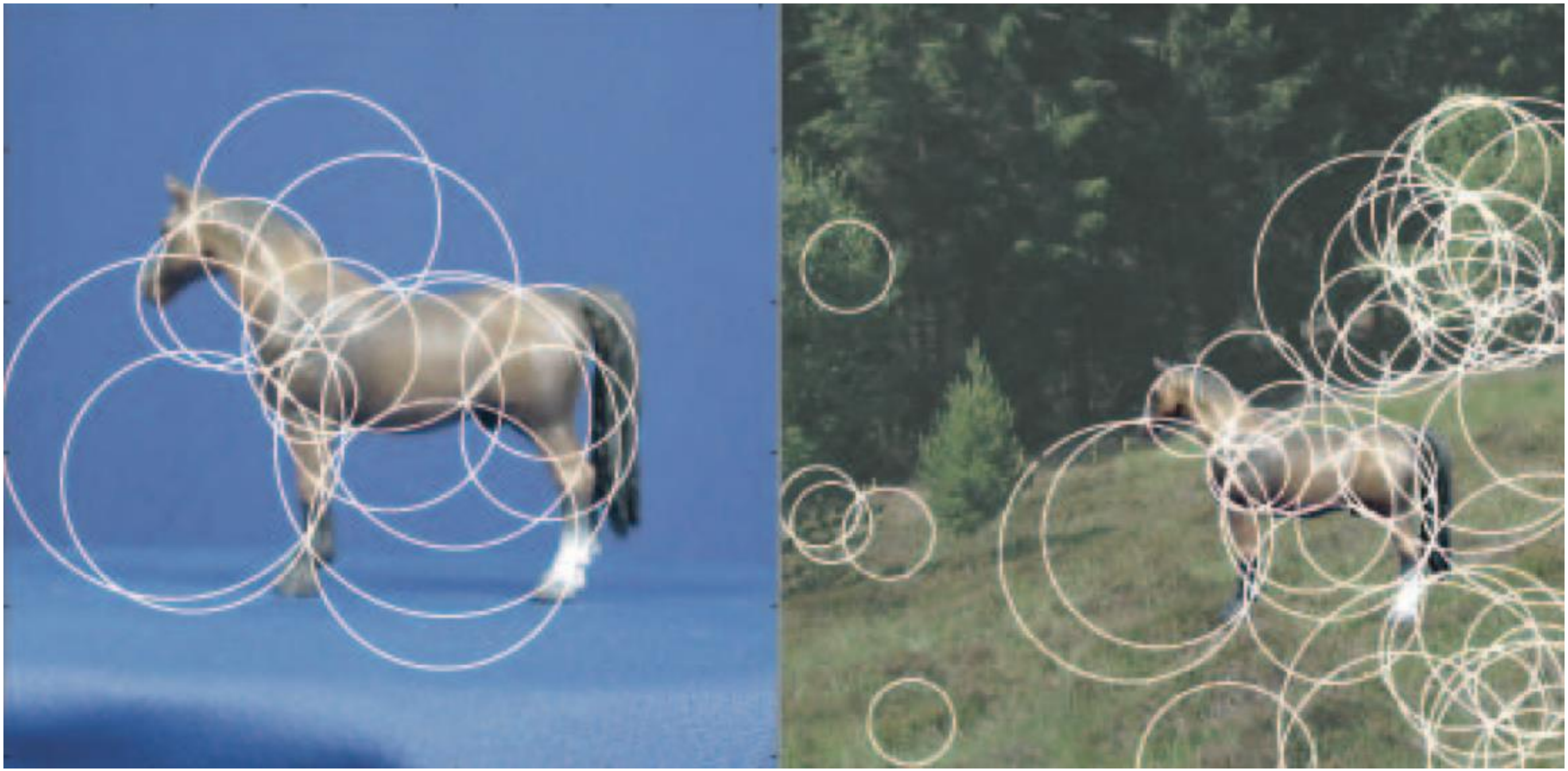


Laplacian circles



Hough-like circles

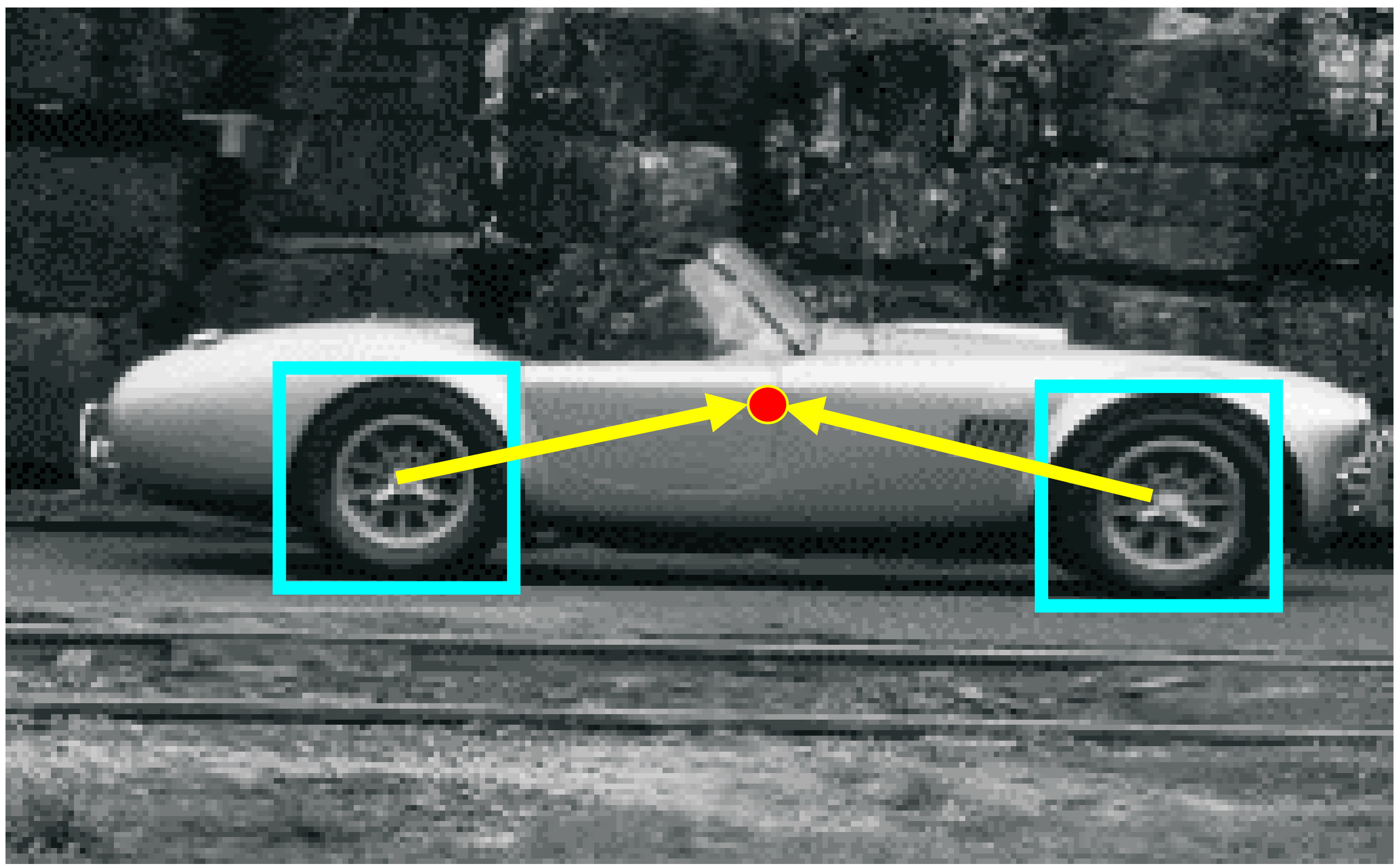
Which feature detector is more consistent?



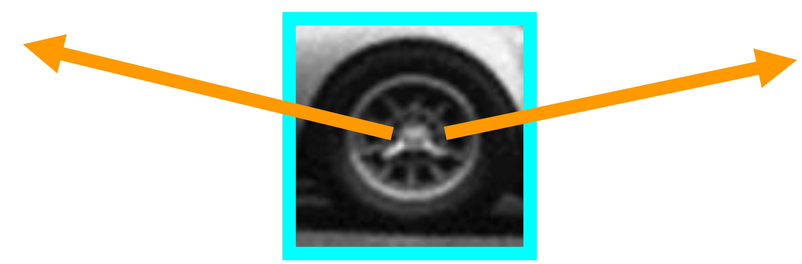
Robustness to scale and clutter

Object detection

Index displacements by “visual codeword”



training image



visual codeword with
displacement vectors

B. Leibe, A. Leonardis, and B. Schiele, Combined Object Categorization and Segmentation with an Implicit Shape Model,
ECCV Workshop on Statistical Learning in Computer Vision 2004

References

Basic reading:

- Szeliski textbook, Sections 4.2, 4.3.