24-505/705: Variational Image Processing
Assignment 4: Level Set Segmentation
Due 29 November 2004

In this assignment we apply the level-set-based *active contours without edges* method of Chan and Vese to segment a brain scan image. We represent the the segementing curve as the zero isocontour of the level set function $\phi(\boldsymbol{x})$, where $\phi > 0$ inside the contour and $\phi < 0$ outside. Associated with $\phi(\boldsymbol{x})$ is a Heaviside function $H(\phi)$, where $H = 1$ inside the zero isocontour and $H = 0$ outside. We work with a smooth approximation to the Heaviside, $H_\alpha(\phi)$, which is represented according to the expression

$$H_\alpha(\phi) := 0.5[1 + \tanh(\alpha\phi)],$$

where the larger α is, the sharper the transition from 0 to 1. Given an image u_0 that we wish to segment into two regions of relatively constant intensity, the active contours without edges model seeks to minimize the following "energy:"

$$\mathcal{F} := \lambda_1 \int_\Omega H_\alpha(\phi)(u_0 - c_1(\phi))^2 \, d\boldsymbol{x} + \lambda_2 \int_\Omega (1 - H_\alpha(\phi))(u_0 - c_2(\phi))^2 \, d\boldsymbol{x} + \mu \int_\Omega \delta_\alpha(\phi))|\nabla\phi| \, d\boldsymbol{x} + \nu \int_\Omega H_\alpha(\phi) \, d\boldsymbol{x},$$

where $\delta_\alpha(\phi)$ is a smooth approximation to the delta function given by $\frac{\partial H(\phi)}{\partial \phi}$, and c_1 and c_2 are average intensities of u_0 inside and outside the zero isocontour, respectively, i.e.

$$\begin{aligned} c_1(\phi) &= \frac{\int_\Omega u_0 H_\alpha(\phi) \, d\boldsymbol{x}}{\int_\Omega H_\alpha(\phi) \, d\boldsymbol{x}}, \\ c_2(\phi) &= \frac{\int_\Omega u_0(1 - H_\alpha(\phi)) \, d\boldsymbol{x}}{\int_\Omega (1 - H_\alpha(\phi)) \, d\boldsymbol{x}}. \end{aligned}$$

1. Using the calculus of variations, derive the weak and strong form of the first order optimality condition. (Denote the variation of ϕ by $\hat{\phi}$.)

2. Show that the first order condition provides insufficient information to determine $\phi(\boldsymbol{x})$.

3. Consider regularization by a Tikhonov functional so that the regularized energy is given by

   $$\mathcal{F}_R := \mathcal{F} + \frac{\varepsilon}{2} \int_\Omega \nabla\phi \cdot \nabla\phi \, d\boldsymbol{x},$$

   with homogeneous Neumann boundary conditions on ϕ.

4. Derive the weak form of an approximate Newton method for ϕ, where the Hessian is approximated by dropping

   - terms that involve the variations of c_1 and c_2,
   - terms that involve the first and second derivatives of $\delta_\alpha(\phi)$, and
   - terms that are close to zero at the optimum (i.e. the misfit term between u_0 and the c_i).

   (Denote the Newton increment of ϕ by $\tilde{\phi}$.) Show that for $\varepsilon > 0$, the Hessian operator is positive definite.

5. Implement the approximate Newton method in `Sundance` and apply it to segment the 256×256 brain scan image (`brain.png, brain.pnm`) in the images directory of the class website. Use $\lambda_1 = \lambda_2 = 1$, $\nu = 0$, and experiment with μ to produce a satisfactory segmentation of the tumor.