Floating Point

15-213: Introduction to Computer Systems
4th Lecture, May 20, 2022

Instructor:

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Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Rounding, addition, multiplication
- Example and properties
- Floating point in C
- Summary

Activity Time

- **Activity: https://www.cs.cmu.edu/afs/cs/academic/class/15213-m22/www/activities/213_lecture4.pdf**

- **Do "Model 1" and "Model 2", then stop.**

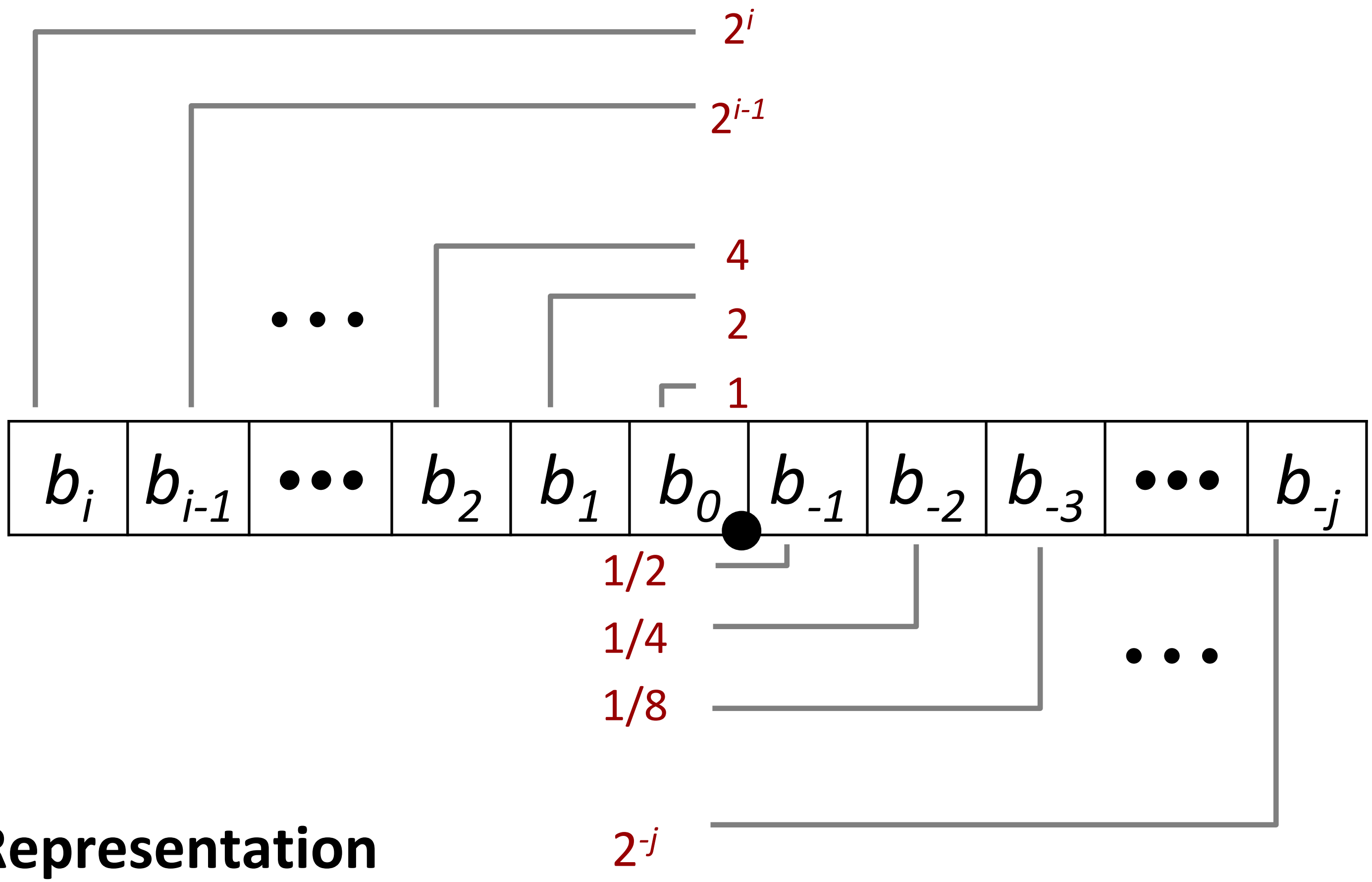
Fractional Binary Numbers

- We want to represent fractions
  - Can we assign integers to fractions? (i.e. 0 -> 0.0, 1 -> 0.1, 2 -> 0.2, …)
    - Multiplication doesn’t work

Fractional Binary Numbers

- What is 1011.101_2?

Fractional Binary Numbers



Representation

- Bits to right of "binary point" represent fractional powers of 2
- Represents rational number:

$$\sum_{k=-j}^{i} b_k \times 2^k$$

Fractional Binary Numbers: Examples

Value / Representation

Value	Representation	
5 3/4	101.11_2	= 4 + 1 + 1/2 + 1/4
2 7/8	10.111_2	= 2 + 1/2 + 1/4 + 1/8
1 7/16	1.0111_2	= 1 + 1/4 + 1/8 + 1/16

Observations

- Divide by 2 by shifting right (unsigned)
- Multiply by 2 by shifting left
- Numbers of form $0.111111\ldots_2$ are just below 1.0
  - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
  - Use notation $1.0 - \varepsilon$

Representable Numbers

Limitation #1

- Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations
- Value Representation
  - 1/3 `0.0101010101[01]`…$_2$
  - 1/5 `0.001100110011[0011]`…$_2$
  - 1/10 `0.0001100110011[0011]`…$_2$

Limitation #2

- Just one setting of binary point within the *w* bits
  - Limited range of numbers (very small values? very large?)

Activity Time

- **Activity:** https://www.cs.cmu.edu/afs/cs/academic/class/15213-m22/www/activities/213_lecture4.pdf
- **Do "Model 3" and "Model 4", then stop.**

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IEEE Floating Point

IEEE Standard 754

- Established in 1985 as uniform standard for floating point arithmetic
  - Before that, many idiosyncratic formats
- Supported by all major CPUs
- Some CPUs don’t implement IEEE 754 in full
  e.g., early GPUs, Cell BE processor

Driven by numerical concerns

- Nice standards for rounding, overflow, underflow
- Hard to make fast in hardware
  - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

Example:
$15213_{10} = (-1)^0 \times 1.1101101101101_2 \times 2^{13}$

- **Numerical Form:**

  $$(-1)^s \, M \, 2^E$$

  - **Sign bit** s determines whether number is negative or positive
  - **Significand** M normally a fractional value in range [1.0,2.0).
  - **Exponent** E weights value by power of two

- **Encoding**
  - MSB s is sign bit s
  - **exp** field encodes E (but is not equal to E)
  - **frac** field encodes M (but is not equal to M)

Precision options

- **Single precision: 32 bits**

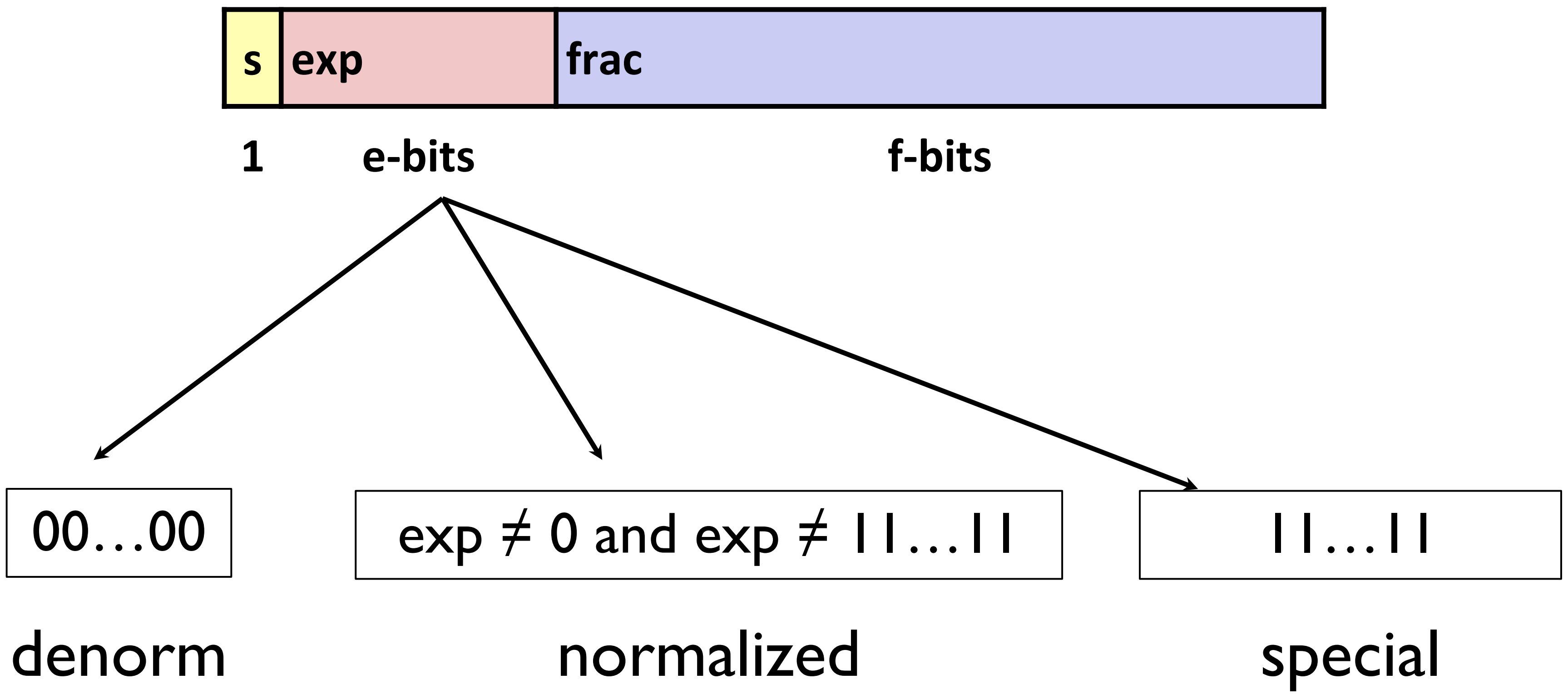
  ≈ 7 decimal digits, $10^{\pm 38}$

- **Double precision: 64 bits**

  ≈ 16 decimal digits, $10^{\pm 308}$

- **Other formats: half precision, quad precision**

Three “kinds” of floating point numbers



"Normalized" Values

$v = (-1)^s M 2^E$

- **When: exp ≠ 000…0 and exp ≠ 111…1**

- **Exponent coded as a** biased **value:** E **=** Exp **–** Bias
  - *Exp*: unsigned value of exp field
  - *Bias* = 2^{k-1} - 1, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- **Significand coded with implied leading 1:** M **=** $1.xxx…x_2$
  - xxx…x: bits of frac field
  - Minimum when frac=000…0 (M = 1.0)
  - Maximum when frac=111…1 (M = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

$v = (-1)^s M 2^E$
$E = Exp - Bias$

- **Value:** `float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- **Significand**

  M = `1.1101101101101`$_2$
  `frac`= `11011011011010000000000`$_2$

- **Exponent**

  E = 13
  $Bias$ = 127
  Exp = 140 = `10001100`$_2$

- **Result:**

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

$v = (-1)^s M 2^E$
$E = 1 - \text{Bias}$

- **Condition:** exp = 000…0
- **Exponent value:** E = **1 – Bias (instead of** E = **0 –** Bias**)**
  - **Why not 0 – Bias?**
    - **Smallest normalized value has Exp = 1**
- **Significand coded with implied leading 0:** M = **0.xxx…x$_2$**
  - `xxx…x`: bits of `frac`
- **Cases**
  - `exp` = 000…0, `frac` = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - `exp` = 000…0, `frac` ≠ 000…0
    - Numbers closest to 0.0
    - Equispaced

Special Values

- **Condition: `exp` = `111`…`1`**

- **Case: `exp` = `111`…`1`, `frac` = `000`…`0`**
  - **Represents value ∞ (infinity)**
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: `exp` = `111`…`1`, `frac` ≠ `000`…`0`**
  - **Not-a-Number (NaN)**
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

C float Decoding Example

$$v = (-1)^s M 2^E$$
$$E = \text{Exp} - \text{Bias}$$

Bias = $2^{k-1} - 1 = 127$

float: 0xC0A00000

binary: 1100 0000 1010 0000 0000 0000 0000 0000

1	1000 0001	010 0000 0000 0000 0000 0000
1	8-bits	23-bits

E = 129 -> Exp = 129 – 127 = 2 (decimal)

S = 1 -> negative number

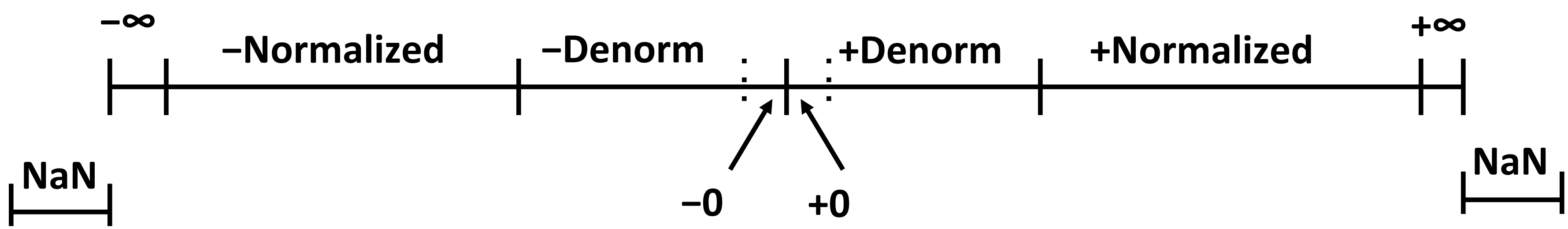
M = 1.010 0000 0000 0000 0000 0000

= 1 + 1/4 = 1.25

$v = (-1)^s M 2^E = (-1)^1 * 1.25 * 2^2 = -5$

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Visualization: Floating Point Encodings



Activity Time

- **Activity: https://www.cs.cmu.edu/afs/cs/academic/class/15213-m22/www/activities/213_lecture4.pdf**

- **Do “Model 5” and “Model 6”, then stop.**

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- IEEE floating point standard: Definition
- **Rounding, addition, multiplication**
- Example and properties
- Floating point in C
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Floating Point Operations: Basic Idea

- $x +_f y = \text{Round}(x + y)$

- $x \times_f y = \text{Round}(x \times y)$

- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	↓ $1	↓ $1	↓ $1	↓ $2	↑ –$1
Round down (−∞)	↓ $1	↓ $1	↓ $1	↓ $2	↓ –$2
Round up (+∞)	↑ $2	↑ $2	↑ $2	↑ $3	↑ –$1
Nearest Even (default)	↓ $1	↑ $2	↑ $2	↓ $2	↓ –$2

Closer Look at Round-To-Even

Default Rounding Mode

- Hard to get any other kind without dropping into assembly
- C99 has support for rounding mode management
- All others are statistically biased
  - Sum of set of positive numbers will consistently be over- or under-estimated

Applying to Other Decimal Places / Bit Positions

- When exactly halfway between two possible values
  - Round so that least significant digit is even
- E.g., round to nearest hundredth

7.8949999	7.89	(Less than half way)
7.8950001	7.90	(Greater than half way)
7.8950000	7.90	(Half way—round up)
7.8850000	7.88	(Half way—round down)

Rounding Binary Numbers

Binary Fractional Numbers

- "Even" when least significant bit is `0`
- "Half way" when bits to right of rounding position = `100`…$_2$

Examples

- Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

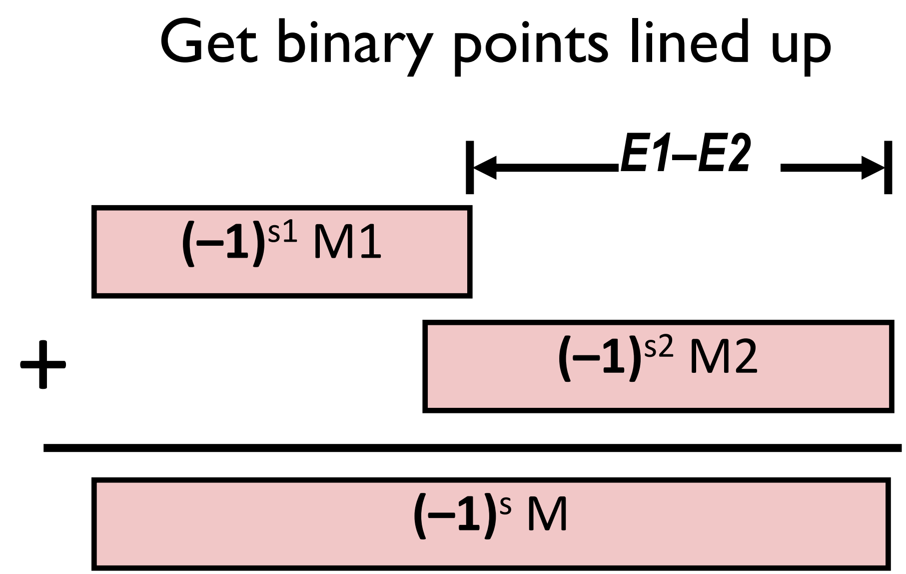
- $(-1)^{s1}$ M1 2^{E1} **x** $(-1)^{s2}$ M2 2^{E2}
- **Exact Result:** $(-1)^{s}$ M 2^{E}
  - Sign *s*: *s1 ^ s2*
  - Significand *M*: *M1* x *M2*
  - Exponent *E*: *E1* + *E2*
- **Fixing**
  - If *M* ≥ 2, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit `frac` precision
- **Implementation**
  - Biggest chore is multiplying significands

4 bit mantissa: $1.010*2^{2}$ **x** $1.110*2^{3} = 10.0011*2^{5}$

$= 1.00011*2^{6} = 1.001*2^{6}$

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1} \ \ + \ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume $E1 > E2$
- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign *s*, significand *M*:
    - Result of signed align & add
  - Exponent *E*: *E1*
- **Fixing**
  - If $M \geq 2$, shift *M* right, increment *E*
  - if $M < 1$, shift *M* left *k* positions, decrement *E* by *k*
  - Overflow if *E* out of range
  - Round *M* to fit `frac` precision



$$1.010 * 2^2 + 1.110 * 2^3 = (1.010 + 11.100) * 2^2$$
$$= 100.110 * 2^2 = 1.0011 * 2^4 = 1.010 * 2^4$$

Mathematical Properties of FP Add

Compare to those of Abelian Group

- Closed under addition? — Yes
  - But may generate infinity or NaN
- Commutative? — Yes
- Associative? — No
  - Overflow and inexactness of rounding
  - `(3.14+1e10)-1e10 = 0, 3.14+(1e10-1e10) = 3.14`
- 0 is additive identity? — Yes
- Every element has additive inverse? — Almost
  - Yes, except for infinities & NaNs

Monotonicity

- $a \geq b \Rightarrow a+c \geq b+c$? — Almost
  - Except for infinities & NaNs

Mathematical Properties of FP Mult

Compare to Commutative Ring

- Closed under multiplication? — Yes
  - But may generate infinity or NaN
- Multiplication Commutative? — Yes
- Multiplication is Associative? — No
  - Possibility of overflow, inexactness of rounding
  - Ex: `(1e20*1e20)*1e-20= inf`, `1e20*(1e20*1e-20)= 1e20`
- 1 is multiplicative identity? — Yes
- Multiplication distributes over addition? — No
  - Possibility of overflow, inexactness of rounding
  - `1e20*(1e20-1e20)= 0.0`, `1e20*1e20 – 1e20*1e20 = NaN`

Monotonicity

- $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? — Almost
  - Except for infinities & NaNs

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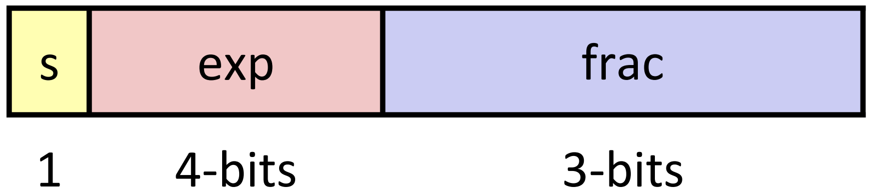
Tiny Floating Point Example



8-bit Floating Point Representation

- the sign bit is in the most significant bit
- the next four bits are the exponent, with a bias of 7
- the last three bits are the **`frac`**

Same general form as IEEE Format

- normalized, denormalized
- representation of 0, NaN, infinity

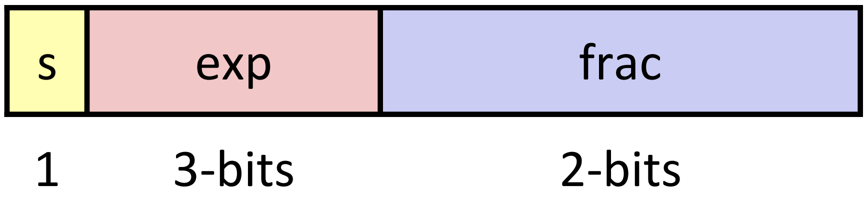
Dynamic Range (Positive Only)

$v = (-1)^s M 2^E$
n: E = Exp – Bias
d: E = 1 – Bias

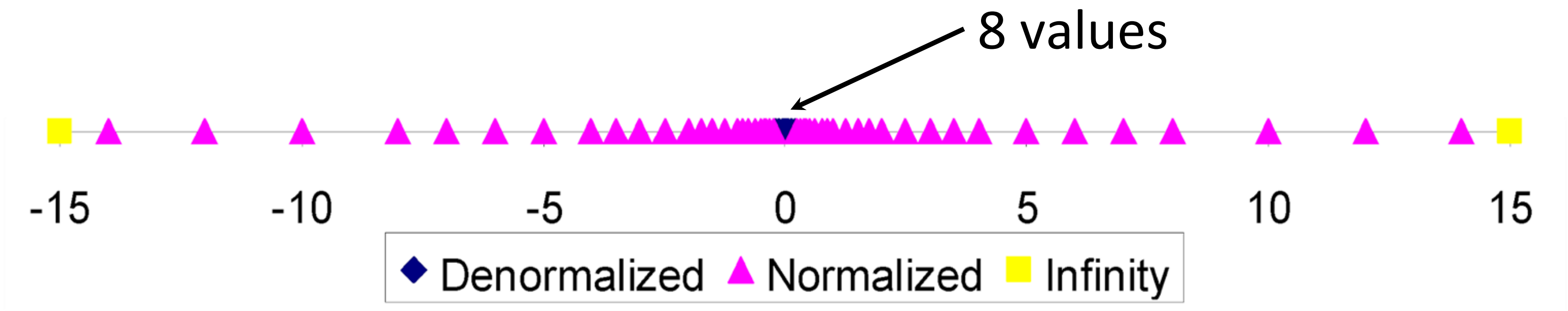
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	$(-1)^0(0+1/4)*2^{-6}$
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	$(-1)^0(1+1/8)*2^{-6}$
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$



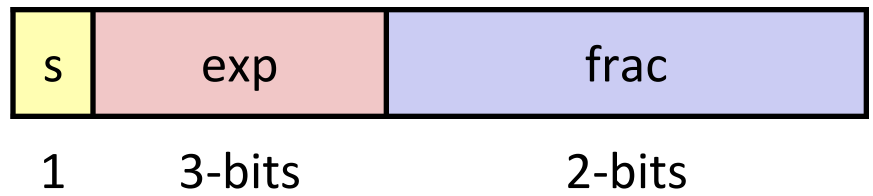
- **Notice how the distribution gets denser toward zero.**

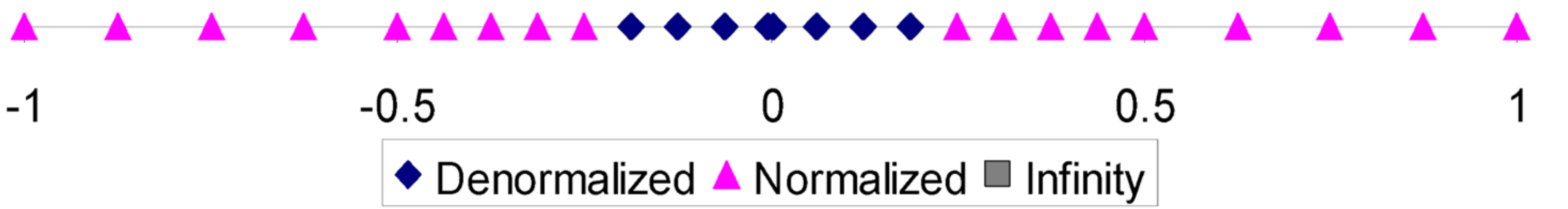


Distribution of Values (close-up view)

6-bit IEEE-like format

- e = 3 exponent bits
- f = 2 fraction bits
- Bias is 3





Special Properties of the IEEE Encoding

FP Zero Same as Integer Zero

- All bits = 0

Can (Almost) Use Unsigned Integer Comparison

- Must first compare sign bits
- Must consider −0 = 0
- NaNs problematic
  - Will be greater than any other values
  - What should comparison yield? The answer is complicated.
- Otherwise OK
  - Denorm vs. normalized
  - Normalized vs. infinity

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- **Floating point in C**
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Floating Point in C

C Guarantees Two Levels
- **`float`** single precision
- **`double`** double precision

Conversions/Casting
- Casting between **`int`**, **`float`**, and **`double`** changes bit representation
- **`double`/`float`** → **`int`**
  - Truncates fractional part
  - Like rounding toward zero
  - Not defined when out of range or NaN: Generally sets to TMin
- **`int`** → **`double`**
  - Exact conversion, as long as **`int`** has ≤ 53 bit word size
- **`int`** → **`float`**
  - Will round according to rounding mode

Activity Time

- **Activity: https://www.cs.cmu.edu/afs/cs/academic/class/15213-m22/www/activities/213_lecture4.pdf**

- **Do "Model 7", then stop.**

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

- `x == (int)(float) x` ✗
- `x == (int)(double) x` ✓
- `f == (float)(double) f` ✓
- `d == (double)(float) d` ✗
- `f == -(-f);` ✓
- `2/3 == 2/3.0` ✗
- `d < 0.0` ⇒ `((d*2) < 0.0)` ✓
- `d > f` ⇒ `-f > -d` ✓
- `d * d >= 0.0` ✓
- `(d+f)-d == f` ✗

Single precision: 32 bits

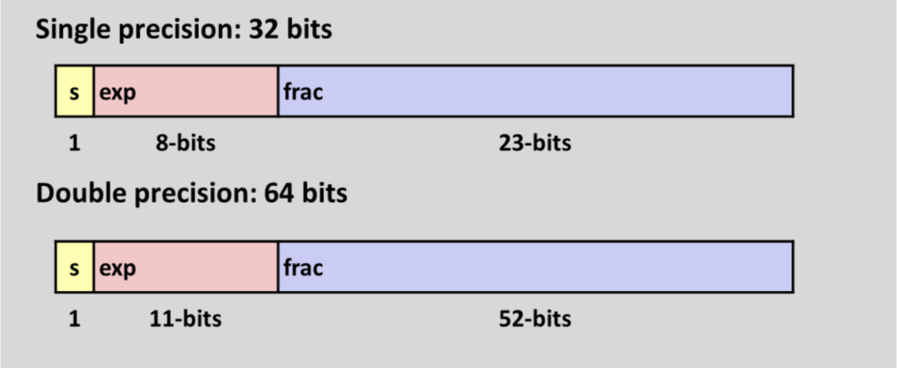
s	exp	frac
1	8-bits	23-bits

Double precision: 64 bits

s	exp	frac
1	11-bits	52-bits

Summary

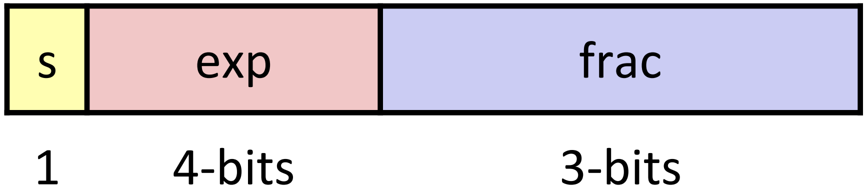
- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form M x 2^E**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers



Additional Slides

Creating Floating Point Number

- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

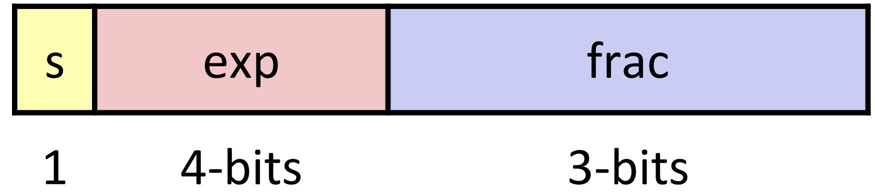


- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

```
128  10000000
 15  00001101
 33  00010001
 35  00010011
138  10001010
 63  00111111
```

Normalize



Requirement

- Set binary point so that numbers of form 1.xxxxx
- Adjust all to have leading one
  - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding

1.BBGRXXX

Guard bit: LSB of result

Sticky bit: OR of remaining bits

Round bit: 1st bit removed

Round up conditions

- Round = 1, Sticky = 1 → > 0.5
- Guard = 1, Round = 1, Sticky = 0 → Round to even

Value	Fraction	GRS	Incr?	Rounded
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

Issue

- Rounding may have caused overflow
- Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Result
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

Interesting Numbers {single,double}

Description	*exp*	*frac*	*Numeric Value*
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$

- **Zero** 00…00 00…00 0.0
- **Smallest Pos. Denorm.** 00…00 00…01 $2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
  - Single ≈ 1.4×10^{-45}
  - Double ≈ 4.9×10^{-324}
- **Largest Denormalized** 00…00 11…11 $(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
  - Single ≈ 1.18×10^{-38}
  - Double ≈ 2.2×10^{-308}
- **Smallest Pos. Normalized** 00…01 00…00 $1.0 \times 2^{-\{126,1022\}}$
  - Just larger than largest denormalized
- **One** 01…11 00…00 1.0
- **Largest Normalized** 11…10 11…11 $(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
  - Single ≈ 3.4×10^{38}
  - Double ≈ 1.8×10^{308}