Chapter 3
Numerical Methods

Part 3

3.4 Differential Algebraic Systems

3.5 Integration of Differential Equations

Outline

- 3.4 Differential Algebraic Systems
  - 3.4.1 Constrained Dynamics
  - 3.4.2 First and Second Order Constrained Kinematic Systems
  - 3.4.3 Lagrangian Dynamics
  - 3.4.4 Constraints
  - Summary
- 3.5 Integration of Differential Equations

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3.4.1.1. Augmented Systems

- Consider a differential equation with n states subject to m constraints:

$$\dot{\underline{x}} = f(\underline{x}, \underline{u})$$
$$\underline{c}(\underline{x}) = \underline{0}$$

- Linear equations could be substituted into the DE. Nonlinear is the case that matters to us.
- What does it mean? Both equations cannot be correct …
  - It means the DE applies in the subspace of $\mathcal{R}^n$ that satisfies the constraints.
  - The subspace is known as the constraint manifold.

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3.4.1.3 Sequential Approach

- First Approach: Integrate the unconstrained DE one time step.

$$\underline{x}_{k+1} = \underline{x}_k + \Delta\underline{x}_k = \underline{x}_k + f(\underline{x}, \underline{u})\Delta t$$

- Use result as initial conditions for a rootfinding problem that enforces constraints $\underline{c}(\underline{x}) = \underline{0}$
- Should work but …
  - What if rootfinding step reverses the DE step?
  - Did it move by Δt or $< \Delta t$ or $> \Delta t$?
  - The two equations can disagree with each other. They need to be decoupled.
    - Idea: Make the DE step satisfy the constraints to first order.

3.4.1.4 Projection Approach

- Second Approach: Remove the component of the state derivative out of the constraint tangent plane.
  - Equivalently, project it into the tangent plane.
- Write step in terms of feasible and infeasible component:

$$\Delta \underline{x} = \Delta \underline{x}_{\perp} + \Delta \underline{x}_{||}$$

- Remove the component out of the tangent plane:

$$\boxed{\Delta \underline{x}_{\perp} = \underline{c}_{\underline{x}}^{T}[\underline{c}_{\underline{x}}\underline{c}_{\underline{x}}^{T}]^{-1}\underline{c}_{\underline{x}}\Delta \underline{x}}$$

- The matrix $P_C(M) = M[M^T M]^{-1} M^T$ performs a projection on the column space of M.
  - Here we project onto colspace of $\underline{c}_{\underline{x}}^{T}$ - which is the rowspace of $\underline{c}_{\underline{x}}$ (the Constraint Jacobian).
  - So this is the component that violates the constraints to first order.

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3.4.2.1 Augmented First Order Systems

- Third approach: Remove the infeasible component right in the differential equation.
- For a feasible perturbation: $\underline{c}_{\underline{x}} \Delta \underline{x} = \underline{0}$
- The infeasible part is some unknown combination of the constraint gradients. Let it be of the form: $\Delta \underline{x}_{\perp} = \underline{c}_{\underline{x}}^T \underline{\lambda} \Delta t$
- Remove the infeasible component with:

$$\Delta \underline{x}_{\parallel} = f(\underline{x}, \underline{u}) \Delta t - \Delta \underline{x}_{\perp} = f(\underline{x}, \underline{u}) \Delta t - \underline{c}_{\underline{x}}^T \underline{\lambda} \Delta t$$

- Divide both equations by Δt and pass to the limit:

$$\dot{\underline{x}} + \underline{c}_{\underline{x}}^T \underline{\lambda} = f(\underline{x}, \underline{u})$$
$$\underline{c}_{\underline{x}} \dot{\underline{x}} = \underline{0}$$

3.4.2.2 Solving The Eqns of Motion

- This can be written in matrix form:

$$\dot{\underline{x}} + \underline{c}_{\underline{x}}^T \underline{\lambda} = \underline{f}(\underline{x}, \underline{u})$$
$$\underline{c}_{\underline{x}} \dot{\underline{x}} = \underline{0}$$

$$\boxed{\begin{bmatrix} I & \underline{c}_{\underline{x}}^T \\ \underline{c}_{\underline{x}} & 0 \end{bmatrix} \begin{bmatrix} \dot{\underline{x}} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{f}(\underline{x}, \underline{u}) \\ \underline{0} \end{bmatrix}}$$

We will see this Again in Lagrangian Dynamics

- To solve, multiply 1st by $\underline{c}_{\underline{x}}$:

$$\underline{c}_{\underline{x}} \dot{\underline{x}} + \underline{c}_{\underline{x}} \underline{c}_{\underline{x}}^T \underline{\lambda} = \underline{c}_{\underline{x}} \underline{f}(\underline{x}, \underline{u})$$

- By 2nd equation $\underline{c}_{\underline{x}} \dot{\underline{x}} = 0$ so solve for λ:

$$\underline{\lambda} = (\underline{c}_{\underline{x}} \underline{c}_{\underline{x}}^T)^{-1} \underline{c}_{\underline{x}} \underline{f}(\underline{x}, \underline{u})$$

Right Pseudo Inverse

3.4.2.2 Solving The Eqns of Motion

- Substitute for λ in first equation:

$$\boxed{\dot{\underline{x}} = [I - \underline{c}_{\underline{x}}^T(\underline{c}_{\underline{x}}\underline{c}_{\underline{x}}^T)^{-1}\underline{c}_{\underline{x}}]f(\underline{x}, \underline{u})}$$

- The matrix:

$$P_N(\underline{c}_{\underline{x}}^T) = I - P_C(\underline{c}_{\underline{x}}^T) = I - \underline{c}_{\underline{x}}^T(\underline{c}_{\underline{x}}\underline{c}_{\underline{x}}^T)^{-1}\underline{c}_{\underline{x}}$$

- Projects the state derivative directly into the nullspace of the constraints – i.e. directly into the tangent plane.
  - By simply removing the component normal to the tangent plane (i.e. a weighted sum of the gradients).

3.4.2.3 Holonomic Constraints

- These are of the form:

$$\underline{c}(\underline{x}) = \underline{0}$$

- It is useful to differentiate constraints sometimes.
- Differentiating wrt time gives our standard form for a velocity constraint:

$$\dot{\underline{c}}(\underline{x}) = \underline{c}_{\underline{x}}\dot{\underline{x}} = \underline{0}$$

- If the DE has a holonomic constraint on $\underline{x}$, this implies that the derivative ($\dot{\underline{x}}$,) must be constrained too:
  - In fact it must be orthogonal to the constraint gradient.

3.4.2.4 Nonholonomic Constraints

- Consider a form that depends on the state derivative:

$$\underline{c}(\underline{x}, \dot{\underline{x}}) = \underline{0}$$

- Differentiate to get:

$$\dot{\underline{c}}(\underline{x}, \dot{\underline{x}}) = \underline{c}_{\underline{x}} \dot{\underline{x}} + \underline{c}_{\dot{\underline{x}}} \ddot{\underline{x}} = \underline{0}$$

- In general, all higher state derivatives are constrained too.
- A special form that is relevant to us is:

$$\underline{c}(\underline{x}, \dot{\underline{x}}) = \underline{w}(\underline{x})\dot{\underline{x}} = \underline{0}$$

- This does not need to be differentiated to use.

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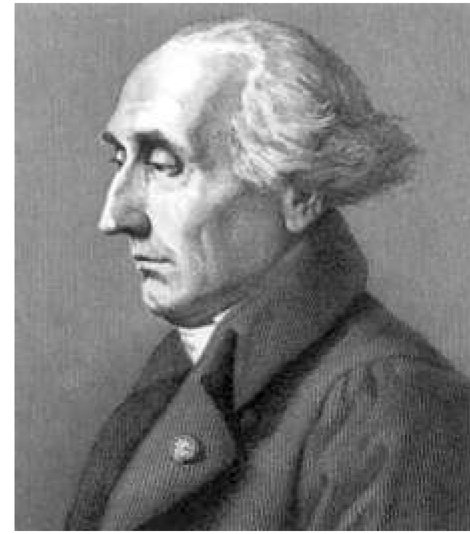
Compte Joseph-Louis Lagrange

- Greatest mathematician of 18 century?
- Advised by Leonhard Euler (who was advised by Bernoulli)
- Notable students
  - Joseph Fourier
  - Simeon Poisson
- Reformulated Newtonian Mechanics Mécanique Analytique (Analytical Mechanics) (1788).
- Invented:
  - Theory of Differential Equations
  - Calculus of variations



1736-1813
Italian-French

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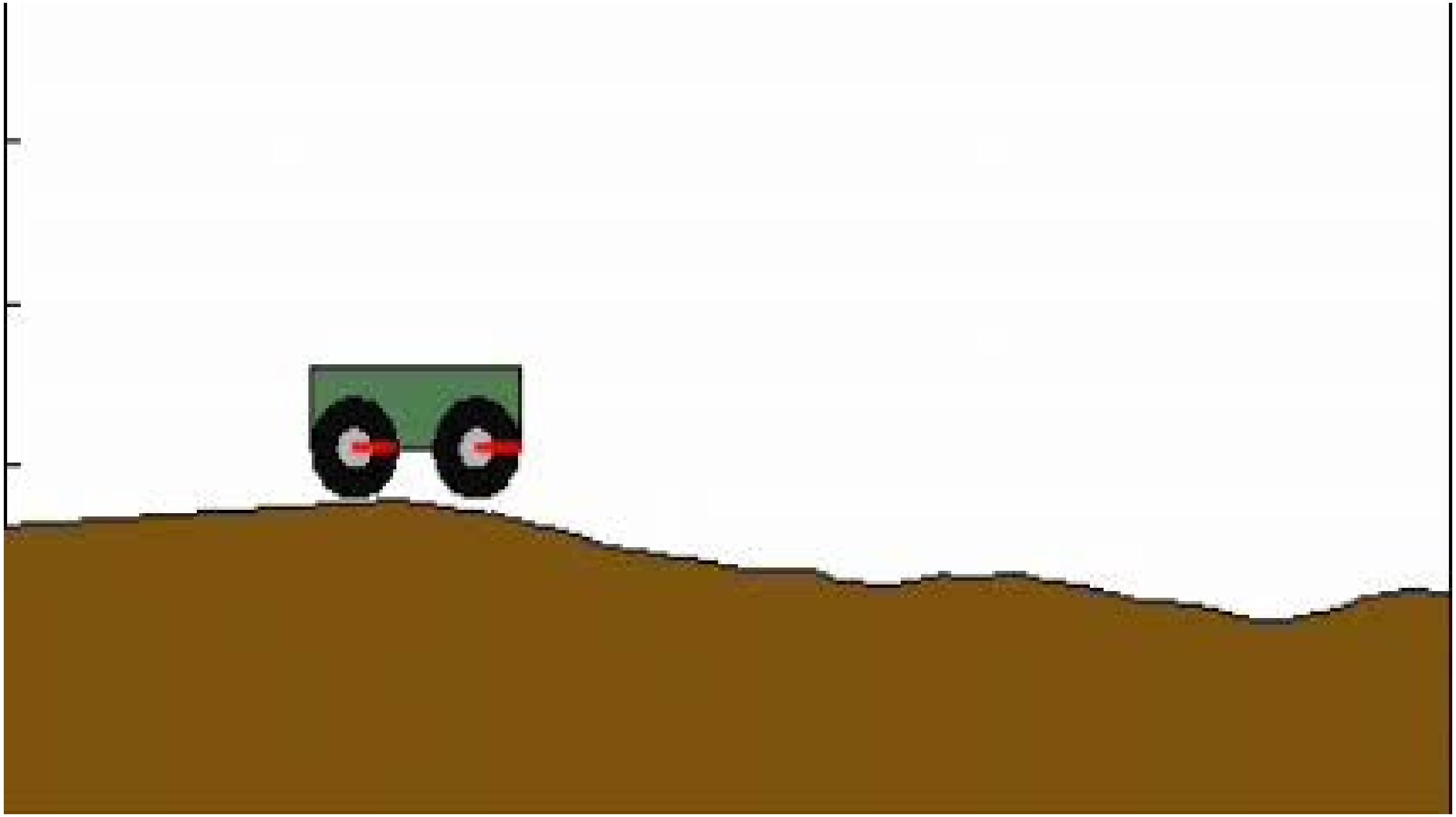
Vehicle on Terrain Video



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3.4.3.1 Equations of Motion

- Embedded Form:
  - Eliminates constraint forces
  - Fewest dof, fewest equations
  - Nonlinear, complex equations
  - Popular for manipulators
- Augmented Form:
  - Redundant coordinates
  - Explicit constraint forces
  - Simpler equations
  - Suitable for automation

Real wheels do slip. Hence nonholonomic “constraints” are not really “no slip” constraints, they are “slip like this” constraints.

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3.4.3.1 Equations of Motion – One Body

- Equations of motion are simple when the generalized coordinates (q) are absolute (inertial).
- Coordinates for one body:

$$\underline{x}_i = \begin{bmatrix} x_i & y_i & \theta_i \end{bmatrix}^T$$

- Equation of motion:

Unknowns

Eqn A

$$M_i \ddot{\underline{x}}_i = \underline{F}_i^{ext} + \underline{F}_i^{con}$$

mass

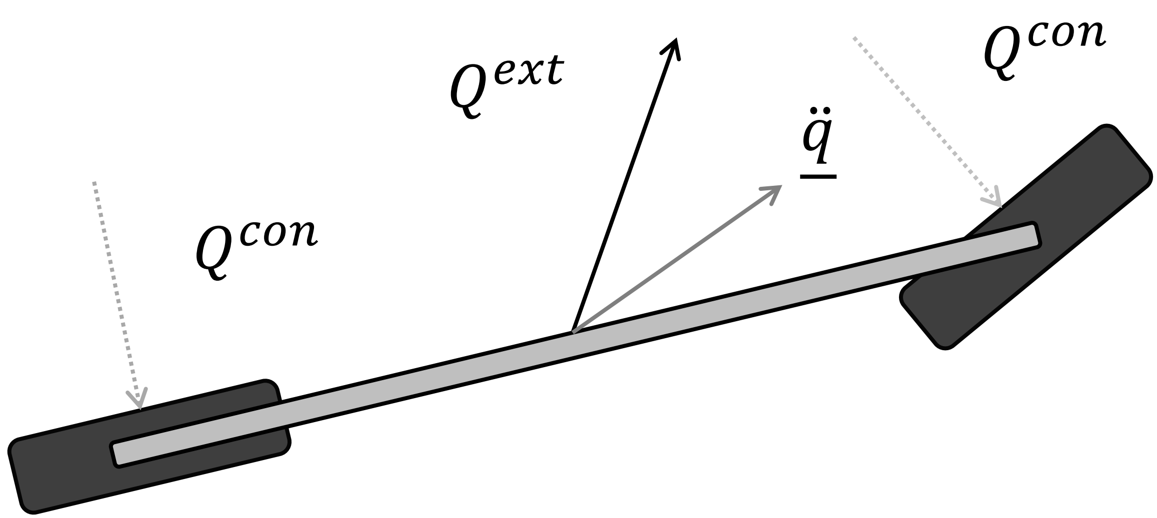
external force

constraint force

Segment forces into those known and those unknown

3.4.3.1 Applied and Constraint Forces

- Constraint forces are generated to oppose motion in the disallowed directions.
- The net force (parallel to acceleration) is therefore not in the direction of the applied force.



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Center of Mass Reference

- Choose Center of mass as the body reference point. Then:

2x2 identity

$$M_i = \begin{bmatrix} m_i I & 0 \\ 0 & J_i \end{bmatrix}$$

Polar moment of inertia

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Underdetermined System – n bodies

- # of equations:
  - 3 n: one for each element of $\underline{\ddot{x}}$
- # of unknowns:
  - 3 n generalized accelerations $\underline{\ddot{x}}$
  - 3 c constraint forces
  - 3 n generalized velocities $\underline{\dot{x}}$
  - 3 n generalized coordinates $\underline{x}$
- Where do the other 3c "constraints" come from?
  - The constraints ☺
- Where do the velocities and positions come from?
  - Integration

3.4.3.2 Differentiated Constraints - Holonomic

- The 2nd derivative of a holonomic constraint is:

$$\ddot{\underline{c}}(\underline{x}) = \underline{c}_{\underline{x}t}\,\dot{\underline{x}} + \underline{c}_{\underline{x}}\,\ddot{\underline{x}} = \underline{0}$$

- Define:

$$\boxed{\underline{F}_d = -\,\underline{c}_{\underline{x}t}\,\dot{\underline{x}}}$$

- Then we have:

$$\underline{c}_{\underline{x}}\,\ddot{\underline{x}} = \underline{F}_d$$

- This makes the differentiated constraint look like Newton's 2nd law.

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3.4.3.2 Differentiated Constraints-Nonholonomic

- Also, for a nonholonomic constraint:

$$\dot{\underline{c}}(\underline{x}, \dot{\underline{x}}) = \underline{c}_{\underline{x}}\, \dot{\underline{x}} + \underline{c}_{\dot{\underline{x}}}\, \ddot{\underline{x}} = \underline{0}$$

- Define:

$$\boxed{\underline{F}_d = -\underline{c}_{\underline{x}}\, \dot{\underline{x}}}$$

- Then the constraint becomes:

$$\underline{c}_{\dot{\underline{x}}}\, \ddot{\underline{x}} = \underline{F}_d$$

- Once again, this makes the differentiated constraint look like Newton's 2nd law.

3.4.3.2 Differentiated Constraints-General

- Both earlier forms are of the form:

Eqn B

$$\boxed{C\,\ddot{\underline{x}} = \underline{F}_d}$$

$$\underline{c}_{\underline{x}}\,\ddot{\underline{x}} = \underline{F}_d$$

holonomic

$$\underline{c}_{\dot{\underline{x}}}\,\ddot{\underline{x}} = \underline{F}_d$$

nonholonomic

3.4.3.3 Principle of Virtual Work

- Credited to Aristotle(!) and/or Bernoulli.
- Work: The product of a force and a displacement in the direction of the force.
- Virtual Work: As above but either force or displacement is not real.

3.4.3.3 Lagrange Multipliers

- We will require the virtual work performed by constraint forces to vanish.
- This is accomplished by writing:

Constraint Jacobian

$$\underline{F}_i^{con} = -C^T \underline{\lambda}$$

Eqn C

- Why?
  - Columns of C^T (rows of C) are prohibited directions.
  - Constraint forces are confined above to those prohibited directions.
  - Dot product of constraint forces with any feasible displacement will be zero. Displacements parallel to the the rows of J are infeasible.

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3.4.3.4 Augmented System

- Recall the original equations of motion:

$$M_i \underline{\ddot{x}}_i = \underline{F}_i^{ext} + \underline{F}_i^{con}$$ Eqn A

- Substitute from Eqn C:

$$M\underline{\ddot{x}} + C^T\underline{\lambda} = \underline{F}^{ext}$$ Eqn A1

Forces in Infeasible Directions

- Combine this with Eqn B:

$$C\,\underline{\ddot{x}} = \underline{F}_d$$ Eqn B

Feasible Accelerations

Augmented System

- We now have c extra equations and have replaced the constraint forces with the Lagrange Multipliers as unknowns.

$$\begin{bmatrix} M & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \ddot{\underline{x}} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{F}^{ext} \\ \underline{F}_d \end{bmatrix}$$

Augmented Mass Matrix

Augmented Force Vector

unknowns

- Solve and then integrate acceleration twice.

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3.4.4.1 Constraint Trim

- Due to inevitable numerical error, enforcing differentiated constraints does not enforce the original constraints.
- Lt f(t) be the state derivative computed by integrating the second order system:

$$\underline{f}(t) = \underline{f}(0) + \int_0^t \ddot{\underline{x}}(t)dt$$

- The first order system is subject to the original constraints:

$$\dot{\underline{x}} = \underline{f}(t)$$
$$C\dot{\underline{x}} = \underline{0}$$

- $\dot{\underline{x}}$ generated by integration will likely not satisfy these constraints, so fix it with the following before integration:

$$\dot{\underline{x}} = [I - C^T(CC^T)^{-1}C]\underline{f}(t)$$

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3.4.4.2 Drift Control

- Constraints will drift over time since only derivatives are enforced.
- Elegant solution is to add compensation pseudoforces in PID loops…

$$\underline{F}_d \leftarrow \underline{F}_d - k_p\ \underline{c}(\underline{x})$$

$$\underline{F}_d \leftarrow \underline{F}_d - k_p\ \underline{c}(\underline{x}, \dot{\underline{x}})$$

- Gains relate to time constants:

$$k_p = \cancel{\frac{\tau}{\Delta t}} \quad \frac{\Delta t}{\tau}$$

3.4.4.4 Initial Conditions

- Differentiated constraints will hold the constraints constant rather than at zero.
  - So they must start at zero to stay at zero.
- Two approaches.
  - 1) Start from zero velocity state which automatically satisfies constraints. Then activate system with forces.
  - 2) Start from moving state but guarantee constraints are satisfied by solving the rootfinding problem.

$$\underline{c}(\underline{q}) = 0$$

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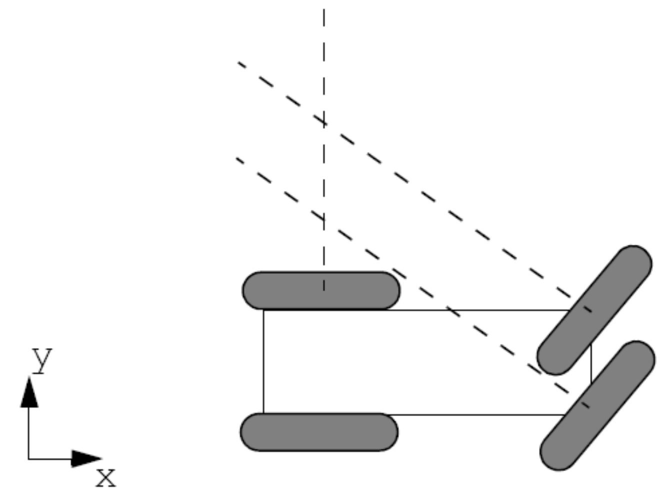
Overconstraint

- Typical of wheeled vehicles.
- Leads to collapse of nullspace.
  - No motion possible.
  - … or constraints do work.
- Few approaches:
  - 1) Use any two independent constraints.
  - 2) Compute and equivalent bicycle model of constraints for each cycle.
  - 3) Use an equation solver that tolerates the situation
- It may or may not be appropriate to let the overconstrained system slow down.

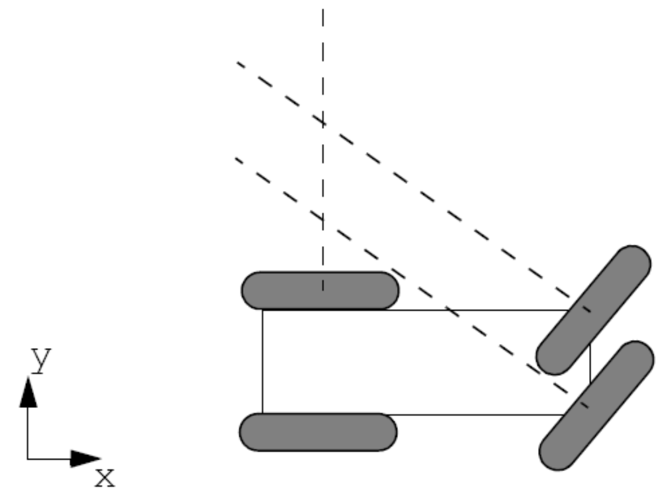


Redundant Constraints

- Example: two rear wheels of car generate the same constraint equation.
- Leads to singularity of the system.
- Good approach is avoid inversion. Compute least residual norm:

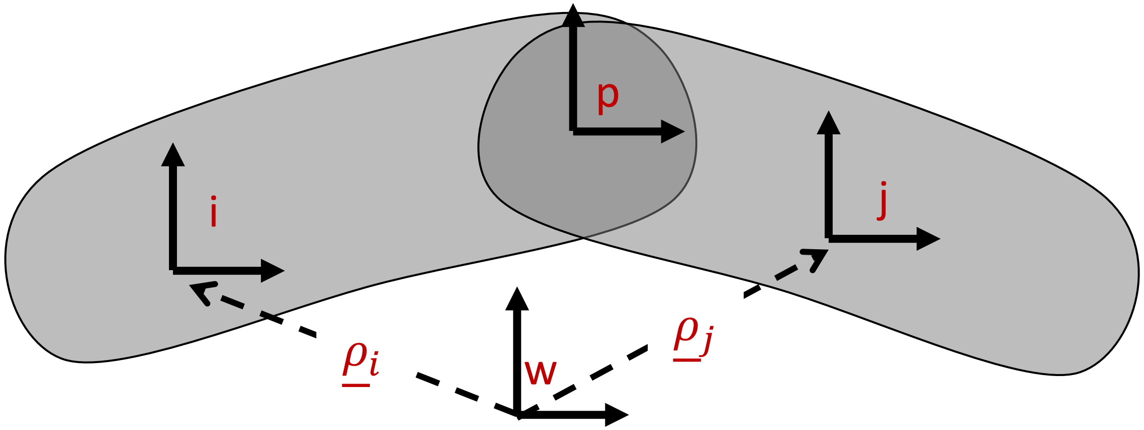
$$\underline{\lambda}^{*} = argmin \left\| C M^{-1} C^{T} \underline{\lambda} - F \right\|_{2}$$

- Conjugate gradient algorithm…



3.4.4.5 Basic Rigid Body Constraint

- Two bodies have poses known with respect to the world frame.
- If there is a rotary constraint, frame p is at the pivot point.



Rigidity Constraint

- Express this in terms of pose composition as follows:

$$\underline{\rho}_i^j = \underline{\rho}_w^j * \underline{\rho}_i^w = (\underline{\rho}_j^w)^{-1} * \underline{\rho}_i^w = const$$

- Holonomic of the form:

$$\underline{g}(\underline{x}) = const$$

- Equivalent to:

$$\underline{c}(\underline{x}) = \underline{g}(\underline{x}) - const = \underline{0}$$

Rigidity Constraint

- Gradient contains two elements:

$$C_{\underline{\rho}_i} = \frac{\partial \underline{\rho}_i^j}{\partial \underline{\rho}_i^w} \qquad C_{\underline{\rho}_j} = \frac{\partial \underline{\rho}_i^j}{\partial \underline{\rho}_j^w}$$

- The first is a right pose Jacobian:

$$C_{\underline{\rho}_i} = \frac{\partial \underline{\rho}_i^j}{\partial \underline{\rho}_i^w} = \begin{bmatrix} c\theta_w^j & -s\theta_w^j & 0 \\ s\theta_w^j & c\theta_w^j & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} c\theta_j^w & s\theta_j^w & 0 \\ -s\theta_j^w & c\theta_j^w & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

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Rigidity Constraint

- The second is more complicated. By the chain rule:

$$C_{\underline{\rho}_j} = \frac{\partial \underline{\rho}_i^j}{\partial \underline{\rho}_j^w} = \left(\frac{\partial \underline{\rho}_i^j}{\partial \underline{\rho}_w^j}\right)\left(\frac{\partial \underline{\rho}_w^j}{\partial \underline{\rho}_j^w}\right)$$

Left Pose Jacobian

Inverse Pose Jacobian

- This is:

$$C_{\underline{\rho}_j} = -\begin{bmatrix} 1 & 0 & -(y_i^j - y_w^j) \\ 0 & 1 & (x_i^j - x_w^j) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_j^w & s\theta_j^w & -y_w^j \\ -s\theta_j^w & c\theta_j^w & x_w^j \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_{\underline{\rho}_j} = \boxed{-\begin{bmatrix} c\theta_j^w & s\theta_j^w & -y_i^j \\ -s\theta_j^w & c\theta_j^w & x_i^j \\ 0 & 0 & 1 \end{bmatrix}}$$

Rigidity Constraint

- Total Constraint Jacobian:

$$\underline{c}_{\underline{x}} = \begin{array}{c} \qquad\quad \text{Posn i} \qquad\qquad\qquad\qquad \text{Posn j} \\ \begin{bmatrix} \dots & c\theta & s\theta & 0 & \dots & -c\theta & -s\theta & y_i^j & \dots \\ \dots & -s\theta & c\theta & 0 & \dots & s\theta & -c\theta & -x_i^j & \dots \\ \dots & 0 & 0 & 1 & \dots & 0 & 0 & -1 & \dots \end{bmatrix} \end{array}$$

Where

$$\theta = \theta_j^w$$

- Time Derivative:

$$\underline{c}_{\underline{x}t} = \begin{bmatrix} \dots & -\omega s\theta & \omega c\theta & 0 & \dots & \omega s\theta & -\omega c\theta & 0 & \dots \\ \dots & -\omega c\theta & -\omega s\theta & 0 & \dots & \omega c\theta & \omega s\theta & 0 & \dots \\ \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots \end{bmatrix}$$

Where

$$\omega = \dot{\theta}$$

Rotary Joint Constraint

- Let p denote a reference frame attached to the point of rotation.
- The constraints for the rotary joint at the front wheel can be expressed as the first two elements of the equation:

$$\underline{\rho}_p^w - \underline{\rho}_p^w = 0$$

"p" means axis of rotation

$$\underline{\rho}_i^w * \underline{\rho}_p^i - \underline{\rho}_j^w * \underline{\rho}_p^j = 0$$

Rewritten in terms of generalized coordinates.

Unknowns

Knowns

Rotary Joint Constraint

- This gives:

$$J_{\underline{\rho}_i} = \frac{\partial \underline{\rho}_p^w}{\partial \rho_i^w} = \begin{bmatrix} 1 & 0 & -(y_p^w - y_i^w) \\ 0 & 1 & (x_p^w - x_i^w) \\ 0 & 0 & 1 \end{bmatrix} \quad J_{\underline{\rho}_j} = -\frac{\partial \underline{\rho}_p^w}{\partial \rho_j^w} = -\begin{bmatrix} 1 & 0 & -(y_p^w - y_j^w) \\ 0 & 1 & (x_p^w - x_j^w) \\ 0 & 0 & 1 \end{bmatrix}$$

Minus sign

- Hence, total constraint Jacobian is:

$$\begin{array}{cc} & \begin{array}{cc} \text{Posn i} & \text{Posn j} \end{array} \\ \underline{c}_{\underline{x}} = & \begin{bmatrix} \ldots & 1 & 0 & -\Delta y_i & \ldots & -1 & 0 & \Delta y_j & \ldots \\ \ldots & 0 & 1 & \Delta x_i & \ldots & 0 & -1 & -\Delta x_j & \ldots \end{bmatrix} \end{array}$$

$$\Delta x_i = (x_p^w - x_i^w)$$

$$\Delta y_i = (y_p^w - y_i^w)$$

$$\Delta x_j = (x_p^w - x_j^w)$$

$$\Delta y_j = (y_p^w - y_j^w)$$

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Rotary Joint Constraint

- Time Derivative:

$$\begin{array}{cc} & \begin{array}{cc}\text{Posn i} & \text{Posn j}\end{array} \\ \underline{c}_{xt} = & \begin{bmatrix} \dots & 0 & 0 & -\Delta x_i \omega_i & \dots & 0 & 0 & \Delta x_j \omega_j & \dots \\ \dots & 0 & 0 & -\Delta y_i \omega_i & \dots & 0 & 0 & \Delta y_j \omega_j & \dots \end{bmatrix} \end{array}$$

$$\omega_i = \omega_i^w = \dot{\theta}_i^w$$

$$\omega_j = \omega_j^w = \dot{\theta}_j^w$$

- Fd vector is:

$$\underline{F}_d = -\underline{c}_{xt}\,\dot{\underline{x}}$$

$$\underline{F}_d = \begin{bmatrix} 0 & 0 & \Delta x_i \omega_i & 0 & 0 & -\Delta x_j \omega_j \\ 0 & 0 & \Delta y_i \omega_i & 0 & 0 & -\Delta y_j \omega_j \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \\ \dot{x}_j \\ \dot{y}_j \\ \dot{\theta}_j \end{bmatrix}$$

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Summary - DAEs

- Simplest formulation for some problems.
- Only practical formulation for some problems.
- Can be really fast for mobile robots.
- Can be written in completely general way to simulate anything.

Outline

- 3.4 Differential Algebraic Systems
- 3.5 Integration of Differential Equations
  – <u>3.5.1 Dynamic Models in State Space</u>
  – 3.5.2 Integration of State Space Models

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3.5.1 State Space

- State space = a minimal set of variables which can be used to predict future state given inputs:
  - Number of initial conditions in a differential equation.

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3.5.1 Dynamic Models in State Space

- Predicting the future involves predicting trajectories caused by motion commands.
- General case:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

Known as the State Space representation of the system

- The "inputs" u are a new addition.
- Known as a "forced" system although the inputs need not be forces.

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Constraints

- The dynamics of wheeled mobile robots are constrained dynamics in 3D of systems of rigid bodies.
- Often must consider:
  - Actuator kinematics
  - Lateral and longitudinal wheel slip (or nonslip).
  - Terrain following.

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3.5.2.1 Euler's Method

- For the nonlinear differential equation:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}, t)$$

- Seems reasonable to use the definition of integration:

$$\underline{x}(t + \Delta t) = \underline{x}(t) + \underline{f}(\underline{x}, t)\Delta t$$

- In discrete time:

$$\underline{x}_{k+1} = \underline{x}_k + \underline{f}(\underline{x}_k, t_k)\Delta t_k$$

- Works well if f() is nearly linear. Errors are 2nd order.

3.5.2.2 Midpoint Method

- Let's try for a 2nd order approximation. A 2nd order Taylor series is:

h= Δ t

$$\underline{x}(t+h) \approx \underline{x}(t) + \underline{f}(\underline{x}, t)h + \frac{d\underline{f}}{dt}(\underline{x}, t)\frac{h^2}{2}$$

- Which can be written as (factor out an h):

$$\underline{x}(t+h) \approx \underline{x}(t) + \left\{\underline{f}(\underline{x}, t) + \frac{d\underline{f}}{dt}(\underline{x}, t)\frac{h}{2}\right\}h \qquad \text{Eqn A}$$

- Now, the part in brackets is the first degree Taylor series for the first time derivative evaluated at the midpoint of the step because:

$$\underline{f}(\underline{x}(t+h/2), t+h/2) \approx \underline{f}(\underline{x}, t) + \frac{d\underline{f}}{dt}(\underline{x}, t)\frac{h}{2}$$

3.5.2.2 Midpoint Method

- The derivative $d\underline{f}(\underline{x},t)/dt$ is typically expensive computationally. Instead, invert the last formula to produce a finite difference approximation:

$$\frac{d\underline{f}(\underline{x},t)}{dt}\frac{h}{2} \approx \underline{f}(\underline{x}(t+h/2), t+h/2) - \underline{f}(\underline{x},t)$$

- Substituting into Eqn A produces:

$$\underline{x}(t+h) \approx \underline{x}(t) + h\underline{f}(\underline{x}(t+h/2), t+h/2)$$

- And, the value of x at the midpoint can be approximated:

$$\underline{x}(t+h/2) \approx \underline{x}(t) + \underline{f}(\underline{x},t)(h/2)$$

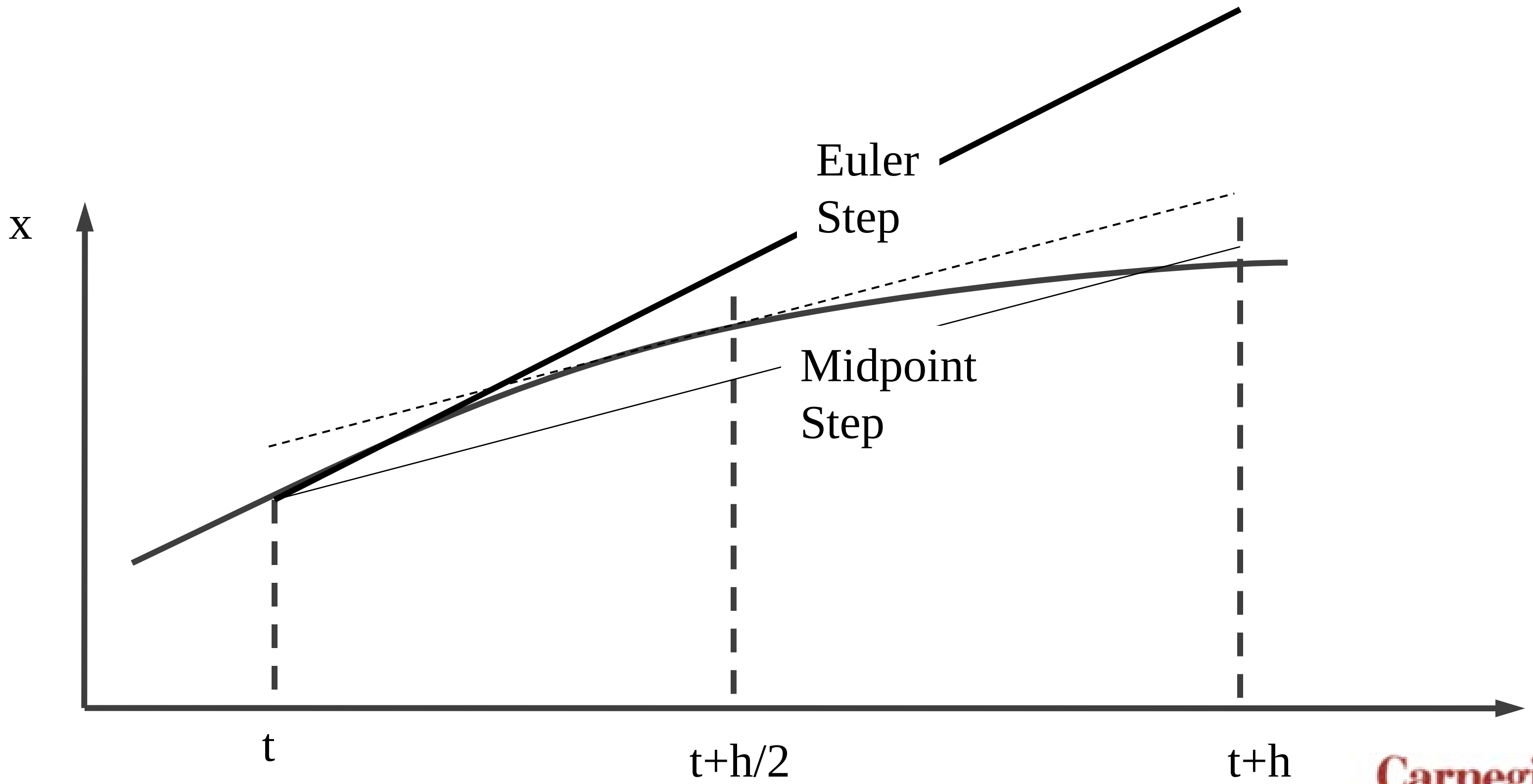
- This gives finally:

$$\underline{x}(t+h) \approx \underline{x}(t) + h\underline{f}[\underline{x}(t) + \underline{f}(\underline{x},t)(h/2), t+h/2]$$

3.5.2.2 Midpoint Method

- For future reference, this is best written as:

$$\underline{k} = \underline{x}(t) + \underline{f}(\underline{x}, t)(h/2)$$
$$\underline{x}(t+h) \approx \underline{x}(t) + h\underline{f}(\underline{k}, t+h/2)$$



Example

- Example, integrate a general curve with respect to distance:

$$\begin{bmatrix} x(s) \\ y(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \\ \theta(0) \end{bmatrix} + \int_0^s \begin{bmatrix} \cos\theta(s) \\ \sin\theta(s) \\ \kappa(s) \end{bmatrix} ds$$

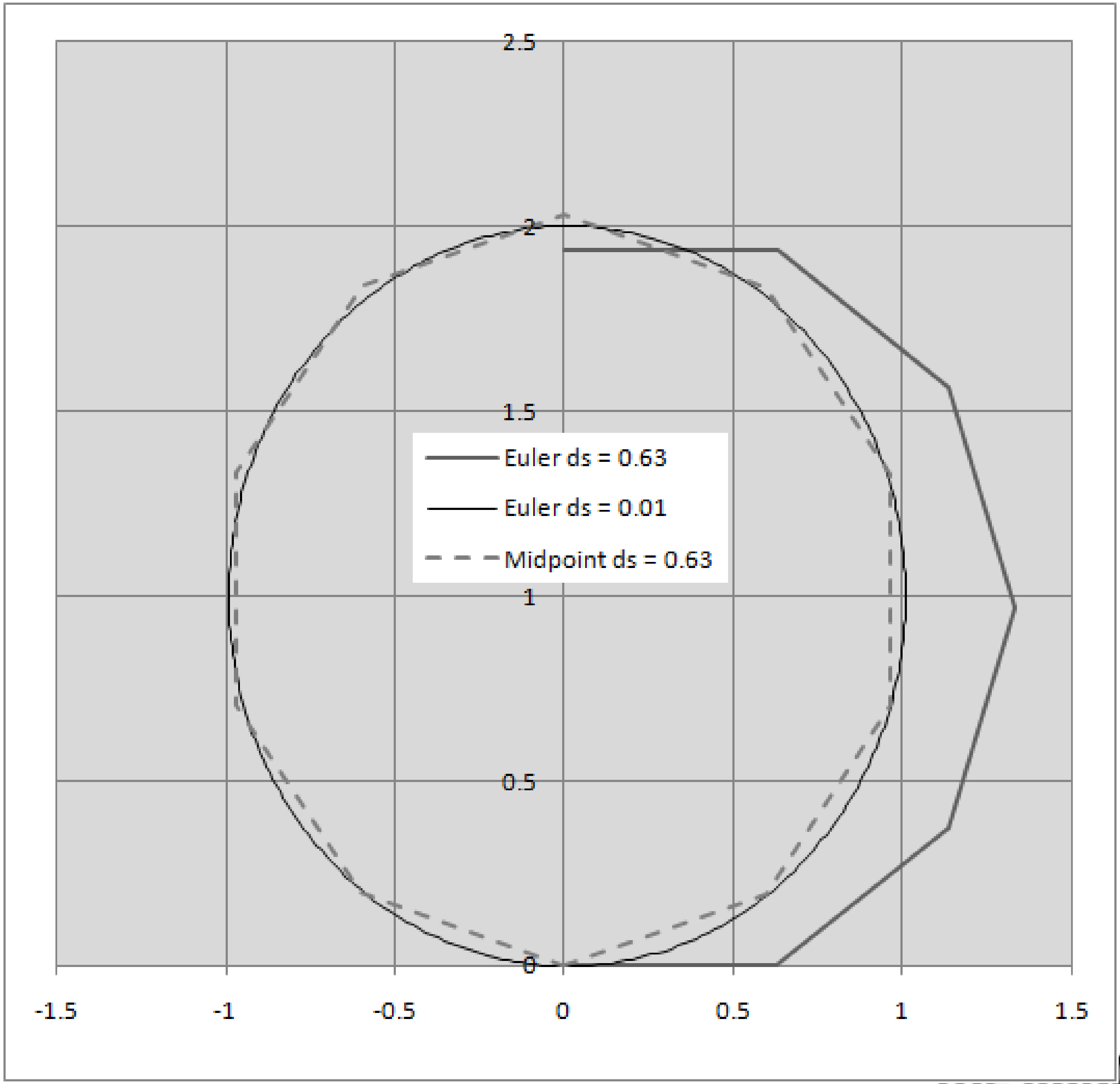
- In discrete time:

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_{k+1} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_k + \begin{bmatrix} \cos\theta \\ \sin\theta \\ \kappa \end{bmatrix}_k \Delta s$$

- Specific curvature profile is constant (arc):

$$\kappa(s) = \kappa_0 = 1$$

Example



Midpoint Algorithm Reduces Error By a Factor Of 20

Carnegie Mellon
THE ROBOTICS INSTITUTE

3.5.2.3 (4th Order) Runge Kutta

- Closest thing to a definitive algorithm for integration.

$$
\begin{aligned}
\underline{k}_1 &= h\underline{f}(\underline{x}, \underline{u}, t)\\
\underline{k}_2 &= h\underline{f}[\underline{x}(t) + \underline{k}_1/2, \underline{u}(t+h/2), t+h/2]\\
\underline{k}_3 &= h\underline{f}[\underline{x}(t) + \underline{k}_2/2, \underline{u}(t+h/2), t+h/2]\\
\underline{k}_4 &= h\underline{f}[\underline{x}(t) + \underline{k}_3, \underline{u}(t+h), t+h]\\
\underline{x}(t+h) &= \underline{x}(t) + \underline{k}_1/6 + \underline{k}_2/3 + \underline{k}_3/3 + \underline{k}_4/6
\end{aligned}
$$

- This can be 1000 times more accurate than Euler’s method.

Outline

- 3.4 Differential Algebraic Systems
- 3.5 Integration of Differential Equations
  - 3.5.1 Dynamic Models in State Space
  - 3.5.2 Integration of State Space Models
  - Summary

Summary

- This is worth knowing about.
- A few lines of code can be the difference between:
  - 100 mm of error after moving 10 meters
  - 0.1 mm of error after moving 10 meters