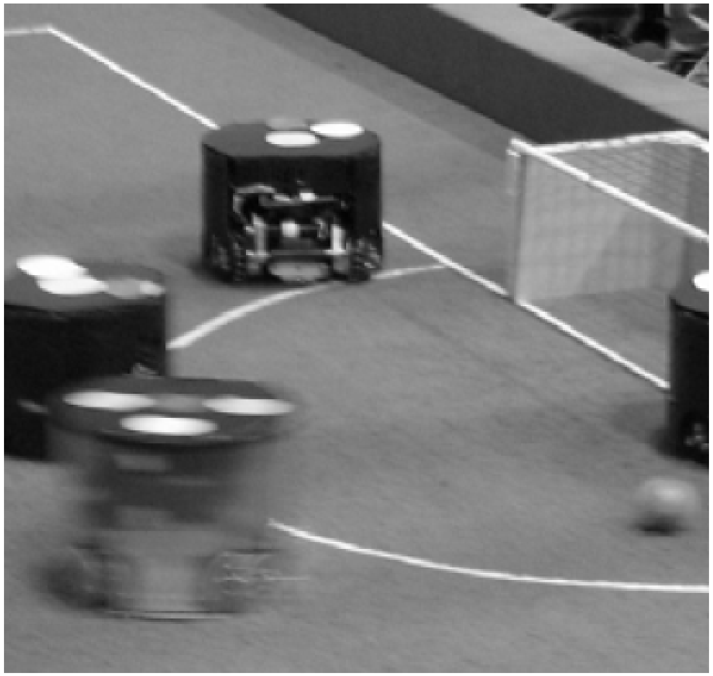

Chapter 4
Dynamics

Part 1

4.1 Moving Coordinate Systems

4.2 Kinematics of WMRs

Outline

- 4.1 Moving Coordinate Systems
  - 4.1.1 Context of Measurement
  - 4.1.2 Change of Reference Frame
  - 4.1.3 Example: Attitude Stability Margin
  - 4.1.4 Recursive Transformation of States of Motion
  - Summary
- 4.2 Kinematics of Wheeled Mobile Robots

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4.1.1 Context of Measurement

- Police says "you were going 30 m/s southbound on I-279".
- Any measurement in physics lacks meaning without several contextual elements:
  - a unit system (e.g. meters, seconds)
  - a number system (e.g. base 10 weighted positional)
  - a coordinate system (e.g. directions north, east)
  - a reference frame to which the measurement is ascribed (e.g. your car).
  - a reference frame with respect to which the measurement is made (e.g. the earth).

Coordinate Systems

- Conventions for representation of physical quantities.
  - Any set of quantities that fixes all degrees of freedom of a system.
  - Cartesian systems represent vectors by projections onto three orthogonal axes.
  - The Euler angle definition expresses the three degrees of rotational freedom.
- Mathematical laws alone govern conversion from coordinate system to coordinate system.
- Conversion of coordinates does not change the magnitude or direction of a measurement - only the way you describe it.

Reference Frames

- Allows us to reconcile the differences in observations of the same property of an object by two observers with different states of motion.
  - Laws of physics are necessary to convert among frames of reference (i.e to predict a measurement made by one observer from those of another).
  - A reference frame is a real physical body. The state of motion of such a body distinguishes it from other frames of reference.
- A phenomenon, when observed from one frame of reference, may or may not look the same when observed from a second frame of reference.
- Two frames are equivalent with respect to a measurement when the measurement is the same in both frames. If they are not equivalent, a method of converting between the frames of reference is often available.

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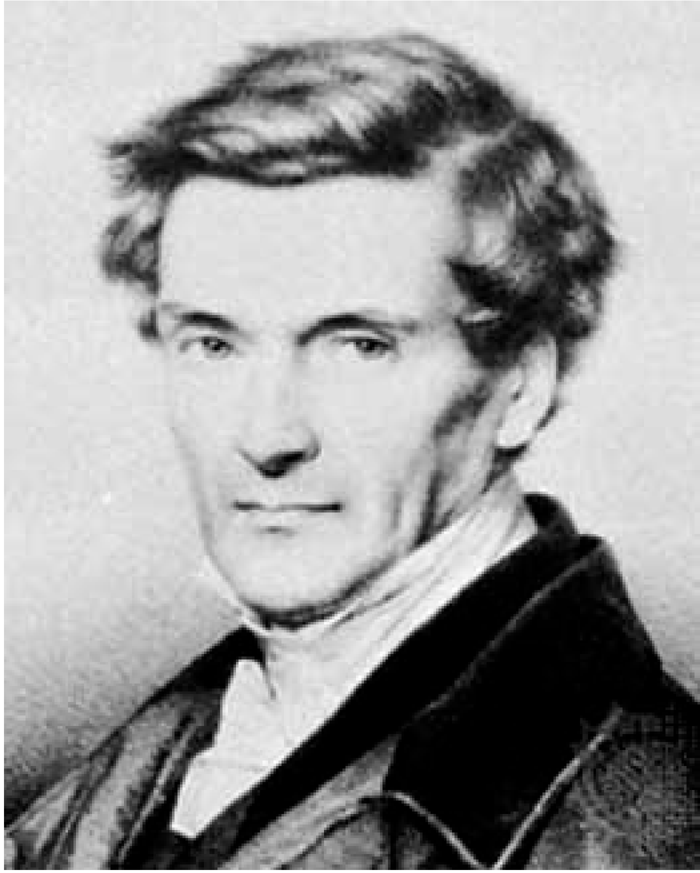
Coriolis

- Cauchy recommended him to a job at Ecole Polytechnique.
- Introduced the terms 'work' and 'kinetic energy' with their present scientific meaning
- Best remembered for *“Sur les équations du mouvement relatif des systèmes de corps* (1835)” which introduced the Coriolis force.
- Also wrote a mathematical theory of billiards!



Gaspard-Gustave de Coriolis
1792-1843 Paris, France

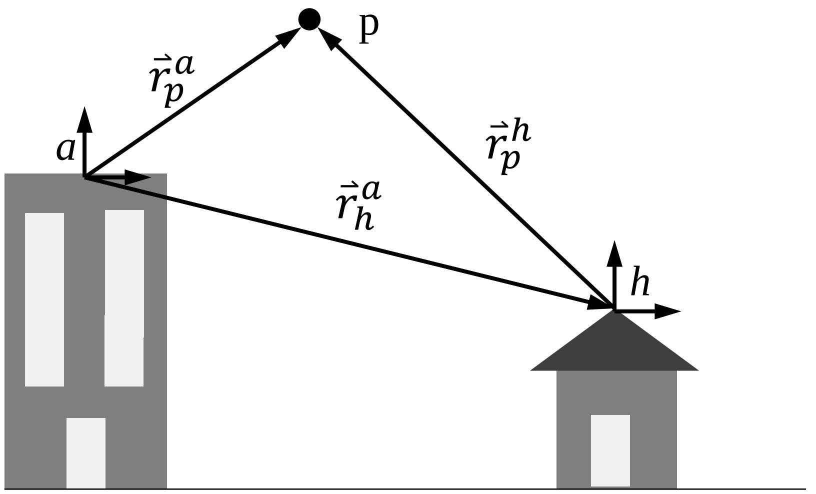
4.1.2.1 Mutually Stationary Frames

- Motion of particle can be expressed wrt either frame.
- Position vectors are related:

$$\vec{r}_p^{\,a} = \vec{r}_p^{\,h} + \vec{r}_h^{\,a}$$

- Differentiate wrt time:

$$\boxed{\vec{v}_p^{\,a} = \vec{v}_p^{\,h}}$$



4.1.2.2 Galilean Transformation

Translating (Const V) Frames

- Motion of particle can be expressed wrt either frame.
- Position vectors are related:

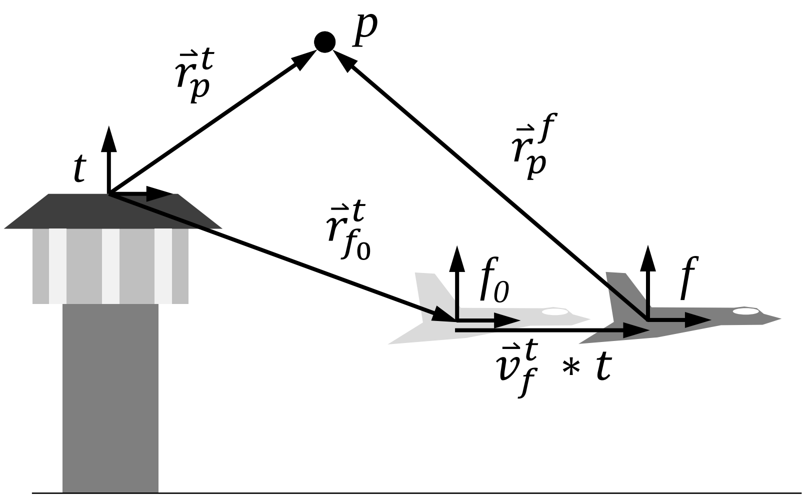
$$\vec{r}_p^{\,t} = \vec{r}_p^{\,f} + \vec{r}_{f0}^{\,t} + \vec{v}_f^{\,t} \cdot t$$

- Differentiate wrt time:

$$\vec{v}_p^{\,t} = \vec{v}_p^{\,f} + \vec{v}_f^{\,t}$$

- Frames are equivalent for acceleration:

$$\boxed{\vec{a}_p^{\,t} = \vec{a}_p^{\,f}}$$

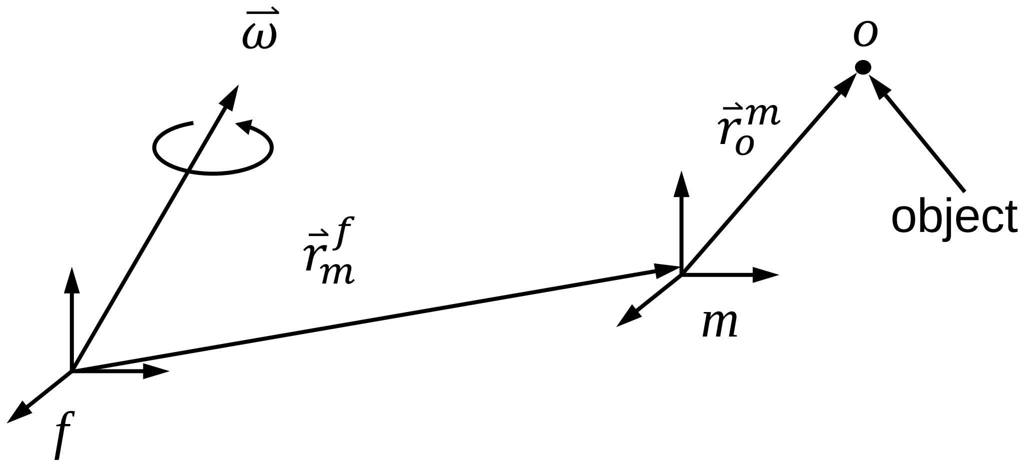


Relates particle velocity for an observer in the control tower to that observed in the airplane.

Called "Galilean Transformation"

4.1.2.3 Rotating Frames

- When two frames are rotating with respect to each other, something must be accelerating.
- Let ω denote angular velocity of m frame with respect to f frame.
- Lets predict measurements of observer in f given those of observer in m.



4.1.2.3 Coriolis Equation

- Coriolis Equation (aka Transport Theorem) relates derivatives of same vector by both observers.

$$\left(\frac{d\vec{u}}{dt}\right)_f = \left(\frac{d\vec{u}}{dt}\right)_m + \vec{\omega} \times \vec{u}$$

- $\vec{u}$ is any vector (position, velocity, acceleration, force)
- $\vec{\omega}$ is angular velocity of moving frame wrt fixed one.

4.1.2.4 Velocity Transformation

- Positions add by vector addition.

$$\vec{r}_o^{\,f} = \vec{r}_m^{\,f} + \vec{r}_o^{\,m}$$

- Time derivative in fixed frame.

$$\left.\frac{d}{dt}\right|_f (\vec{r}_o^{\,f}) = \left.\frac{d}{dt}\right|_f (\vec{r}_m^{\,f} + \vec{r}_o^{\,m}) = \left.\frac{d}{dt}\right|_f (\vec{r}_m^{\,f}) + \left.\frac{d}{dt}\right|_f (\vec{r}_o^{\,m})$$

Note:

$$\left.\frac{d}{dt}\right|_x (\vec{r}_y^{\,x}) = \vec{v}_y^{\,x}$$

- Apply Coriolis Equation to 2nd term on right.

$$\vec{v}_o^{\,f} = \vec{v}_m^{\,f} + \vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m} + \vec{v}_o^{\,m}$$

4.1.2.4 General Velocity Relation

Extra Component 1

Extra Component 2

$$\vec{v}_o^{\,f} = \vec{v}_m^{\,f} + \vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m} + \vec{v}_o^{\,m}$$

$\vec{v}_o^{\,f}$: velocity of particle relative to fixed observer

$\vec{v}_o^{\,m}$: velocity of particle relative to moving observer

$\vec{v}_m^{\,f}$: linear velocity of moving observer relative to fixed

$\vec{\omega}_m^{\,f}$: angular velocity of moving observer relative to fixed

$\vec{r}_o^{\,m}$: position of particle relative to moving observer

4.1.2.5 General Acceleration Relation

- Apply to velocity relation

From Last Slide

$$\vec{v}_o^{\,f} = \vec{v}_m^{\,f} + \vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m} + \vec{v}_o^{\,m}$$

$$\left.\frac{d}{dt}\right|_f (\vec{v}_o^{\,f}) = \left.\frac{d}{dt}\right|_f (\vec{v}_m^{\,f} + \vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m} + \vec{v}_o^{\,m})$$

$$\left.\frac{d}{dt}\right|_f (\vec{v}_o^{\,f}) = \left.\frac{d}{dt}\right|_f (\vec{v}_m^{\,f}) + \left.\frac{d}{dt}\right|_f (\vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m}) + \left.\frac{d}{dt}\right|_f (\vec{v}_o^{\,m})$$

$$\vec{a}_o^{\,f} = \vec{a}_m^{\,f} + \vec{\alpha}_m^{\,f} \times \vec{r}_o^{\,m} + \vec{\omega}_m^{\,f} \times [\vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m}] + 2\vec{\omega}_m^{\,f} \times \vec{v}_o^{\,m} + \vec{a}_o^{\,m}$$

4.1.2.5 General Acceleration Relation

$$\vec{a}_o^{\,f} = \vec{a}_m^{\,f} + \vec{\alpha}_m^{\,f} \times \vec{r}_o^{\,m} + \vec{\omega}_m^{\,f} \times [\vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m}] + 2\vec{\omega}_m^{\,f} \times \vec{v}_o^{\,m} + \vec{a}_o^{\,m}$$

Extra Component 1

Extra Component 2

Extra Component 3

Extra Component 4

$\vec{a}_o^{\,f}$: acceleration of particle relative to fixed observer

$\vec{a}_o^{\,m}$: acceleration of particle relative to moving observer

$\vec{a}_m^{\,f}$: Einstein acceleration (of moving frame wrt fixed)

$2\vec{\omega}_m^{\,f} \times \vec{v}_o^{\,m}$: Coriolis acceleration

$\vec{\alpha}_m^{\,f} \times \vec{r}_o^{\,m}$: Euler acceleration

$\vec{\omega}_m^{\,f} \times [\vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m}]$: Centripetal acceleration

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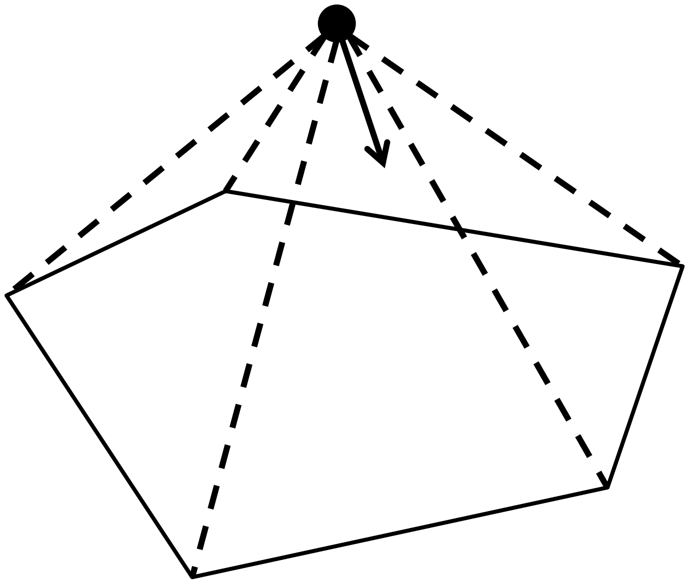
4.1.3 Attitude Stability Margin Estimation

- Staying upright
- Keeping contact with terrain.
- Important when:
  - Lifting heavy loads
  - Turning at speed
  - Operating on sloped terrain
- Many vehicles do one or more of these things.

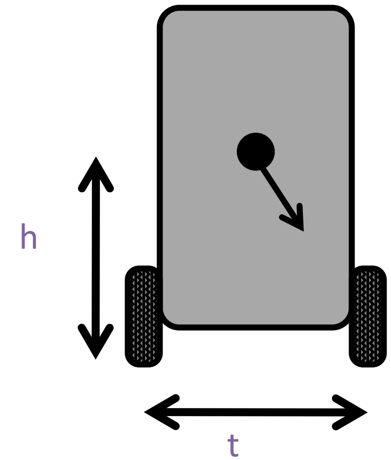
Liftoff Criterion

- Preventing liftoff will prevent rollover.
- For liftoff, issue is the direction of the net noncontact force vector acting at the cg
  - Any unbalanced moment about any tipover axis.



4.1.3.1 Proximity to Wheel Liftoff

- Place a 2 axis accel right at the cg.
- BUT:
  - CG may not be accessible.
  - It may move due to:
    - articulations
    - payload changes
    - changing human passengers.

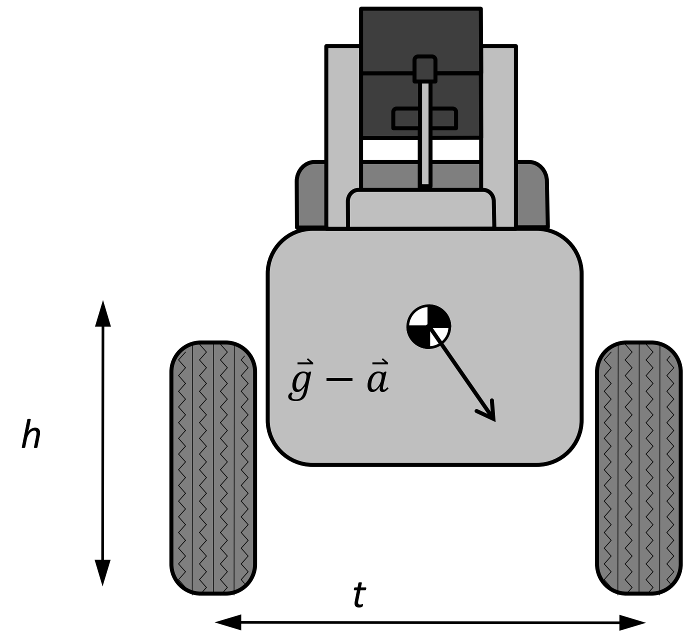


4.1.3.1 Proximity to Wheel Liftoff

- Define the specific force acting at the cg:

$$\vec{t} = \vec{a} - \vec{g}$$

- An accelerometer can measure specific force but it cannot usually be placed at the cg.
  - Therefore transform it.



Vehicle is viewed from rear when executing a hard left turn. For the specific force direction shown, the inner (left) wheels are about to lift off.

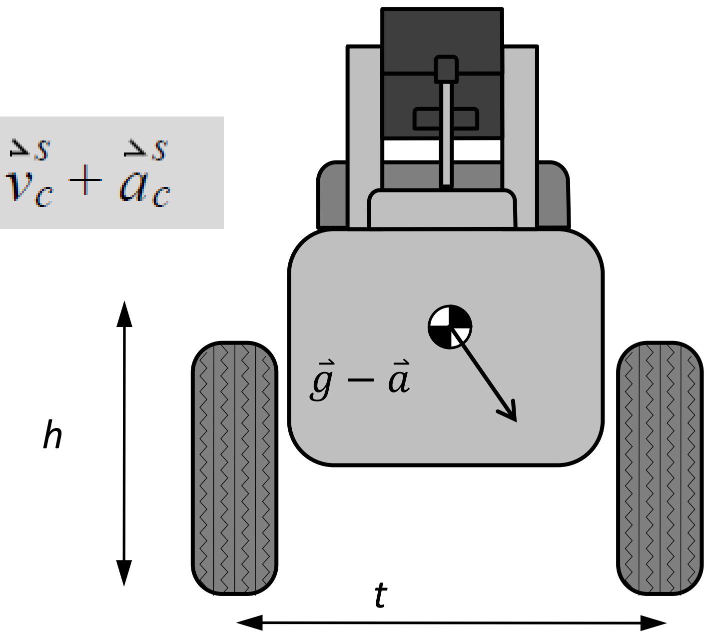
4.1.3.1 Proximity to Wheel Liftoff

Transformation

- Use earlier result:
  - f frame in inertial (i) frame.
  - m frame is sensor (s) frame.
  - o frame is cg (c) frame
- Simply substitute the letters to get:

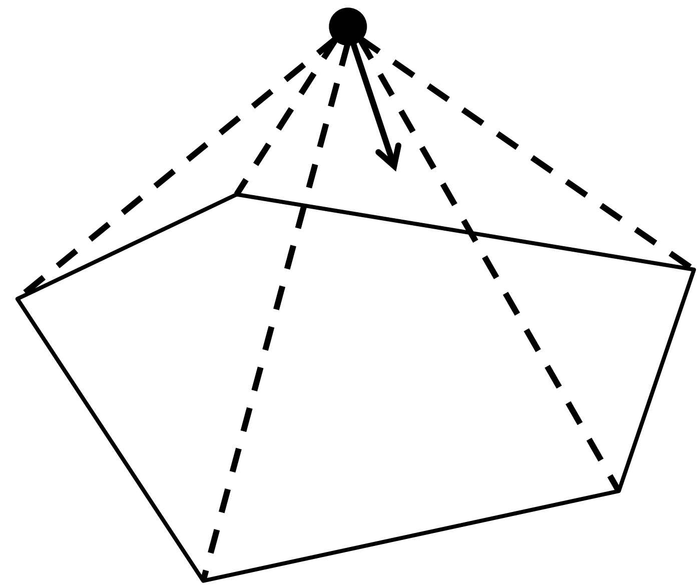
$$\vec{a}_c^{\,i} = \vec{a}_s^{\,i} + \vec{\alpha}_s^{\,i} \times \vec{r}_c^{\,s} + \vec{\omega}_s^{\,i} \times [\vec{\omega}_s^{\,i} \times \vec{r}_c^{\,s}] + 2\vec{\omega}_s^{\,i} \times \vec{v}_c^{\,s} + \vec{a}_c^{\,s}$$

- Subtract the gravity vector from both sides to get the real (s) and transformed (c) accelerometer readings



4.1.3.3 Computational Requirements

- Geometry
  - Location of the center of gravity (cg).
  - Convex polygon formed by the wheel contact points.
- Forces
  - Gravity vector.
  - Inertial forces being experienced due to accelerated motion.



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4.1.4 Recursive Transformation of States of Motion

- Suppose we have a sequence of frames numbered:
  - 1,2…k,k+1…n
- Their motions can be related by the results just derived…

$$\underline{r}_o^k = \underline{r}_{k+1}^k + \underline{r}_o^{k+1}$$

Eqn 4.18

$$\underline{v}_o^k = \underline{v}_{k+1}^k + \underline{\omega}_{k+1}^k \times \underline{r}_o^{k+1} + \underline{v}_o^{k+1}$$

$$\underline{a}_o^k = \underline{a}_{k+1}^k + \underline{\alpha}_{k+1}^k \times \underline{r}_o^{k+1} + \underline{\omega}_{k+1}^k \times [\underline{\omega}_{k+1}^k \times \underline{r}_o^{k+1}] + 2\underline{\omega}_{k+1}^k \times \underline{v}_o^{k+1} + \underline{a}_o^{k+1}$$

4.1.4.1 Conversion to Coordinatized Form

- Recall: $$\vec{\omega} \times \vec{u} \Rightarrow \underline{\omega} \times \underline{u} = [\underline{\omega}]^{\times}\underline{u} = -[\underline{u}]^{\times}\underline{\omega}$$
- Where:

$$[\underline{u}]^{\times} \cong \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix}$$

- Also, we can write the transport theorem in matrix form:

$$\left(\frac{d\underline{u}}{dt}\right)_f = \left(\frac{d\underline{u}}{dt}\right)_m + [\underline{\omega}]^{\times}\underline{u}$$

4.1.4.1 Conversion to Coordinatized Form

- Now, use this to rewrite Equation 4.18.

$$\underline{r}_o^k = \underline{r}_{k+1}^k + \underline{r}_o^{k+1}$$

$$\underline{v}_o^k = \begin{bmatrix} I & | & -[\underline{r}_o^{k+1}]^{\times} \end{bmatrix} \begin{bmatrix} \underline{v}_{k+1}^k \\ \underline{\omega}_{k+1}^k \end{bmatrix} + \underline{v}_o^{k+1}$$

$$\underline{a}_o^k = \begin{bmatrix} I & | & -[\underline{r}_o^{k+1}]^{\times} \end{bmatrix} \begin{bmatrix} \underline{a}_{k+1}^k \\ \underline{\alpha}_{k+1}^k \end{bmatrix} + \begin{bmatrix} [\underline{\omega}_{k+1}^k]^{\times\times} & | & 2[\underline{\omega}_{k+1}^k]^{\times} & | & I \end{bmatrix} \begin{bmatrix} \underline{r}_o^{k+1} \\ \underline{v}_o^{k+1} \\ \underline{a}_o^{k+1} \end{bmatrix}$$

4.1.4.1 Conversion to Coordinatized Form

- Define the notation:

$$\underline{\rho} \cong \left[\underline{r}\ ;\ \underline{v}\ ;\ \underline{a}\right] \qquad \underline{x} \cong \left[\underline{r}\ ;\ \underline{v}\ ;\ \underline{\omega}\ ;\ \underline{a}\ ;\ \underline{\alpha}\right]$$

- Then, previous position, velocity and acceleration results can be written as:

$$\underline{\rho}_o^k = H(\underline{\rho}_o^{k+1})\underline{x}_{k+1}^k + \Omega(\underline{x}_{k+1}^k)\underline{\rho}_o^{k+1}$$

- Typically
  - $\underline{x}_{k+1}^{k}$ represent articulations
  - $\underline{\rho}_o^k$ represent state of motion of each frame

4.1.4.2 General Recursive Forms

Velocity Transform

- Consider just the velocity transform:

$$\underline{v}_o^k = \begin{bmatrix} I \mid -[\underline{r}_o^{k+1}]^{\times} \end{bmatrix} \begin{bmatrix} \underline{v}_{k+1}^k \\ \underline{\omega}_{k+1}^k \end{bmatrix} + \underline{v}_o^{k+1} \qquad \textbf{(4.23)}$$

- We will write this compactly as:

$$\underline{v}_o^k = H(\underline{r}_o^{k+1})\underline{\dot{x}}_{k+1}^k + \underline{v}_o^{k+1} \qquad \textbf{(4.24)}$$

- Where H is defined as it occurs in Equation 4.23.

4.1.4.2 General Recursive Forms

Acceleration Transform

- Consider just the acceleration transform:

$$\underline{a}_o^k = \left[I \mid -[\underline{r}_o^{k+1}]^\times \right] \begin{bmatrix} \underline{a}_{k+1}^k \\ \underline{\alpha}_{k+1}^k \end{bmatrix} + \left[[\underline{\omega}_{k+1}^k]^{\times\times} \mid 2[\underline{\omega}_{k+1}^k]^\times \right] \begin{bmatrix} \underline{r}_o^{k+1} \\ \underline{v}_o^{k+1} \end{bmatrix} + \underline{a}_o^{k+1} \quad \textbf{(4.25)}$$

- We will write this compactly as:

$$\underline{a}_o^k = H(\underline{r}_o^{k+1})\ddot{\underline{x}}_{k+1}^k + \Omega(\underline{\omega}_{k+1}^k)\underline{\rho}_o^{k+1} + \underline{a}_o^{k+1} \quad \textbf{(4.26)}$$

- Where H, Ω etc. are defined as they occur in Equation 4.25.

4.1.4.3 The Articulated Wheel

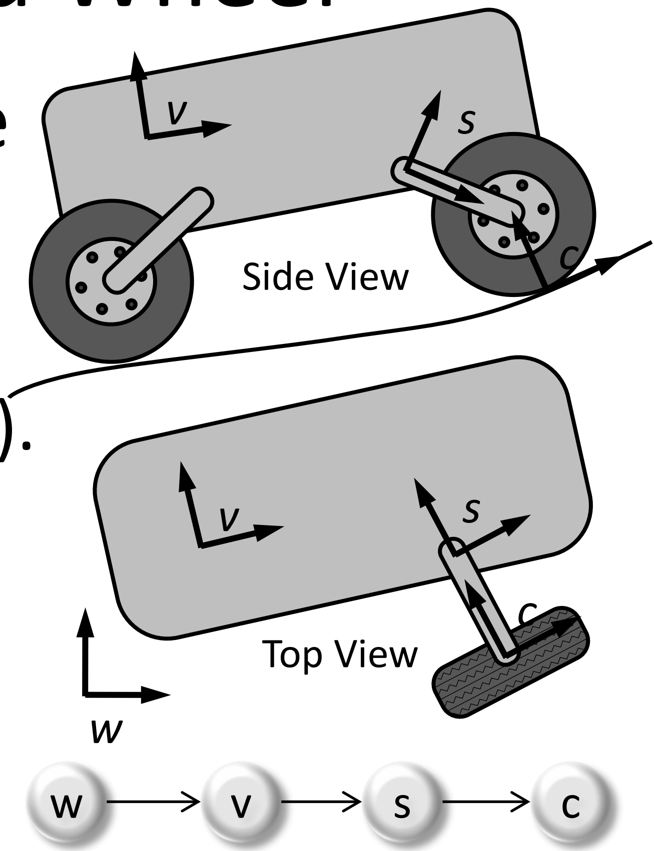
- Let n=k+1 have a maximum value of 2.
  - Two intermediate frames relate zeroth frame (0) to object frame (o).
- Write Equation 4.24 twice:

$$\underline{v}_o^0 = H(\underline{r}_o^1)\underline{\dot{x}}_1^0 + \underline{v}_o^1$$

$$\underline{v}_o^1 = H(\underline{r}_o^2)\underline{\dot{x}}_2^1 + \underline{v}_o^2$$

- Substitute second into first:

$$\underline{v}_o^0 = H(\underline{r}_o^1)\underline{\dot{x}}_1^0 + H(\underline{r}_o^2)\underline{\dot{x}}_2^1 + \underline{v}_o^2 \qquad \textbf{(4.27)}$$

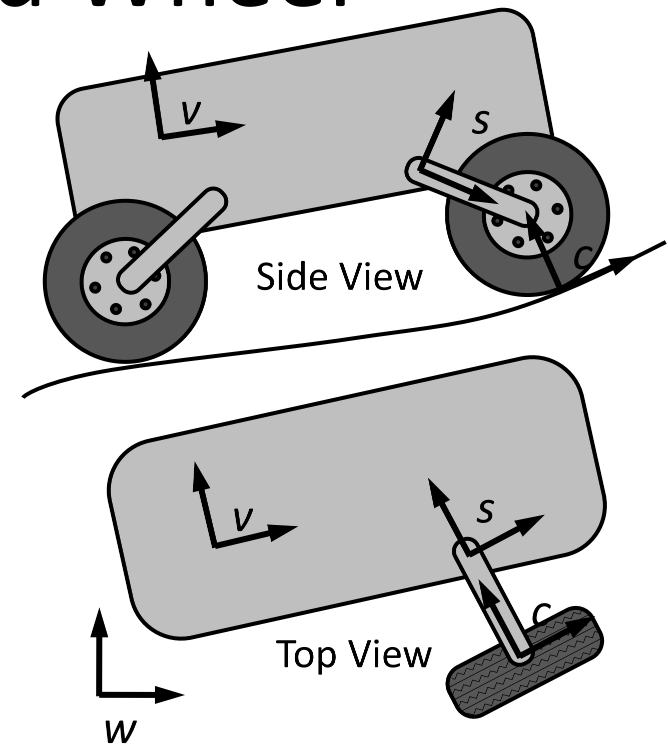


4.1.4.3 The Articulated Wheel

- For acceleration, write Equation 4.26 twice:

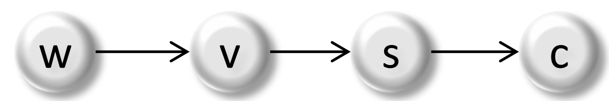
$$\underline{a}_o^0 = H(\underline{r}_o^1)\ddot{\underline{x}}_1^0 + \Omega(\underline{\omega}_1^0)\underline{\rho}_o^1 + \underline{a}_o^1$$

$$\underline{a}_o^1 = H(\underline{r}_o^2)\ddot{\underline{x}}_2^1 + \Omega(\underline{\omega}_2^1)\underline{\rho}_o^2 + \underline{a}_o^2$$



- Substitute second into first:

$$\underline{a}_o^0 = H(\underline{r}_o^1)\ddot{\underline{x}}_1^0 + \Omega(\underline{\omega}_1^0)\underline{\rho}_o^1 + H(\underline{r}_o^2)\ddot{\underline{x}}_2^1 + \Omega(\underline{\omega}_2^1)\underline{\rho}_o^2 + \underline{a}_o^2$$



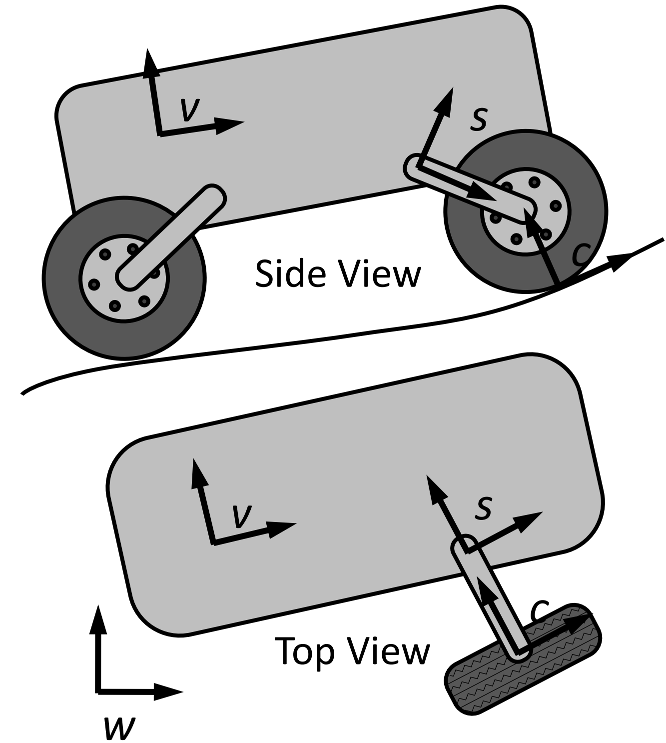
(4.29)

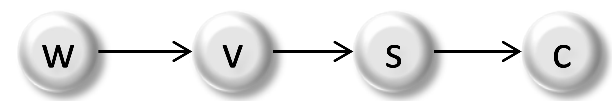
4.1.4.4 Velocity Transforms for Articulated Wheel

- Let these frames be defined:
  - 0: world frame (w)
  - 1: body frame (v)
  - 2: suspension/steering (s)
  - o: wheel contact pt (c)
- Then equation 4.27 becomes:

$$\underline{v}_c^w = H(\underline{r}_c^v)\underline{\dot{x}}_v^w + H(\underline{r}_c^s)\underline{\dot{x}}_s^v + \underline{v}_c^s$$

$$\underline{v}_c^w = \left[I \mid -[\underline{r}_c^v]^\times \right] \begin{bmatrix} \underline{v}_v^w \\ \underline{\omega}_v^w \end{bmatrix} + \left[I \mid -[\underline{r}_c^s]^\times \right] \begin{bmatrix} \underline{v}_s^v \\ \underline{\omega}_s^v \end{bmatrix} + \underline{v}_c^s \qquad (4.30)$$





Articulated Wheel Velocity Kinematics

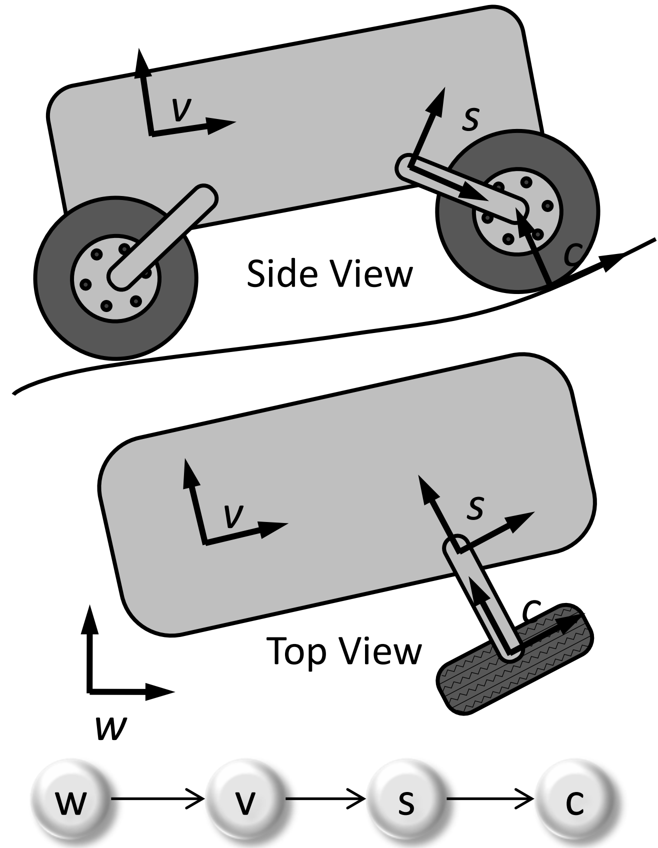
4.1.4.4 Velocity Transforms for Articulated Wheel

- Under the same substitutions Equation 4.29 becomes:

$$\underline{a}_c^w = H(\underline{r}_c^v)\,\underline{\ddot{x}}_v^w + \Omega(\underline{\omega}_v^w)\underline{\rho}_c^v + H(\underline{r}_c^s)\,\underline{\ddot{x}}_s^v + \Omega(\underline{\omega}_s^v)\underline{\rho}_c^s + \underline{a}_c^s$$

$$\underline{a}_c^w = \left[I \;|\; -[\underline{r}_c^v]^\times \right] \begin{bmatrix} \underline{a}_v^w \\ \underline{\alpha}_v^w \end{bmatrix} + \left[[\underline{\omega}_v^w]^{\times\times} \;|\; 2[\underline{\omega}_v^w]^\times \right] \begin{bmatrix} \underline{r}_c^v \\ \underline{v}_c^v \end{bmatrix}$$

$$+ \left[I \;|\; -[\underline{r}_c^s]^\times \right] \begin{bmatrix} \underline{a}_s^v \\ \underline{\alpha}_s^v \end{bmatrix} + \left[[\underline{\omega}_s^v]^{\times\times} \;|\; 2[\underline{\omega}_s^v]^\times \right] \begin{bmatrix} \underline{r}_c^s \\ \underline{v}_c^s \end{bmatrix} + \underline{a}_c^s \qquad \textbf{(4.31)}$$



Articulated Wheel Acceleration Kinematics

Outline

- 4.1 Moving Coordinate Systems
  - 4.1.1 Context of Measurement
  - 4.1.2 Change of Reference Frame
  - 4.1.3 Example: Attitude Stability Margin
  - 4.1.4 Recursive Transformation of States of Motion
  - Summary
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Summary

- Measurements require a context to be precisely meaningful.
- A coordinate system and a reference frame are different.
- Two frames may or may not be equivalent for measuring velocity and higher derivatives.
- The Coriolis Equation provides a general coordinate-free mechanism to differentiate any vector attached to a moving frame of reference.
  - General transformations of position, velocity, and acceleration can be derived from it.
- Basic stability margin estimation is based on lift-off and an acceleration transformation.
- A two step recursion is sufficient to model the velocity and acceleration kinematics relating the wheel contact point motion to the motion and articulation of a WMR.

Outline

- 4.1 Moving Coordinate Systems
- 4.2 Kinematics of Wheeled Mobile Robots
  - 4.2.1 Aspects of Rigid Body Motion
  - 4.2.2 WMR Velocity Kinematics for Fixed Contact Point
  - 4.2.3 Common Steering Configurations
  - Summary

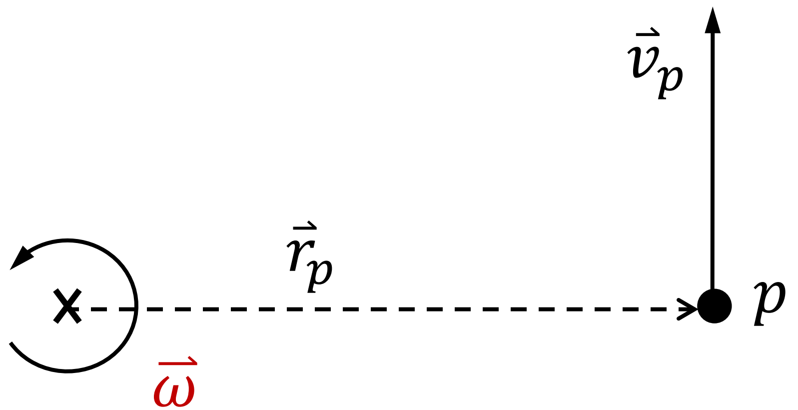
4.2.1.1 Pure Rotation of a Point

- Suppose particle p moves in **pure rotation.**

$$\vec{r}_p = r[\cos(\psi)\hat{i} + \sin(\psi)\hat{j}]$$

- Differentiate:

$$\vec{v}_p = r\omega[-\sin(\psi)\hat{i} + \cos(\psi)\hat{j}]$$

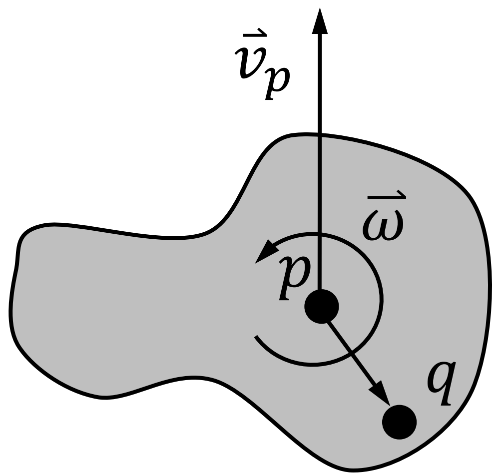


- … orthogonal to $\vec{r}_p$
- In other "words" $\vec{v}_p = \vec{\omega} \times \vec{r}_p$
- Magnitudes are: $v_p = r_p \omega$ Eqn A

Pure Rotation of a Particle on a Body

- Now consider a particle p on a body executing general planar motion.
  - Not pure rotation…
- The body has some some V and ω at the position of p.
- For some world frame W, define the ratio.

$$r = v_p^w / \omega$$



Pure Rotation of a Particle on a Body

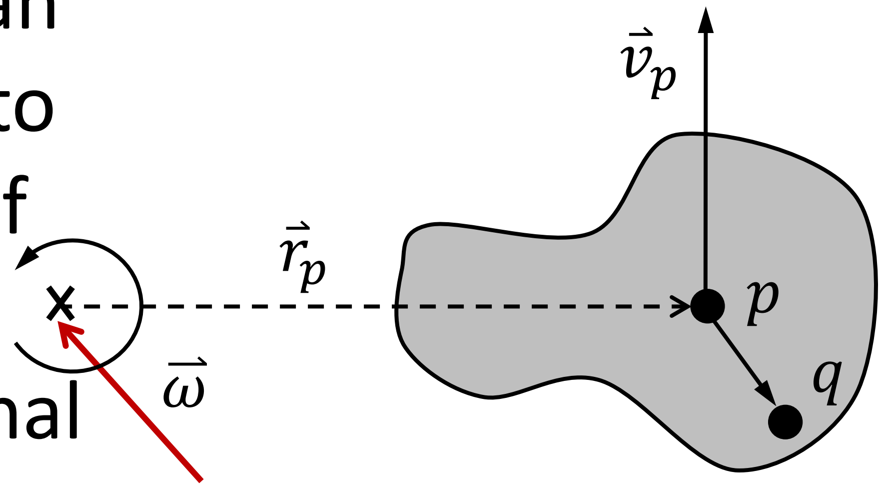
- Rewrite this as:

$$v_p^w = r\omega$$

$$r = v_p^w / \omega$$

From Last Slide

- This is Eqn A! Hence we can interpret 'r' as the radius to an instantaneous center of rotation (ICR) for point p located r units in orthogonal direction to v.



- In vector terms:

$$\vec{v}_p^{\,icr} = \vec{\omega} \times \vec{r}_p^{\,icr}$$

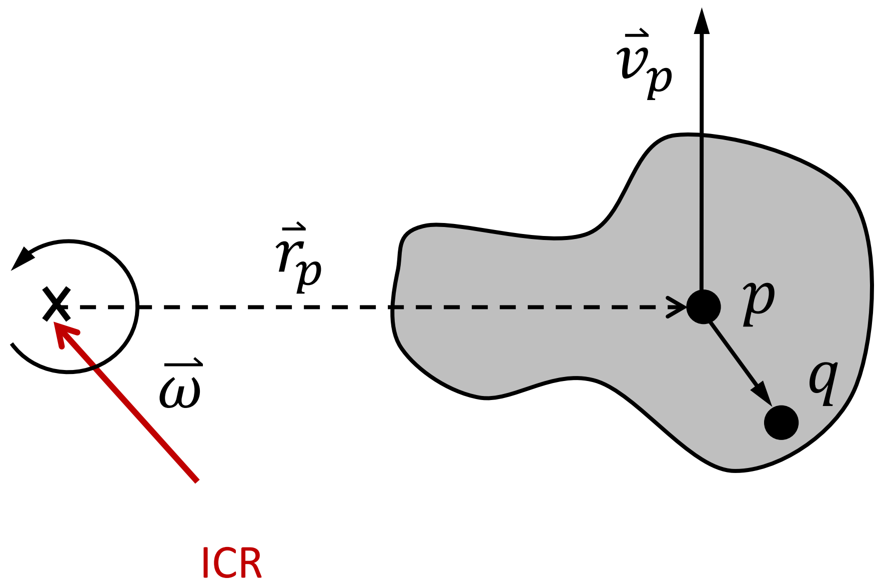
Pure Rotation of a Particle on a Body

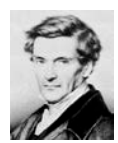
- Consider a neighboring point q:

$$\vec{r}_q^{\,icr} = \vec{r}_p^{\,icr} + \vec{r}_q^{\,p}$$

- Differentiate in the world frame:

$$\left.\frac{d}{dt}\right|_w (\vec{r}_q^{\,icr}) = \left.\frac{d}{dt}\right|_w (\vec{r}_p^{\,icr}) + \left.\frac{d}{dt}\right|_w (\vec{r}_q^{\,p})$$



- But the last term is: 

$$\left.\frac{d}{dt}\right|_w (\vec{r}_q^{\,p}) = \underbrace{\left.\frac{d}{dt}\right|_b (\vec{r}_q^{\,p})}_{0} + \vec{\omega} \times \vec{r}_q^{\,p} = \vec{\omega} \times \vec{r}_q^{\,p}$$

Pure Rotation of a Particle on a Body

- Substituting:

$$\frac{d}{dt}\bigg|_w (\vec{r}_q^{\,p}) = \frac{d}{dt}\bigg|_b (\vec{r}_q^{\,p}) + \vec{\omega} \times \vec{r}_q^{\,p} = \vec{\omega} \times \vec{r}_q^{\,p}$$

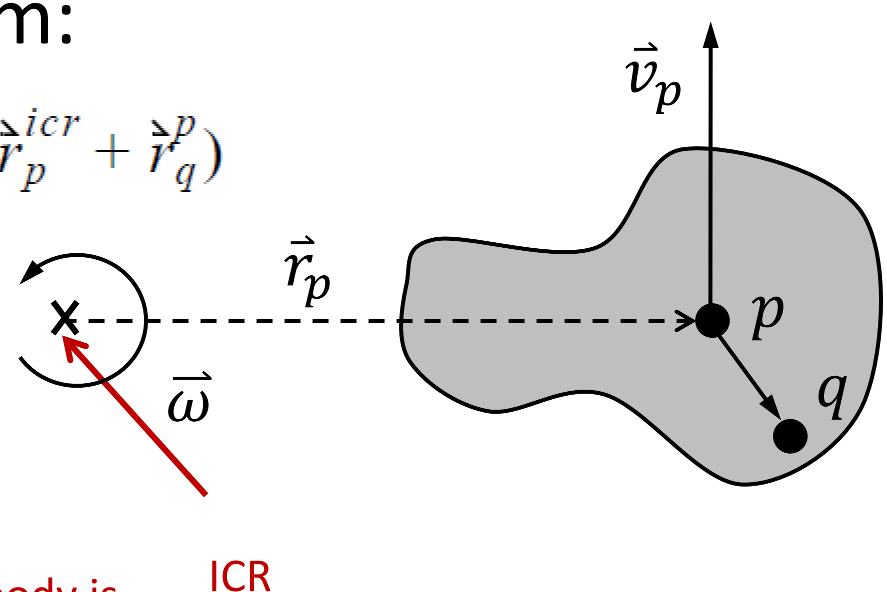
0

From Last Slide

$$\vec{v}_q^{\,icr} = \vec{v}_p^{\,icr} + \vec{\omega} \times \vec{r}_q^{\,p}$$

- Substitute for the first term:

$$\vec{v}_q^{\,icr} = \vec{\omega} \times \vec{r}_p^{\,icr} + \vec{\omega} \times \vec{r}_q^{\,p} = \vec{\omega} \times (\vec{r}_p^{\,icr} + \vec{r}_q^{\,p})$$



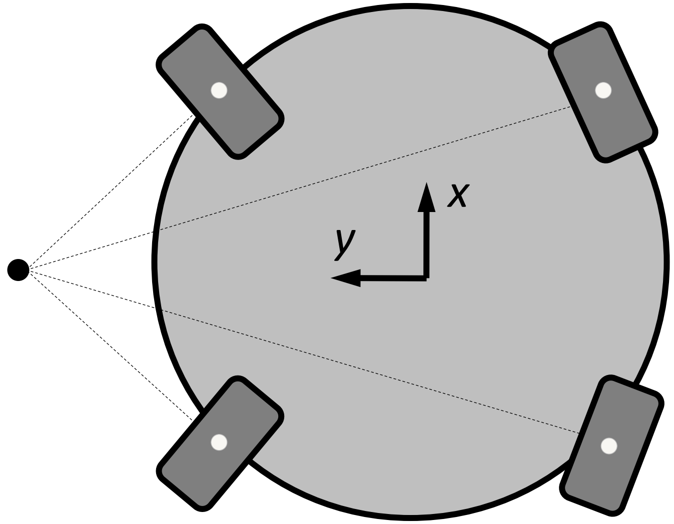
- That is:

$$\boxed{\vec{v}_q^{\,icr} = \vec{\omega} \times \vec{r}_q^{\,icr}}$$

Every point on the body is executing a pure rotation about the ICR.

4.2.1.2 Jeantaud Diagrams

- Fixing just the directions of the velocities of two points on a body determines the ICR.
- Hence, all steered wheels of a vehicle must be consistent to avoid wheel slip and energy loss.
- Wheels do not slip if they move along the normal to the line to the ICR.
- If all wheels are consistent, any two directions and one velocity can be used to predict the motion.



This is called a Jeantaud Diagram.

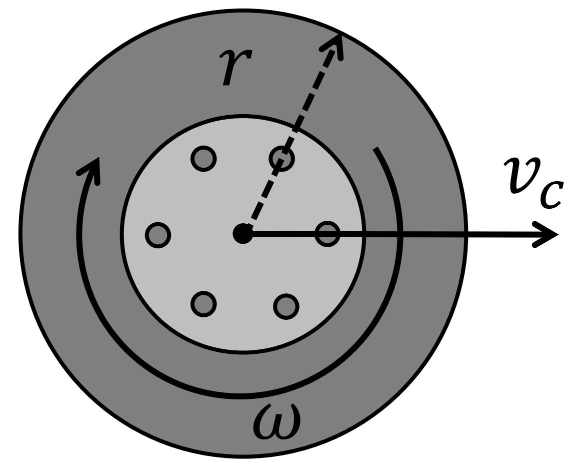
This vehicle can "crab steer".

4.2.1.3 Rolling Contact

- Wheels normally have up to two degrees of freedom.
  - steer
  - drive
- Angular and linear velocity are related as follows …

$$v_c = r\omega$$

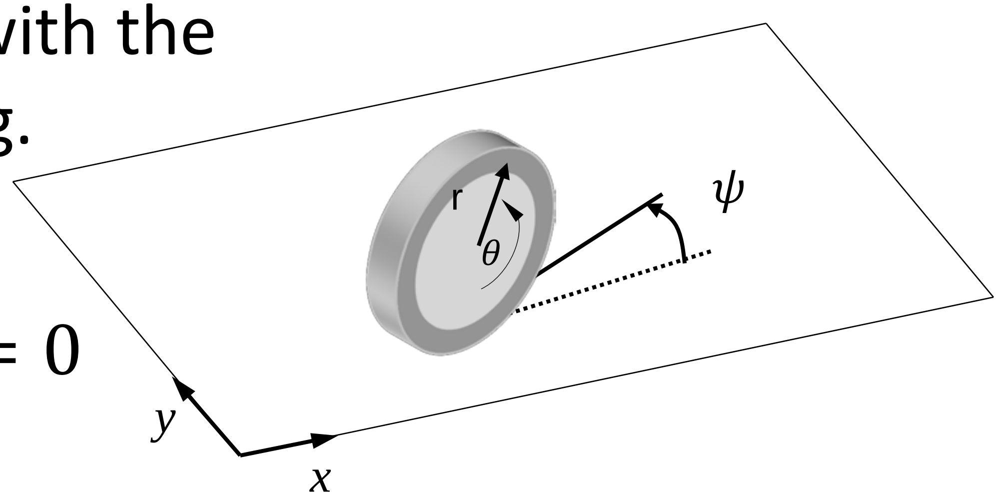
  - under a no slip assumption:



$$v_c = r\omega$$

4.2.1.4 Rolling without Slipping

- This constraint means $\dot{x}$ and $\dot{y}$ are not independent.
  - They must be aligned with the direction of pure rolling.
- The dot product …
  - $[\dot{x} \quad \dot{y}] \cdot [\mathrm{s}\,\psi \quad -\mathrm{c}\,\psi] = 0$
- Written out …
  - $\dot{x}\,\mathrm{s}\,\psi - \dot{y}\,\mathrm{c}\,\psi = 0$



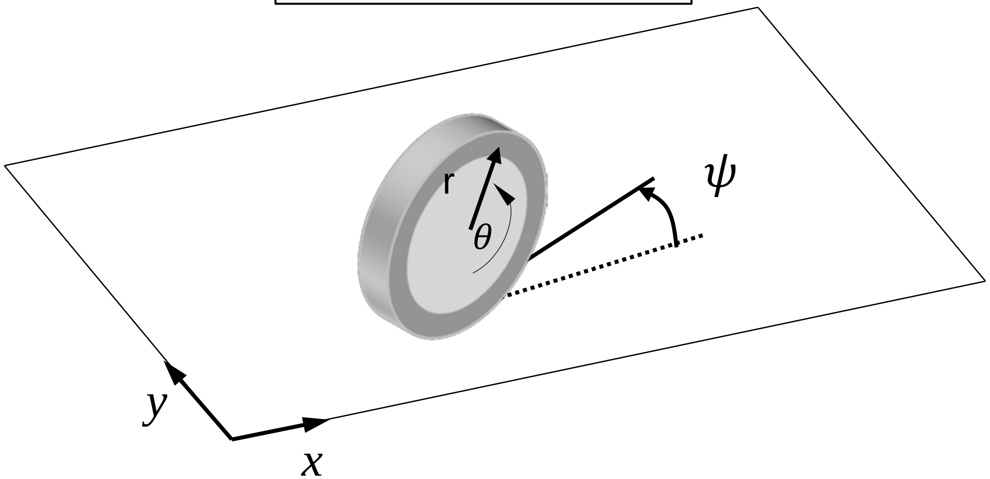
Disallowed Direction

Pfaffian Constraints

- Define the wheel configuration vector:

$$\underline{x} = \begin{bmatrix} x & y & \psi & \theta \end{bmatrix}^T$$

$$\dot{x} s\psi - \dot{y} c\psi = 0$$



- And the weight vector:

$$\underline{w}(\underline{x}) = \begin{bmatrix} sin\psi & -cos\psi & 0 & 0 \end{bmatrix}$$

- The constraint in Pfaffian form is:

$$\underline{w}(\underline{x})\dot{\underline{x}} = 0$$

Nonholonomic Constraints

- Typically (not always) wheels cannot move sideways (without slipping).
- Creates severe mathematical difficulties.
- Most wheels, and therefore most WMR's, are subject to these nonholonomic constraints.

Definition

- Such constraints are "nonholonomic" because they cannot be put in the form:

$$c(\underline{x}) = 0$$

- The integral would be:

$$\int_0^t \underline{w}(\underline{x})\dot{\underline{x}}dt = \int_0^t (\dot{x}\sin\psi(t) - \dot{y}\cos\psi(t))dt$$

- Even when $\psi(t) = t^2$, these integrals are the well-known Fresnel integrals which have no closed form solution.
  - And hence cannot be reduced to the form $c(\underline{x}) = 0$.

Outline

- 4.1 Moving Coordinate Systems
- 4.2 Kinematics of Wheeled Mobile Robots
  - 4.2.1 Aspects of Rigid Body Motion
  - 4.2.2 WMR Velocity Kinematics for Fixed Contact Point
  - 4.2.3 Common Steering Configurations
  - Summary

4.2.2 Character of WMR Models

- Unlike manipulators, the simplest models of how mobile robots move are differential equations that are:
  - Nonlinear
  - Underactuated
  - Constrained

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

$$w(\underline{x})\,\dot{\underline{x}} = 0$$

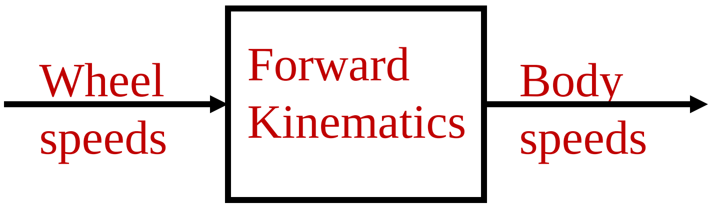
- Much of the difficulty of mobile robots can be traced to this fact.

Motion Prediction

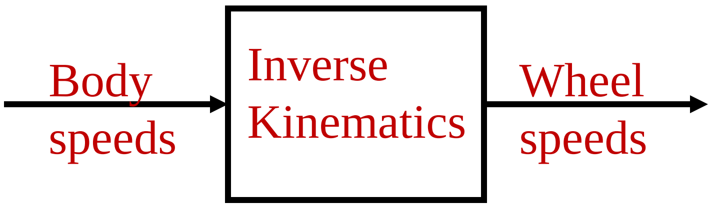
- The process of integrating the differential equations for known inputs can be called motion prediction. It is important for:
  - estimating state in odometry, Kalman filter system models, and more generally in pose estimation of any kind.
  - predicting state in predictive control
  - simulating motion in simulators.

Rate Kinematics

- For (WMRs), we care about the rate kinematics.
- Of basic interest are two questions..
  - For state estimation

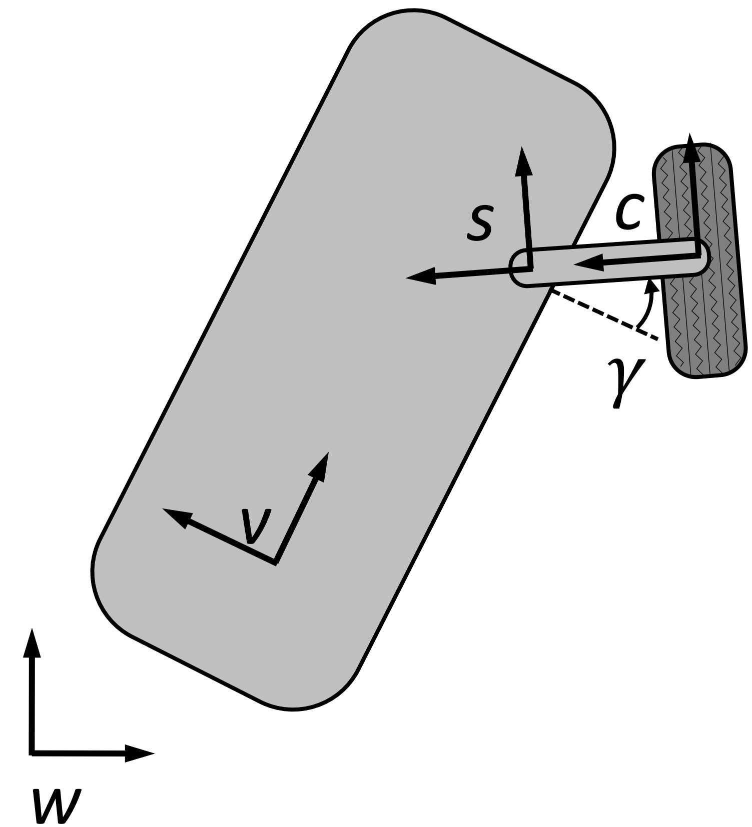


  - For control



Frame Conventions

- w: world
- v: vehicle
- s: steer
- c: contact point.
- Regard vehicles as rigid bodies (no suspension).
  - Except for steering and wheel rotation.
- Contact point moves on wheel and on floor but it is fixed in wheel frame.



Offset Wheel Equation

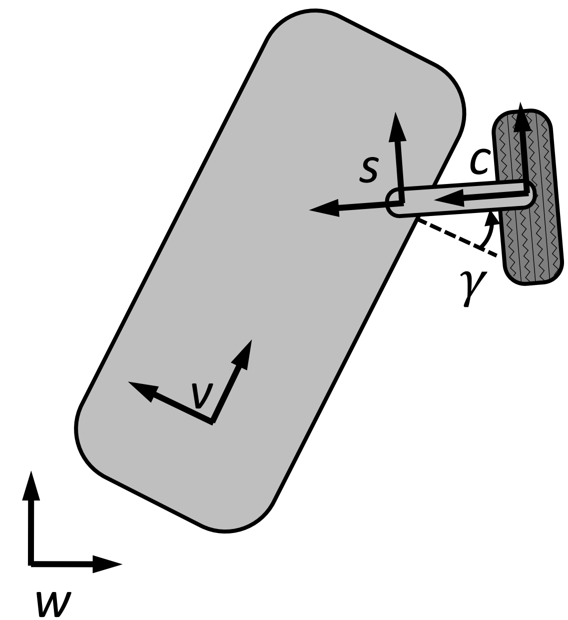
- Key assumption: wheel contact point is fixed to wheel. So... $\underline{v}_s^v = \underline{v}_c^s = 0$

$$\underline{v}_c^w = \left[I \mid -[\underline{r}_c^v]^\times \right] \begin{bmatrix} \underline{v}_v^w \\ \underline{\omega}_v^w \end{bmatrix} + \left[I \mid -[\underline{r}_c^s]^\times \right] \begin{bmatrix} \underline{v}_s^v \\ \underline{\omega}_s^v \end{bmatrix} + \underline{v}_c^s \quad \textbf{(4.30)}$$

- Eqn 4.30 becomes

$$\underline{v}_c^w = \underline{v}_v^w - [\underline{r}_c^v]^\times \underline{\omega}_v^w - [\underline{r}_c^s]^\times \underline{\omega}_s^v \quad \textbf{(4.39)}$$

Offset Wheel Equation



- When s and c frames are coincident

$$\underline{v}_c^w = \underline{v}_v^w - [\underline{r}_c^v]^\times \underline{\omega}_v^w \quad \textbf{(4.40)}$$

Wheel Equation

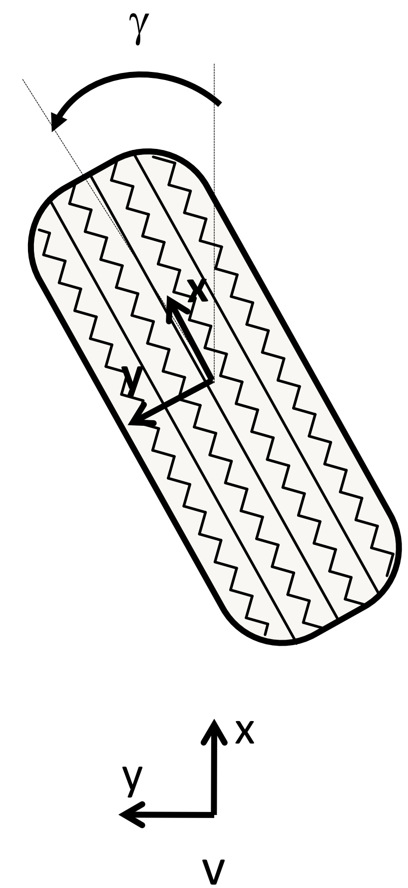
4.2.2.1.1 Wheel Steering Control

- For steering, note that direction (not magnitude) of s frame and c frame velocities must be parallel.
- So, propagate velocity from v frame to s frame:

$$\underline{v}_s^w = \underline{v}_v^w - [{}^v\underline{r}_s]^\times \underline{\omega}_v^w$$

- Express in vehicle coordinates and extract steer angle:

$$\gamma = atan2[({}^v\underline{v}_s^w)_y, ({}^v\underline{v}_s^w)_x]$$



4.2.2.1.2 Wheel Speed Control

- Assuming
  - a) the wheels are steered appropriately
  - b) no slip
- Then, the magnitude of wheel speed is also the component in the forward direction.
- Compute it in vehicle coordinates where posn vectors are easy to get:

$$^v\underline{v}_c^w = {}^v\underline{v}_v^w - [\underline{r}_c^v]^{\times}\, {}^v\underline{\omega}_v^w - [R_s^v \underline{r}_c^s]^{\times}\, {}^v\underline{\omega}_s^v$$

- That gives the wheel speed as

$$v_c^w = \sqrt{({}^v\underline{v}_c^w)_x^2 + ({}^v\underline{v}_c^w)_y^2} \qquad \omega_{wheel} = v_c^w / r_{wheel}$$

4.2.2.2.1 Wheel Sensing

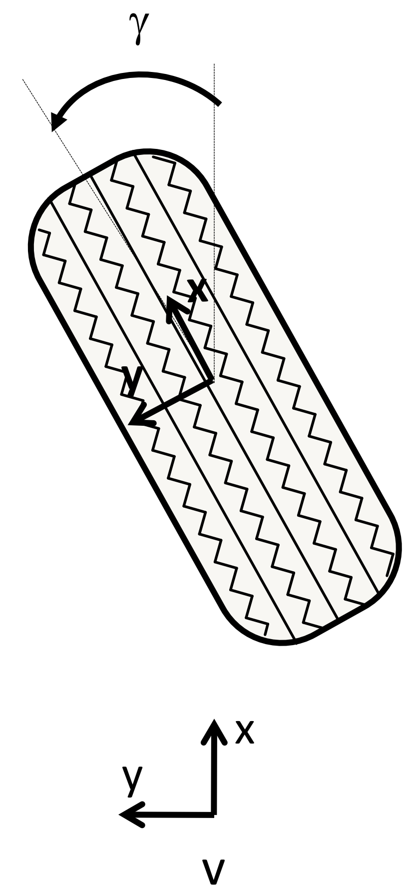
- Wheel linear speed:

$$v_k = r_k \omega_k$$

- Wheel speed components:

$$(v_k)_x = v_k cos(\gamma_k)$$

$$(v_k)_y = v_k sin(\gamma_k)$$



4.2.2.2.2 Multiple Offset Wheels

- Write the offset wheel equation in vehicle coordinates:

$$ {}^{v}\underline{v}_c^{w} = {}^{v}\underline{v}_v^{w} - [\underline{r}_c^{v}]^{\times}\,{}^{v}\underline{\omega}_v^{w} - [R_s^{v}\underline{r}_c^{s}]^{\times}\,{}^{v}\underline{\omega}_s^{v} $$

- This is of the form:

$$ {}^{v}\underline{v}_c^{w} = H_c^{v}(\gamma)\begin{bmatrix} {}^{v}\underline{v}_v^{w} \\ \underline{\omega}_v^{w} \end{bmatrix} + Q_c^{s}(\gamma)\,{}^{v}\underline{\omega}_s^{v} $$

- If we write one of these for each wheel, stack em up, the result looks like:

$$ {}^{v}\underline{v}_c^{w} = H_c^{v}(\underline{\gamma})\dot{\underline{x}}_v^{w} + Q_c^{s}(\underline{\gamma})\dot{\underline{\gamma}} $$

4.2.2.2.2 Multiple Offset Wheels (Inv)

- Result from last slide again is: ${}^{v}\underline{v}_c^{w} = H_c^v(\underline{\gamma})\underline{\dot{x}}_v^w + Q_c^s(\underline{\gamma})\underline{\dot{\gamma}}$
- The LHS and steer angles are known, and this is normally overdetermined, so use the left pseudoinverse:

$$\underline{\dot{x}}_v^w = [H_c^v(\underline{\gamma})^T H_c^v(\underline{\gamma})]^{-1} H_c^v(\underline{\gamma})^T [\underline{v}_c^w - Q_c^s(\underline{\gamma})\underline{\dot{\gamma}}]$$

Robot linear and angular velocity

Steer Angles

Wheel Speeds

Steer Angle Rates

WMR Kinematics

Box 4.2: WMR Forward Kinematics: Offset Wheels

Offset wheel equations for all wheels can be grouped together to produce

$$\underline{v}_c^w = H_c^v(\underline{\gamma})\,\underline{\dot{x}}_v^w + Q_c^s(\underline{\gamma})\,\underline{\dot{\gamma}}$$

where each pair of rows of H_c^v and Q_c^s comes from an offset equation expressed in body coordinates, $\underline{v}_c^w$ is the wheel velocities, $\underline{\dot{x}}_v^w$ is the linear and angular velocity of the vehicle, and $\underline{\gamma}$ is the steer angles.

The inverse mapping (for two or more wheels) can be computed with:

$$\underline{\dot{x}}_v^w = [H_c^v(\underline{\gamma})^T H_c^v(\underline{\gamma})]^{-1} H_c^v(\underline{\gamma})^T [\underline{v}_c^w - Q_c^s(\underline{\gamma})\underline{\dot{\gamma}}]$$

For nonoffset wheels H_c^v simplifies, and Q_c^s disappears.

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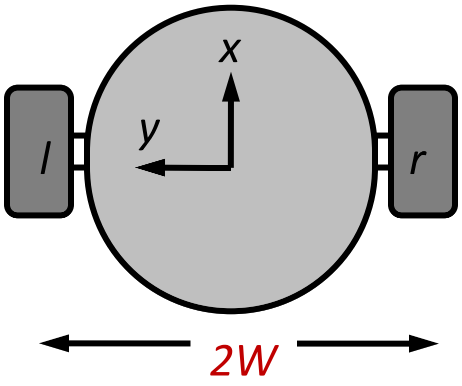
Example: Differential Steer (Inv)

- Let ‘l’ and ‘r’ denote left and right wheel frames.
- The dimensions are:



$$ {}^{v}\underline{r}_l = \begin{bmatrix} 0 & W \end{bmatrix}^T \qquad {}^{v}\underline{r}_r = \begin{bmatrix} 0 & -W \end{bmatrix}^T $$

- In body frame, velocities have only an x component. Equation 4.40 reduces to:

$$ \begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} v_x + \omega W \\ v_x - \omega W \end{bmatrix} = \begin{bmatrix} 1 & W \\ 1 & -W \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix} $$

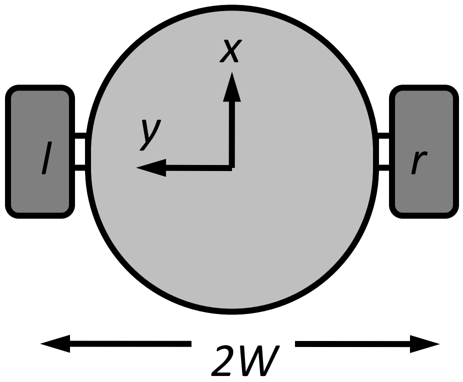
Can solve for 2 dof of 3 dof motion. Other dof is zero <u>in body frame</u> (for this choice of body frame).

- Two equations giving sideways wheel velocities were of the form $v_y=0$, so these were not written.

Example: Differential Steer (Fwd)

- Inverse kinematics again are:

$$\begin{bmatrix} v_r \\ v_l \end{bmatrix} = \begin{bmatrix} 1 & W \\ 1 & -W \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$



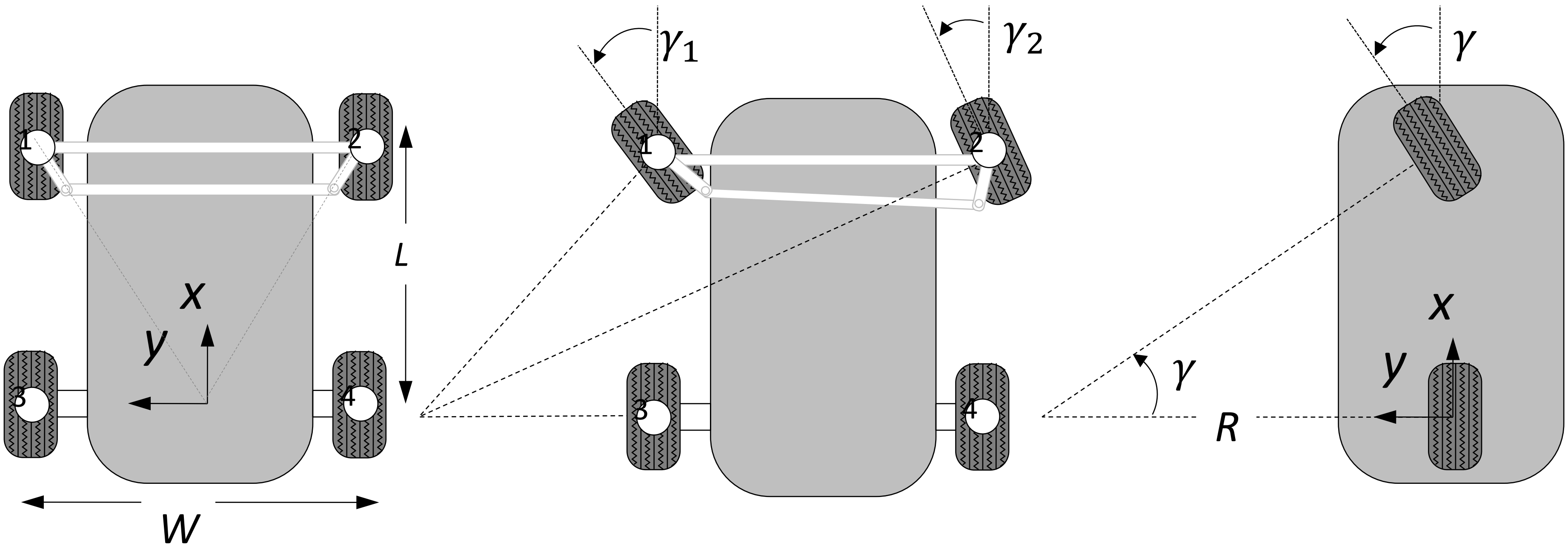
- This is easy to invert:

Again, other dof is zero <u>in body frame</u> due to nonholonomic constraints

$$\begin{bmatrix} v_x \\ \omega \end{bmatrix} = \frac{1}{2W} \begin{bmatrix} W & W \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_r \\ v_l \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} v_r \\ v_l \end{bmatrix}$$

Example: Ackerman Steer

- Special mechanism ensures wheels are lined up properly.



Example: Ackerman Steer (Inverse)

- Position vector to front wheel in body (vehicle) frame:

$$\underline{r}^{v}_{f} = \begin{bmatrix} L & 0 \end{bmatrix}^T$$

- Cross product skew matrix:

$$[\underline{r}^{v}_{f}]^{\times} = \begin{bmatrix} 0 & -(\underline{r}^{v}_{f})_z & (\underline{r}^{v}_{f})_y \\ (\underline{r}^{v}_{f})_z & 0 & -(\underline{r}^{v}_{f})_x \\ -(\underline{r}^{v}_{f})_y & (\underline{r}^{v}_{f})_x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L \\ 0 & L & 0 \end{bmatrix}$$

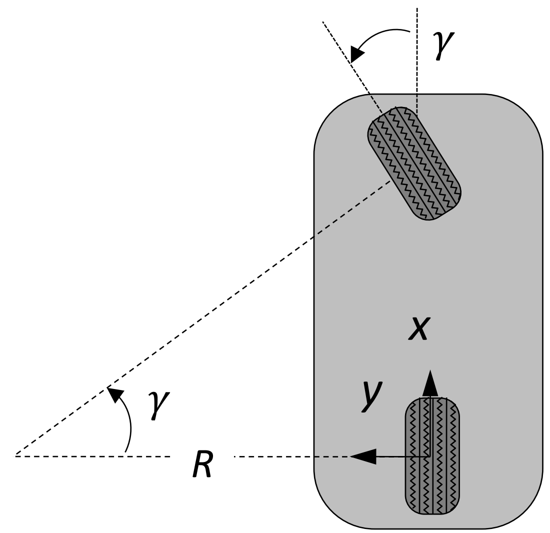
- Wheel equation in body frame reduces to:

$${}^{v}\underline{v}^{w}_{f} = {}^{v}\underline{v}^{w}_{v} - [{}^{v}\underline{r}_{c}]^{\times}\, {}^{v}\underline{\omega}^{w}_{v} \Rightarrow v_f = \begin{bmatrix} v_x & \omega L \end{bmatrix}^T$$

BTW rear wheel velocity is trivial

$$\underline{v}^{w}_{c} = \underline{v}^{w}_{v} - [\underline{r}^{v}_{c}]^{\times}\, \underline{\omega}^{w}_{v}$$

Wheel Equation



Bicycle Model

Example: Ackerman Steer (Inverse)

$$\underline{v}_f = \begin{bmatrix} v_x & \omega L \end{bmatrix}^T$$

From Last Page

- Last result is of the form:

$$^{v}\underline{v}_f^{w} = H_c^{v}\,\underline{\dot{x}}_v^{w}$$

Jacobian does not depend on the steer angle

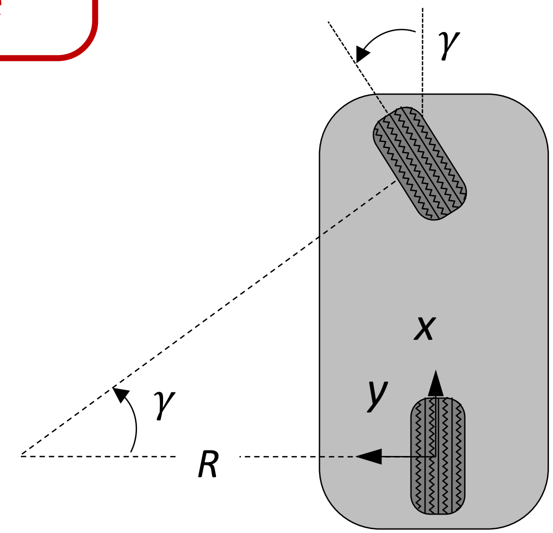
- Written out:

$$\begin{bmatrix} v_{fx} \\ v_{fy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

- So, the angle of the front wheel can be computed:

$$\tan(\gamma) = \frac{\omega L}{v_x} = \kappa L = \frac{L}{R}$$

The "Car" Equation



Bicycle Model

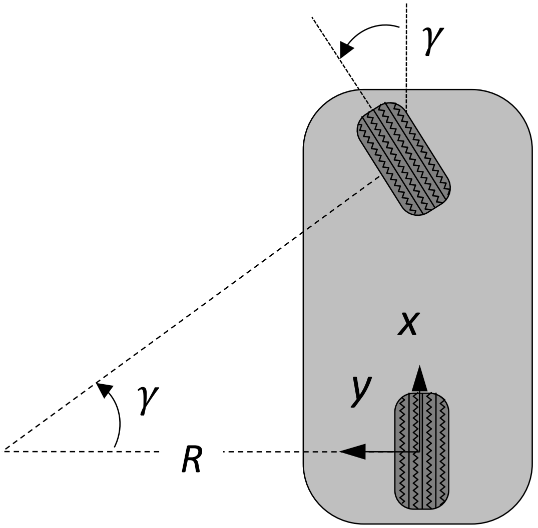
Example: Ackerman Steer (Fwd)

- Inverse kinematics again are:

$$\begin{bmatrix} v_{fx} \\ v_{fy} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} v_x \\ \omega \end{bmatrix}$$

- This is easy to invert:

$$\begin{bmatrix} v_x \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & L \end{bmatrix}^{-1} \begin{bmatrix} v_{fx} \\ v_{fy} \end{bmatrix} = \begin{bmatrix} v_{fx} \\ v_{fy}/L \end{bmatrix}$$



Bicycle Model

Example: Generalized Bicycle

- Models any vehicle whose wheels do not slip.

$$ {}^{v}\underline{v}_{i}^{w} = {}^{v}\underline{v}_{v}^{w} - [{}^{v}\underline{r}_{i}]^{\times}\, {}^{v}\underline{\omega}_{w}^{v} $$

Wheel Equation (in vehicle coordinates)

- Wheel position vectors:

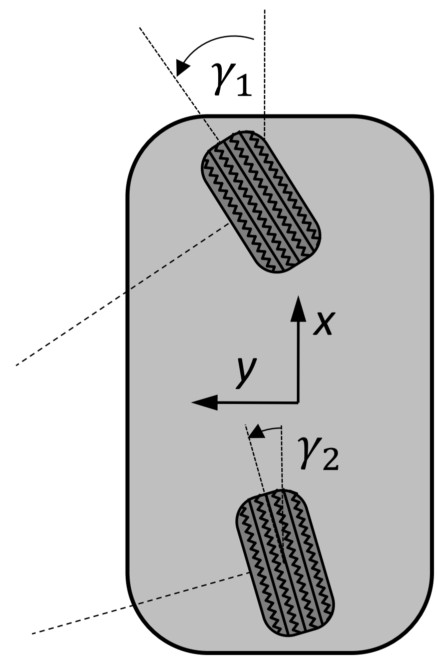
$$ {}^{v}\underline{r}_{1} = \left[x_1 \; y_1\right]^T \qquad {}^{v}\underline{r}_{2} = \left[x_2 \; y_2\right]^T $$

- Skew matrix for wheel i:

$$ [{}^{v}\underline{r}_{i}]^{\times} = \begin{bmatrix} 0 & 0 & y_i \\ 0 & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} $$

- For a single wheel (i):

$$ v_i = \left[(V_x - \omega y_i) \; (V_y + \omega x_i)\right]^T $$



Example: Generalized Bicycle

- Models any vehicle whose wheels do not slip.

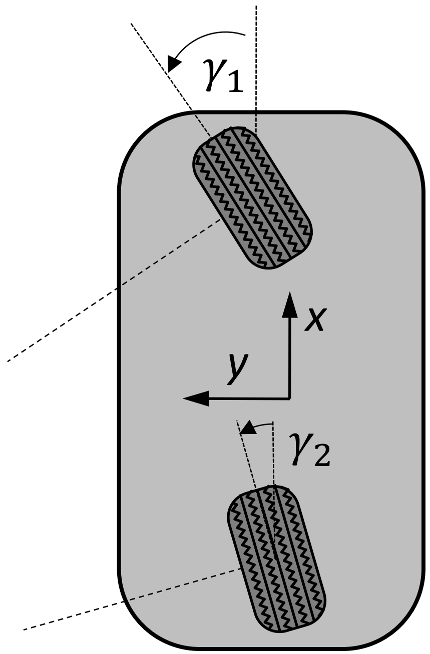
$$v_i = \left[(V_x - \omega y_i)\ (V_y + \omega x_i)\right]^T$$

Wheel Equation (This case) (from last slide)

- Gather equations for both wheels:

$$\underline{v}_c^w = H_c^v \, \underline{\dot{x}}_v^w$$

$$\begin{bmatrix} v_{1x} \\ v_{1y} \\ v_{2x} \\ v_{2y} \end{bmatrix} = \begin{bmatrix} v_x - \omega y_1 \\ v_y + \omega x_1 \\ v_x - \omega y_2 \\ v_y + \omega x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y_1 \\ 0 & 1 & x_1 \\ 1 & 0 & -y_2 \\ 0 & 1 & x_2 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$



- Forward kinematics is again LPI:

$$\underline{\dot{x}}_v^w = \left[(H_c^v)^T (H_c^v)\right]^{-1} (H_c^v)^T \underline{v}_c^w$$

Example: 4 Wheel Steer (Inv)

- Position vectors in body (vehicle) frame:

$$ {}^{v}\underline{r}_{s1} = \left[L \;\; W\right]^T \quad {}^{v}\underline{r}_{s2} = \left[L \;\; -W\right]^T \quad {}^{v}\underline{r}_{s3} = \left[-L \;\; W\right]^T \quad {}^{v}\underline{r}_{s4} = \left[-L \;\; -W\right]^T $$

- Offset vectors in body frame:

$$ {}^{s1}\underline{r}_{c1} = d\left[-s\gamma_1 \;\; c\gamma_1\right]^T \qquad {}^{s2}\underline{r}_{c2} = d\left[s\gamma_2 \;\; -c\gamma_2\right]^T $$

$$ {}^{s3}\underline{r}_{c3} = d\left[-s\gamma_3 \;\; c\gamma_3\right]^T \qquad {}^{s4}\underline{r}_{c4} = d\left[s\gamma_4 \;\; -c\gamma_4\right]^T $$

$$ {}^{sk}\underline{r}_{ck} = \left[a_k \;\; b_k\right]^T $$

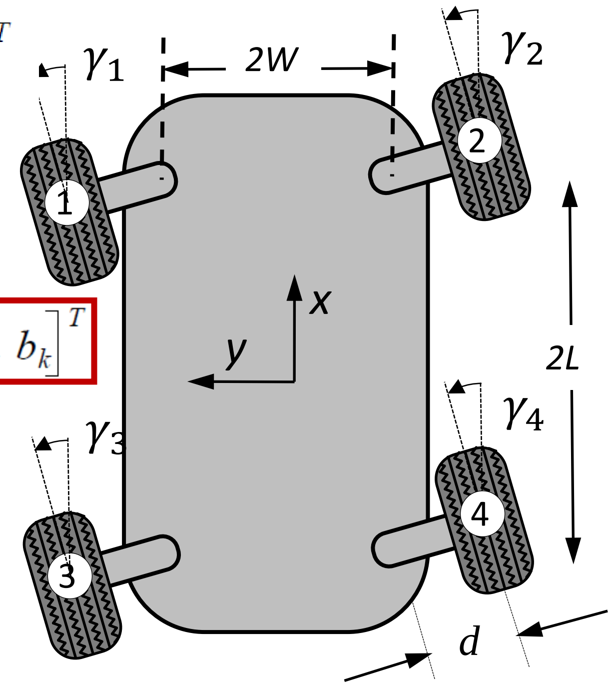
(4.62)

NB: There are typos in the book for these eqns.

- Equations for each wheel are:

$$ {}^{v}\underline{v}_c^{w} = {}^{v}\underline{v}_v^{w} - [{}^{v}\underline{r}_c]^{\times}\,\underline{\omega}_v^{w} - [{}^{v}\,{}^{s}\underline{r}_c]^{\times}\,{}^{v}\underline{\omega}_s $$

$$ \begin{bmatrix} v_{ix} \\ v_{iy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(y^v_{si} + b_i) \\ 0 & 1 & (x^v_{si} + a_i) \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} + \begin{bmatrix} -b_i \\ a_i \end{bmatrix} \dot{\gamma}_i $$



4 Wheel Steer

Example: 4 Wheel Steer (Inv)

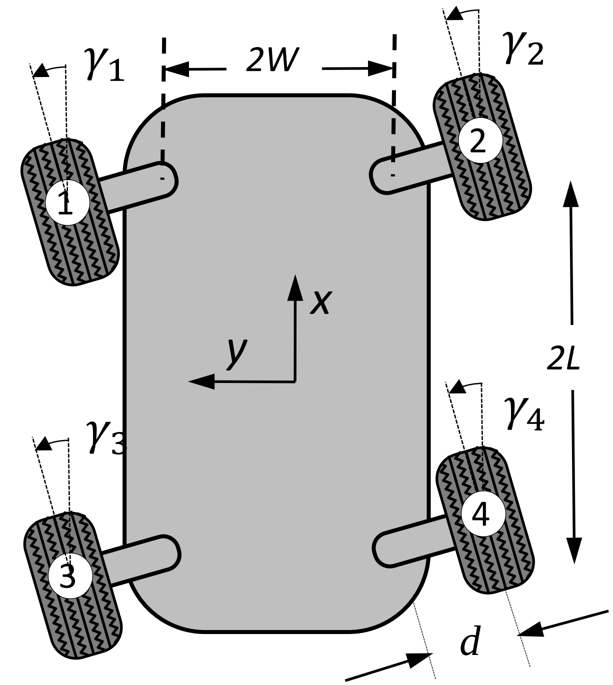
- All together, these are of the form:

$$\underline{v}_c^w = H_c^v(\underline{\gamma})\,\underline{\dot{x}}_v^w + Q_c^s(\underline{\gamma})\,\underline{\dot{\gamma}}$$

$$\underline{\dot{x}}_v^w = \begin{bmatrix} v_x & v_y & \omega \end{bmatrix}^T \qquad \underline{\dot{\gamma}} = \begin{bmatrix} \dot{\gamma}_1 & \dot{\gamma}_2 & \dot{\gamma}_3 & \dot{\gamma}_4 \end{bmatrix}^T$$

$$\underline{v} = \begin{bmatrix} v_{1x} & v_{1y} & v_{2x} & v_{2y} & v_{3x} & v_{3y} & v_{4x} & v_{4y} \end{bmatrix}^T$$

$$H_c^v(\underline{\gamma}) = \begin{bmatrix} 1 & 0 & -(y_1+b_1) \\ 0 & 1 & x_1+a_1 \\ 1 & 0 & -(y_2+b_2) \\ 0 & 1 & x_2+a_2 \\ 1 & 0 & -(y_3+b_3) \\ 0 & 1 & x_3+a_3 \\ 1 & 0 & -(y_4+b_4) \\ 0 & 1 & x_4+a_4 \end{bmatrix} \qquad Q_c^s(\underline{\gamma}) = \begin{bmatrix} -b_1 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & -b_2 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & -b_3 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & -b_4 \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$



4 Wheel Steer

- Forward kinematics is simple:

$$\underline{\dot{x}}_v^w = [H_c^v(\underline{\gamma})^T H_c^v(\underline{\gamma})]^{-1} H_c^v(\underline{\gamma})^T [\underline{v}_c^w - Q_c^s(\underline{\gamma})\underline{\dot{\gamma}}]$$

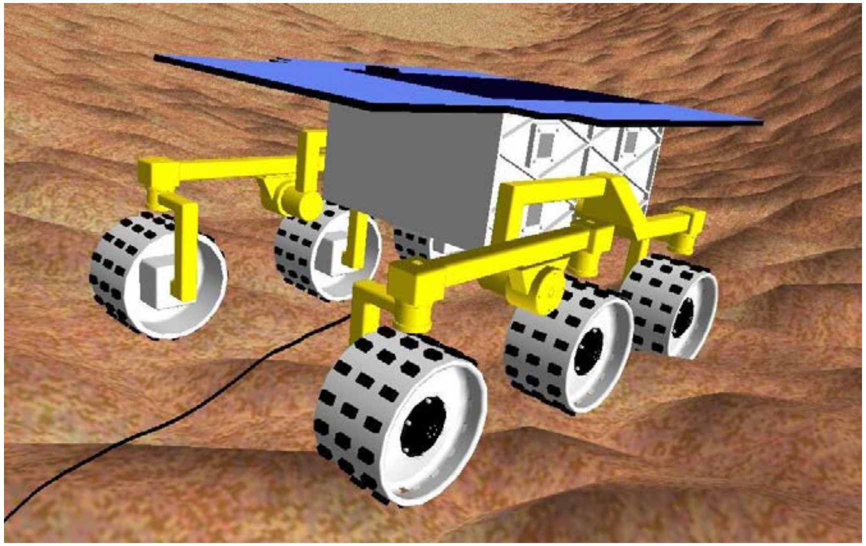
3D Case

- Works even if some vectors are out of the plane.

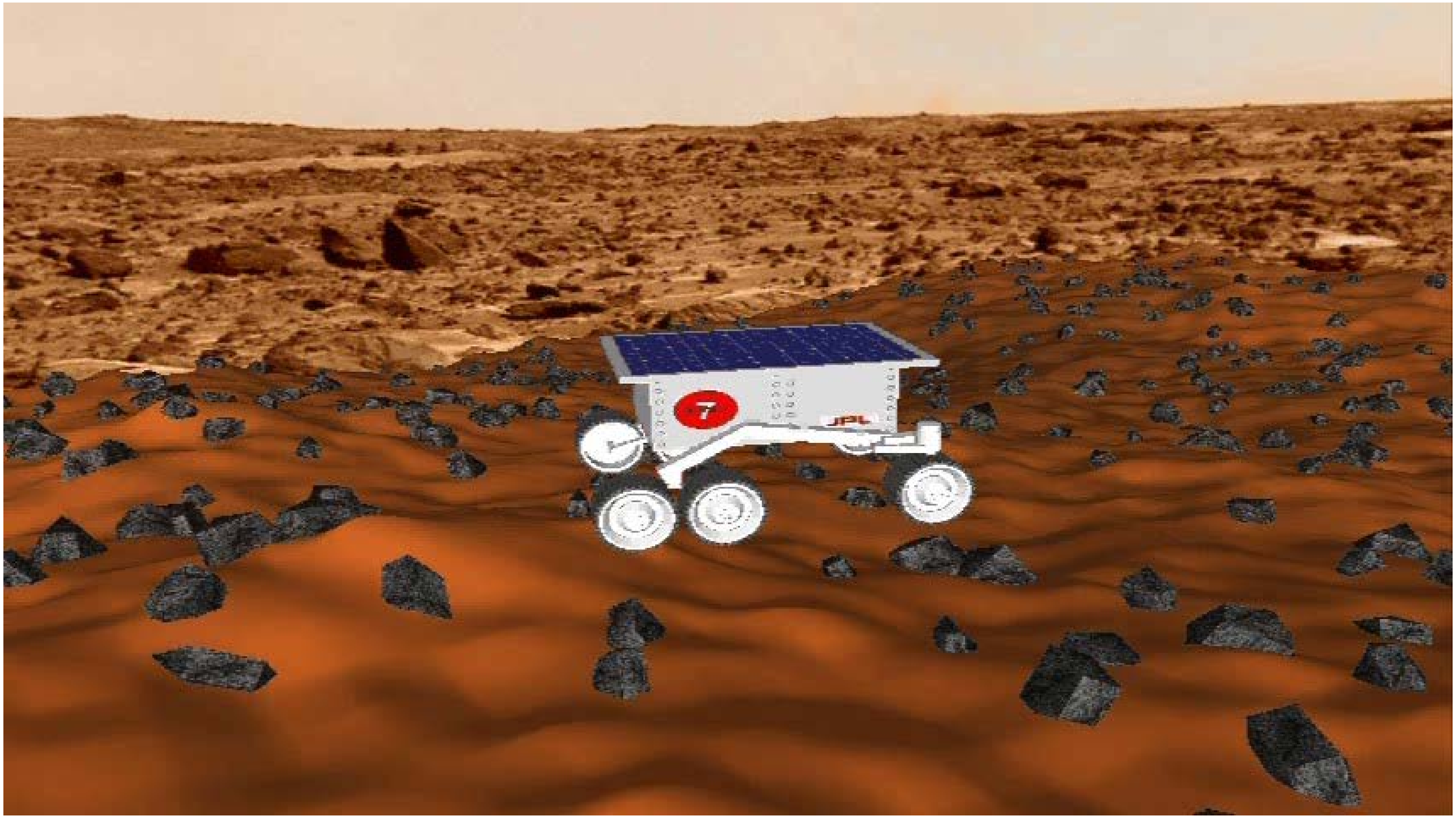
$$\underline{v}_c^w = \underline{v}_v^w - [\underline{r}_c^v]^\times \underline{\omega}_v^w$$

Wheel Equation

Out of plane



Video



Outline

- 4.1 Moving Coordinate Systems
- 4.2 Kinematics of Wheeled Mobile Robots
  - 4.2.1 Aspects of Rigid Body Motion
  - 4.2.2 WMR Velocity Kinematics for Fixed Contact Point
  - 4.2.3 Common Steering Configurations
  - <u>Summary</u>

Summary

- The kinematic equations governing the motion of wheeled vehicles are those of planar rigid bodies.
  - Its all about the ICR.
- Rate kinematics for wheeled mobile robots are pretty straightforward
  - in the general case in 3D.
- The inverse problem is often overdetermined.
  - This is solved like any overdetermined system.