Priority Queues and Heaps

15-211
Fundamental Data Structures and Algorithms

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Based on lectures given by Peter Lee, Avrim Blum, Danny Sleator, William Scherlis, Ananda Guna & Klaus Sutner

Definition of Priority Queue

Definition: An *abstract data type* to efficiently support finding the item with the highest priority across a series of operations. The basic operations are: insert, find-minimum (or maximum), and delete-minimum (or maximum).

Priority Queue

- **P-queue is a data structure that allows:**
  - **Insertion and deleteMin in O(logn)**
  - **O(1) findMin operation**
- **Applications**
  - **Operating System Design – resource allocation**
  - **Data Compression -Huffman algorithm**
  - **Discrete Event simulation**
    - **1) *Insertion* of time-tagged events (time represents a priority of an event -- low time means high priority)**
    - **(2) *Removal* of the event with the smallest time tag**
- **Implementation**
  - **Linked Lists**
  - **Using a binary Heap – a special binary tree with heap property**

Priority queue Operations

- new
  - Create a new priority queue.
- insertItem(x)
  - Insert object x into the p-queue.
- minElement()
  - Return the minimum element from the p-queue.
- removeMin()
  - Return and remove the minimum element from the p-queue.

Some questions

- How do we ensure that there is a concept of a "minimum"?
- What should happen in minElement() and removeMin() if the priority queue is empty?
- How long does it take to perform operations like insertItem(x) and removeMin()?

Comparing Objects

Comparing objects

- In Java, objects can be compared for equality:

```
public void doSomething (Person x, Person y) {
    …
    if (x == y) { … }
    …
}
```

What does it mean for two objects to be equal?

Comparing objects

- Note that this is an issue only for objects.
- Values of base type (such as int, float, char, etc.) have built-in comparison operations ==, <, <=, ...
- But javac can't possibly know how to compare objects.
    - E.g., Is a>b where a and b are objects

The *Comparable* interface

- Suppose we want to put objects of class Person into our priority queue.
- What we can do is require that *every* Person object has a method that computes whether it is bigger, smaller, or equal to another Person object.
- The JDK has a built-in interface just for this purpose, called Comparable.

The *Comparable* interface

```
public interface Comparable {
    public int compareTo (Object obj);
}
```

Returns <0 if object is less than obj,
=0 if object is equal to obj,
>0 if object is greater than obj.

The *Comparable* interface

```
public class Person implements Comparable {
    …
}

…
    Person a = new Person("Klaus");
    Person b = new Person("Peter");

    if (a.compareTo(b)) { … }
…
```

A caution

- Note that the compareTo() method takes *any* object (not just Person objects, for example).

```
Gorilla a = new Gorilla ("Freddy");
Person b  = new Person ("Matt");

if (a.compareTo(b)) { … }
…
```

- If a comparison makes no sense at all, then by convention the exception ClassCastException is raised.

Exceptional Conditions

Some questions

- How do we ensure that there is a concept of a "minimum"?
- What should happen in minElement() and removeMin() if the priority queue is empty?
- How long does it take to perform operations like insertItem(x) and removeMin()?

One possibility

- If removeMin() is applied to an empty priority queue, it could return null.
  - Pro: Simple.
  - Con: May require that all calls to removeMin() check for null.

An alternative

- A common approach is to raise an *exception*.

```
public class PriorityQueue {
…
    public int removeMin() throws
        PriorityQueueEmptyException {
        …
        if (isEmpty())
            throw new PriorityQueueException(
                "Empty priority queue in removeMin()");
        …
    …
}
```

Exception classes

```
public class PriorityQueueException
    extends Exception {

    public PriorityQueueException() {
        super();
    }

    public PriorityQueueException(String s) {
        super(s);
    }
}
```

More on this later…

Implementation p-queue

Using Binary Trees

- We expect the *find* and *insert* operations to take $O(\log N)$ time.
- In fact, operations like *find* take time *d*, where *d* is the depth of the item in the tree.
- Since the depth is not expected to be larger than log(N), and each step down the tree requires constant time, we get $O(\log N)$. More later..

Analysis of BSTs

- If all insertion sequences are equally likely (that is, the insertion order is random), then on average a binary search tree has depth $O(\log_2(N))$.
- Define D(N) to be the sum of the depths of all nodes in a tree with N nodes.
  - D(1) = 0.

Analysis, cont'd

- For a tree with N>1 nodes:
  - i nodes in left subtree,
  - N-i-1 nodes in the right subtree,
  - and one node at root. (for 0<=i<N)

Analysis, cont'd

- So,
  - D(N) = D(i) + D(N-i-1) + N - 1
- The average value of D(i) and D(N-i-1) is

$$\sum_{j=0}^{N} D(j)/N$$

- So, D(N) $= 2(\sum_{j=0}^{N} D(j))/N + N - 1$

Analysis, cont'd

- There are methods for solving such *recurrence equations.*
- We shall see later that this equation has the solution $O(N \times \log N)$.
- Thus, on average the depth of any particular node is $O(\log N)$.

Priority queue implementation

- Linked list
  - removeMin $O(1)$ } *or* { $O(N)$
  - insertItem $O(N)$ | $O(1)$
- Heaps

	avg	worst
deleteMin	$O(\log N)$	$O(\log N)$
insert	2.6	$O(\log N)$
special case: build	$O(N)$	$O(N)$

i.e., insert*N

Heaps

- A binary tree.
- Representation invariant
  1. Structure property
     - **Complete binary tree**
  2. Heap order property
     - **Parent keys less than children keys**

Heaps

- Representation invariant
  1. Structure property
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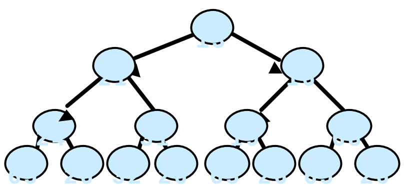
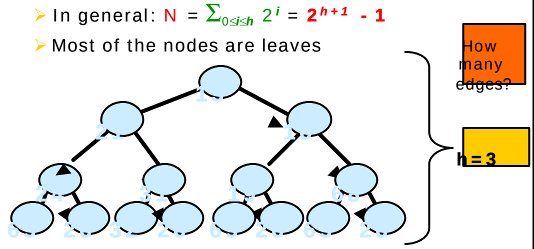
     - *Hence*: efficient compact representation
  2. Heap order property
     - **Parent keys less than children keys**
     - *Hence*: rapid insert, findMin, and deleteMin
       - $O(\log(N))$ for insert and deleteMin
       - $O(1)$ for findMin

Perfect binary trees



Perfect binary trees

- How many **nodes**?
  - N = $2^4 - 1$ = 15
  - In general: N = $\sum_{0 \le i \le h} 2^i = 2^{h+1} - 1$
  - Most of the nodes are leaves

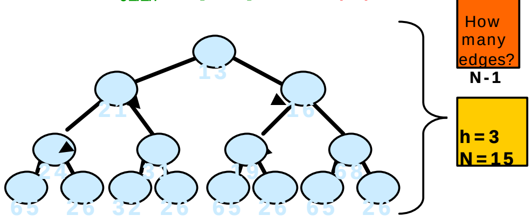


Perfect binary trees

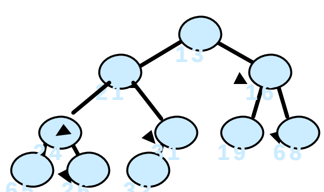
- What is the **sum of the heights**?

$$S = \sum_{0 \le i \le h} 2^i (h-i) = O(N)$$

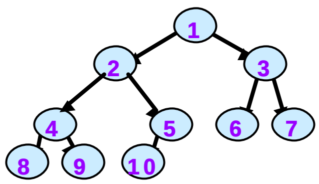
prove this



Complete binary trees

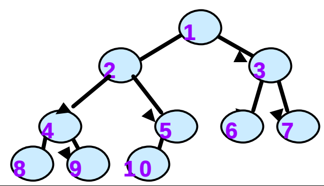


Complete binary trees



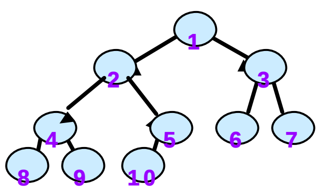
Representing complete binary trees

- Linked structures? *No!*
- Arrays!



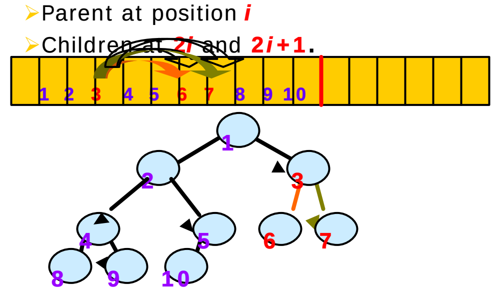
Representing complete binary trees

- Arrays
  - Parent at position ***i***
  - Children at **2*i*** and **2*i*+1**.



Representing complete binary trees

- Arrays (1-based)
  - Parent at position ***i***
  - Children at **2*i*** and **2*i*+1**.



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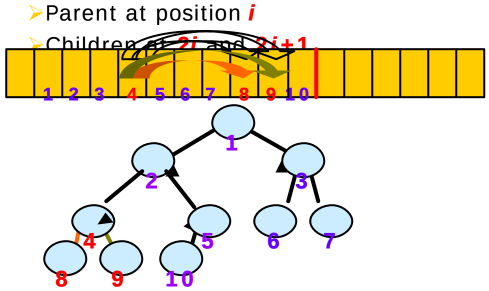

Representing complete binary trees

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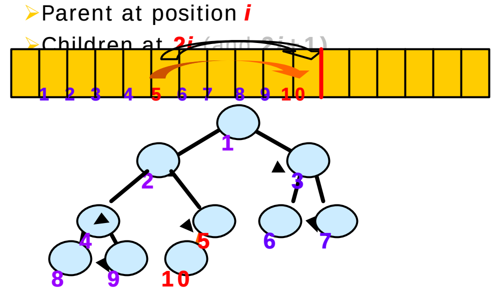


Representing complete binary trees

- Arrays (1-based)
  - Parent at position ***i***
  - Children at **2*i*** and **2*i*+1**.

```
public class BinaryHeap {
    private Comparable[] heap;
    private int size;
    public BinaryHeap(int capacity) {
        size=0;
        heap = new Comparable[capacity+1];
    }
    . . .
```

Representing complete binary trees

- Arrays
  - Parent at position ***i***
  - Children at **2*i*** and **2*i*+1**.
- Example: find the leftmost child

```
int left=1;
for(; left<size; left*=2);
return heap[left/2];
```

- Example: find the rightmost child

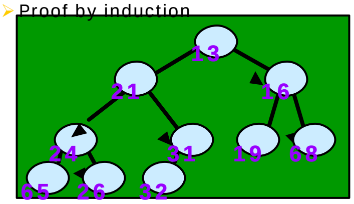
```
int right=1;
for(; right<size; right=right*2+1);
return heap[(right-1)/2];
```

Heaps

- Representation invariant
  1. Structure property
     - **Complete binary tree**
     - *Hence*: efficient compact representation
  2. Heap order property
     - **Parent keys less than children keys**
     - *Hence*: rapid insert, findMin, and deleteMin
       - *O*(log(*N*)) for insert and deleteMin
       - *O*(1) for findMin

The heap order property

- *Each parent is less than each of its children.*
- *Hence*: Root is less than every other node.
  - Proof by induction



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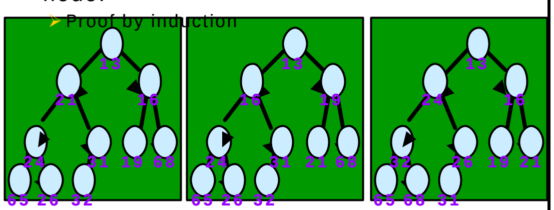


Operating with heaps

Representation invariant:

- All methods **must:**
  1. Produce complete binary trees
  2. Guarantee the heap order property
- All methods **may assume**
  1. The tree is initially complete binary
  2. The heap order property holds

findMin ()

- The code

```
public boolean isEmpty() {
   return size == 0;
   }
public Comparable findMin() {
   if(isEmpty()) return null;
   return heap[1];
   }
```

- Does not change the tree
  - Trivially preserves the invariant

insert (Comparable x)

- Process
  1. Create a "*hole*" at the next tree cell for **x**.
     heap[size+1]
     This preserves the completeness of the tree.
  2. *Percolate* the hole *up* the tree until the heap order property is satisfied.
     This assures the heap order property is satisfied.

insert (Comparable x)

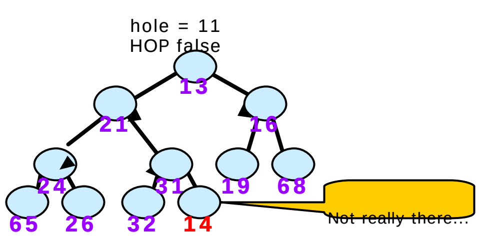
- Process
  1. Create a "*hole*" at the next tree cell for x.
     heap[size+1]
     This preserves the ***completeness*** of the tree *assuming it was complete to begin with.*
  2. *Percolate* the hole *up* the tree until the heap order property is satisfied.
     This assures the ***heap order property*** is satisfied *assuming it held at the outset*.

Percolation up

```
public void insert(Comparable x)
  throws Overflow
  {
    if(isFull()) throw new Overflow();
    int hole = ++size;
    for(;
        hole>1 && x.compareTo(heap[hole/2])<0;
        hole/=2)
      heap[hole] = heap[hole/2];
    heap[hole] = x;
    }
```

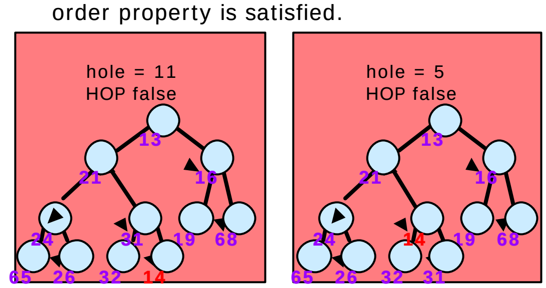
Percolation up

- Bubble the hole ***up the tree*** until the heap order property is satisfied.



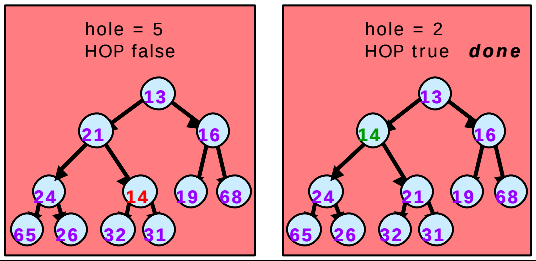
Percolation up

- Bubble the hole up the tree until the heap order property is satisfied.



Percolation up

- Bubble the hole up the tree until the heap order property is satisfied.



deleteMin()

```
/**
* Remove the smallest item from the priority queue.
* @return the smallest item, or null, if empty.
*/
public Comparable deleteMin( )
{
  if(isEmpty()) return null;
  Comparable min = heap[1];
  heap[1] = heap[size--];
  percolateDown(1);
  return min;
 }
```

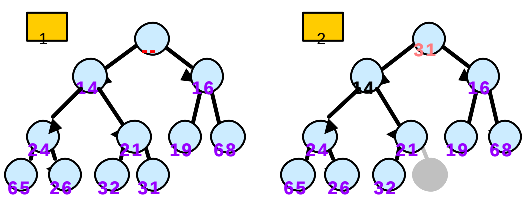
Grab min element

Temporarily place last element at top

!!!

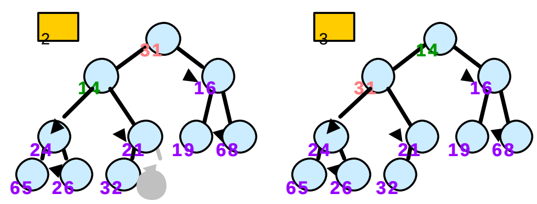
Percolation down

- Bubble the transplanted leaf value ***down the tree*** until the heap order property is satisfied.



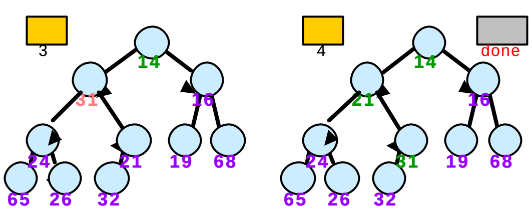
Percolation down

- Bubble the transplanted leaf value ***down the tree*** until the heap order property is satisfied.



Percolation down

- Bubble the transplanted leaf value ***down the tree*** until the heap order property is satisfied.



deleteMin ()

- Observe that both components of the **representation invariant** are preserved by **deleteMin**.
  1. Completeness
     -
     -
  2. Heap order property

deleteMin ()

- Observe that both components of the **representation invariant** are preserved by **deleteMin**.
  1. Completeness
     - The **last cell** (**heap[size]**) **is vacated**, providing the value to percolate down.
     - This assures that the tree remains complete.
  2. Heap order property

deleteMin ()

- Observe that both components of the **representation invariant** are preserved by **deleteMin**.
  1. Completeness
     - The **last cell** (**heap[size]**) **is vacated**, providing the value to percolate down.
     - This assures that the tree remains complete.
  2. Heap order property
     - The percolation algorithm assures that the orphaned value is **relocated to a suitable position**.

buildHeap

- Equivalent to a sequence of inserts

```
for(int i=0;i<N;i++)
    insert(input[i]);
```

- Two steps:
  1. Fill the array (in no particular order).
  2. percolateDown, bottom up.

```
for(int i=size/2; i>0; i--)
    percolateDown(i);
```

  - This does a linear number of comparisons

Thursday

- We will talk about Greedy Algorithms
- Read Chapter 7
- HW3 is online now
  - **You must read homework assignment before recitation tomorrow**
- Start Early
- Ask Questions Early
- Go to Recitation Tomorrow