42-705/505: Variational Image Processing
Assignment 1: FEM and variational methods review
Due 27 September 2004

Consider a 2D image of dimension $l \times l$ with given intensity $u_0(x, y)$. We would like to find a smoother image $u(x, y)$ by minimizing the energy

$$\frac{1}{2}\int_\Omega k(x,y)\,\nabla u \cdot \nabla u\,dx\,dy + \frac{1}{2}\int_\Omega (u-u_0)^2\;dx\,dy$$

with boundary condition $u = 0$ on all four sides. Here Ω represents the interior of the image. The parameter $k(x, y)$ is a "diffusion" coefficient whose magnitude controls how much smoothing occurs. We can easily accommodate spatially variable smoothing by permitting k to vary in (x, y).

1. Using variational calculus, derive the weak and strong forms of the boundary value problem governing this problem. The following Green's identity will prove useful: for scalar functions $a(x, y)$, $u(x, y)$, and $v(x, y)$,
$$\int_\Omega v\nabla\cdot(a\nabla u)\;dx\,dy = -\int_\Omega a\nabla u\cdot\nabla v\,dx\,dy + \int_\Gamma va\nabla u\cdot\boldsymbol{n}\,ds,$$
where $\boldsymbol{n}$ is the outward unit normal to the boundary Γ, and s is the arc length.

2. Application of the Ritz method to this problem, using the finite element approximation
$$u_h(x,y) = \sum_{i=1}^{N} u_i\phi_i(x,y),$$
produces a linear algebraic system of the form $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$. Give expressions for typical elements K_{ij} and f_i of this system. Show that $\boldsymbol{K}$ is symmetric positive semidefinite, by showing that $\boldsymbol{u}^\top\boldsymbol{K}\boldsymbol{u} \geq 0$ for $\boldsymbol{u} \neq \boldsymbol{0}$.

3. Prove that the Ritz method minimizes the square of the error measured in the energy norm for this problem; i.e. if $e(x, y) \equiv u - u_h$ is the error, then prove that the Ritz method minimizes
$$\|e(x,y)\|_E^2 = \frac{1}{2}\int_\Omega \left\{k\,\nabla e\cdot\nabla e + e^2\right\}\;dx\,dy.$$

4. Show that the Galerkin method produces the same algebraic system $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$ as does the Ritz method (and therefore the two methods produce identical approximations).

5. Use Sundance to compute finite element solutions for the case $k = 1$, $l = 4$, with given image u_0 given by
$$u_0(x,y) = \frac{4\sigma - x^2 - y^2 + 4\sigma^2}{16\pi\sigma^3}e^{-(x^2+y^2)/(16\pi\sigma^2)} \qquad -2 < (x,y) < 2.$$
Here, the exact solution $u(x, y)$ is just a Gaussian with variance $\sigma > 0$,
$$u(x,y) = \frac{1}{4\pi\sigma}e^{-(x^2+y^2)/(4\sigma)} \qquad -2 < (x,y) < 2.$$
Verify the asymptotic convergence rate estimates in the L^2 and H^1 norms for $\sigma = 0.2$ (a "smooth" Gaussian). Use both linear and quadratic elements, and tabulate the mesh size against the error in each norm. Does the error decrease as expected when you refine the mesh? Repeat for $\sigma = 0.05$ (a "rough" Gaussian). How does this case differ from the smooth Gaussian?