Chapter 6
State Estimation

Part 3

6.3 Inertial Navigation Systems

Outline

- 6.3 Sensors for State Estimation
  - 6.3.1 Introduction
  - 6.3.2 Mathematics of Inertial Navigation
  - 6.3.3 Errors and Aiding in Inertial Navigation
  - 6.3.4 Example: Simple Odometry Aided AHRS
  - Summary

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History

- Historical roots in German Peenemunde Group.
- Modern form credited to Charles Draper et al. @MIT.
- 1940s Germany:
  - V2 program, gyroscopic guidance
- 1950s Draper Labs, MIT:
  - Shuler tuned INS
  - Floated rate integrating gyros (0.01 deg/hr)
- 1960s DTGs
  - not floated or temp compensated
- 1970s RLGs, USA
- 1980s Strapdown INS
- 1990s GPS

V2

RLG Experiment

Introduction

- Advantages
  - Most accurate dead reckoning available.
  - Useful in wide excursion (outdoor) missions.
  - Work anywhere where gravity is known.
  - Are jamproof - require no external information.
  - Radiate nothing - exhibit perfect stealth.
- Disadvantages
  - Cannot sense accelerations of unpowered space flight.
  - Most errors exhibit Schuler oscillation (advantage?).
  - Most errors are time dependent.
  - Requires input of initial conditions.

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6.3.2 Mathematics of Inertial Navigation
(Concept)

- Use Inertial Properties of Matter
  - Accelerometers
  - Gyros
- Do "Dead Reckoning"
  - Integrate acceleration twice

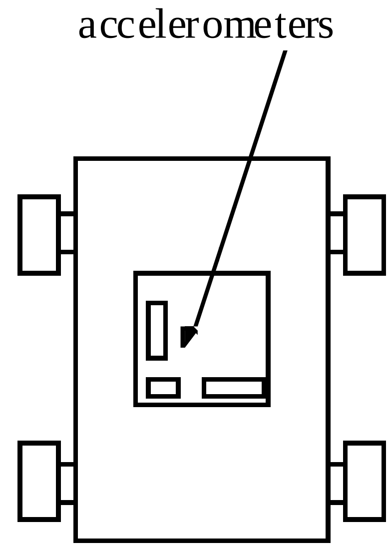
IMU

$$\vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}\,dt$$

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}\,dt$$

Computation

6.3.2 Mathematics of Inertial Navigation (Naïve Concept)



- Just integrating 3 accels will not work for a lot of reasons:
  - Accelerometers measure wrong quantity.
  - They measure it in wrong reference frame.
  - They represent it in wrong coordinate system.
- The quest for ever better engineering solutions to these problems is the primary reason for the complexity of the modern INS.

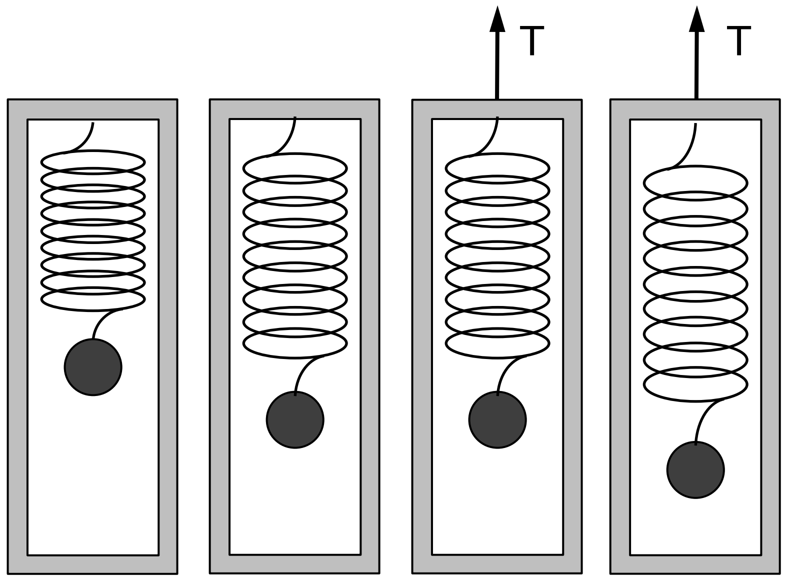
$$\vec{f} \neq \vec{a}$$

$$\vec{a}_v^{\,i} \neq \vec{a}_v^{\,e}$$

$${}^{v}\bar{a}_e^{\,v} \neq {}^{e}\bar{a}_e^{\,v}$$

6.3.2 Problem 1: Equivalence

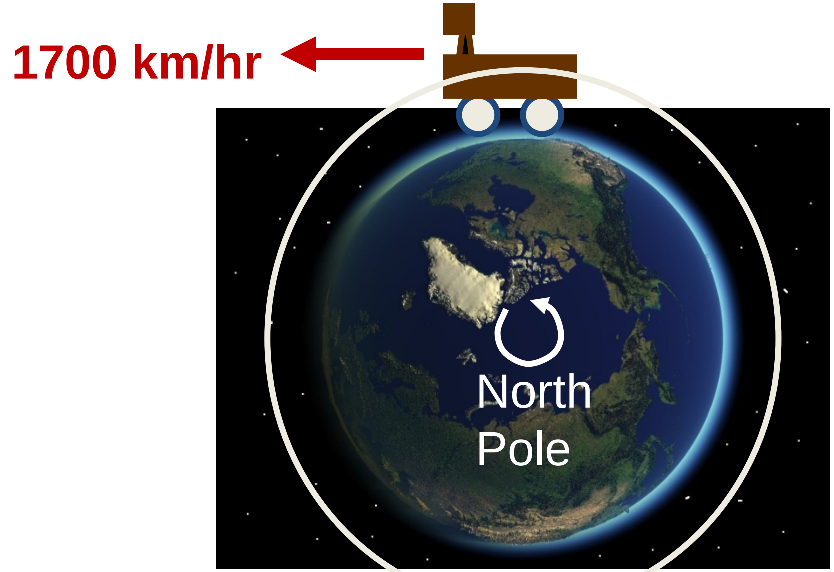
- Accelerometers don't measure acceleration.
- Specific force is: $\vec{f} = \vec{a} - \vec{g}$
- Fix: must know gravity, then: $\vec{a} = \vec{f} + \vec{g}$

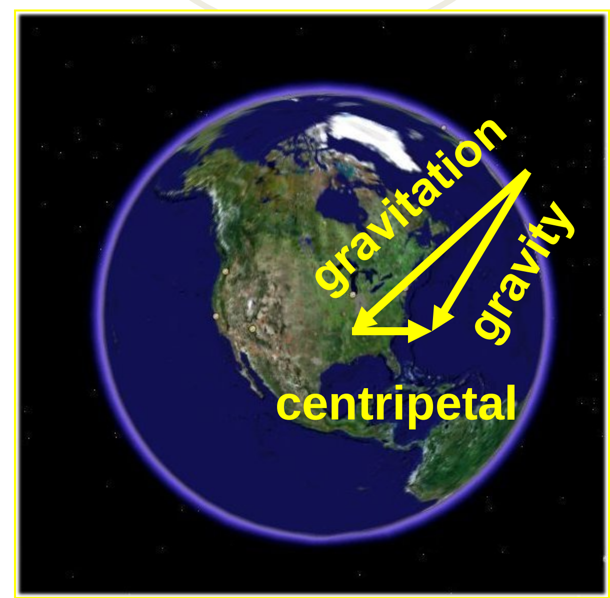


Weight on Earth

Problem 2: Inertial Frame of Reference

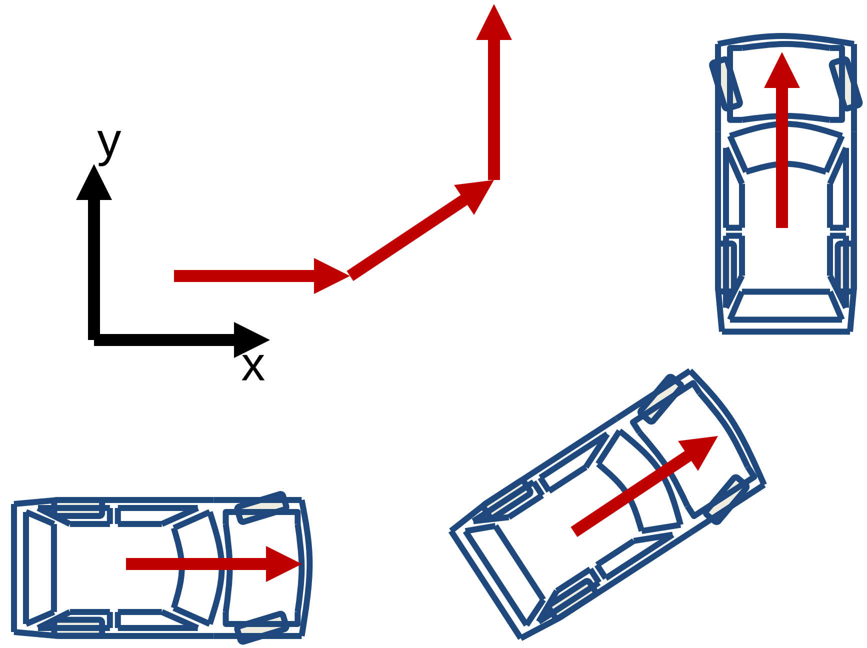
- Now have inertial acceleration.
  - Want earth-referenced acceleration.
- Fix: account for earth angular velocity:
  - “Apparent forces”.
  - “gravity”, not gravitation





Problem 3: Body Coordinates

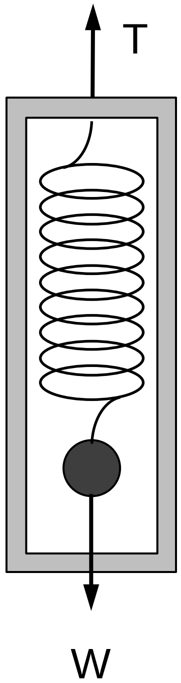
- Accelerometers are fixed to vehicle.
  - Want to integrate in the world frame.
- Need to know instantaneous heading.
- So…, track orientation
  - Use gyros.



$$\theta(t) = \int_0^t \omega(t)dt$$

6.3.2.1 First Fix: Specific Force to Acceleration

- We know specific force is not acceleration.
- The fundamental equation of inertial navigation is Newton's 2nd law applied to the accelerometers:



$$\sum F = \vec{T} + \vec{W} = m\vec{a}^{i}$$

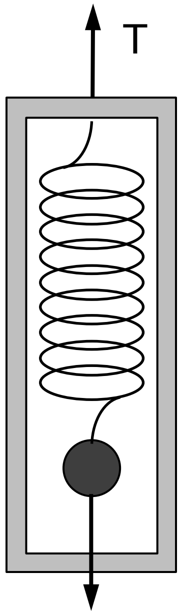
- Need to solve for acceleration….

6.3.2.1 First Fix: Specific Force to Acceleration

- Solving for acceleration:

$$\vec{a}^{\,i} = \frac{\vec{T}}{m} + \frac{\vec{W}}{m} = \vec{t} + \vec{w}$$

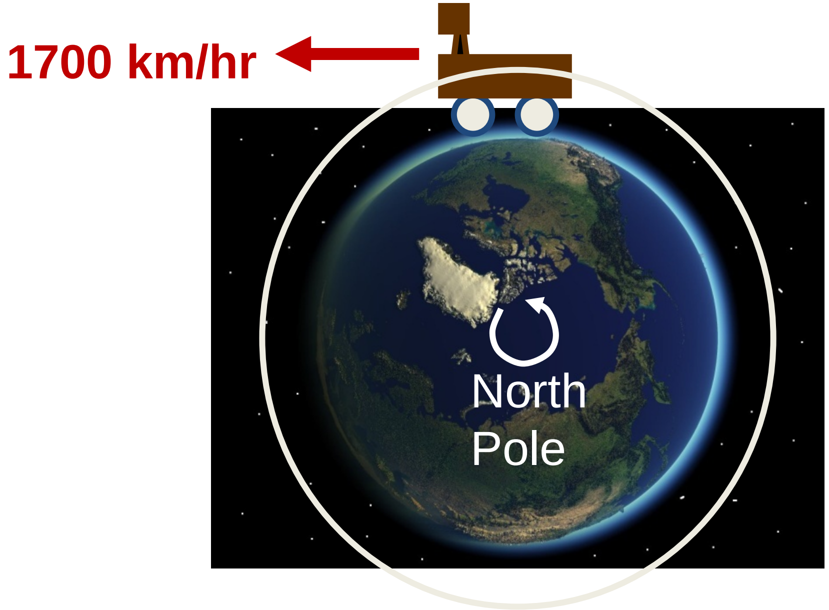
Gravitational field



- Note: you need to know the gravitational field anywhere you want to do inertial navigation.

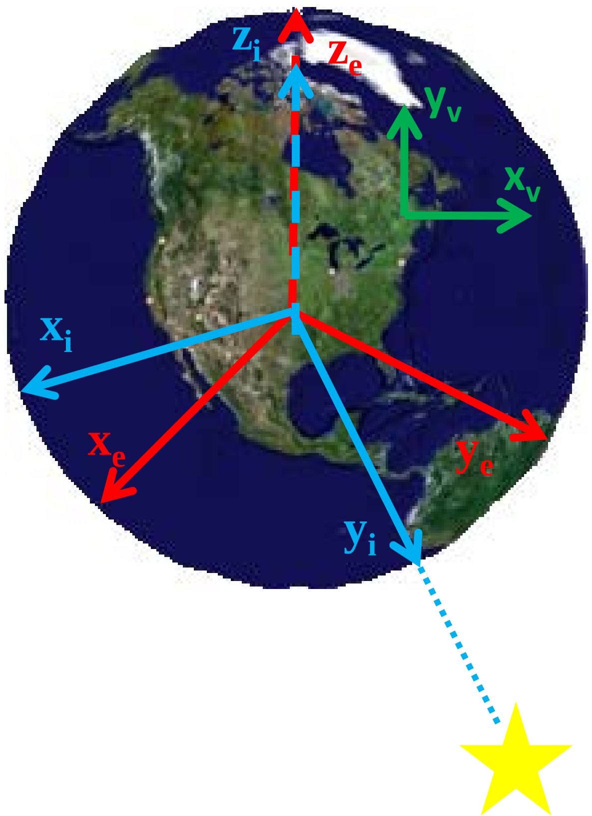
6.3.2.2 Second Fix: Remove Apparent Forces

- Moving vehicle is a moving reference frame.
  - Hence, sensors on-board will sense apparent forces.
  - Remove them with Coriolis law.



6.3.2.2 Second Fix: Remove Apparent Forces

- Define Frames:
  - i: “inertial”, geocentric nonrotating.
  - e: “earth”, geocentric, rotating.
  - v: “vehicle”, fixed to accels. Also known as body frame.



6.3.2.2 Second Fix: Remove Apparent Forces

- Define:

$\vec{r}_v^{\,x}$	Position of vehicle measured in frame x
$\vec{v}_v^{\,x}$	Velocity of vehicle measured in frame x
$\vec{a}_v^{\,x}$	Acceleration of vehicle measured in frame x

6.3.2.2 Second Fix: Remove Apparent Forces

- Basic acceleration transformation under negligible angular acceleration:

$$\vec{a}_o^{\,f} = \vec{a}_o^{\,m} + \vec{a}_m^{\,f} + 2\vec{\omega}_m^{\,f} \times \vec{v}_o^{\,m} + \vec{\omega}_m^{\,f} \times [\vec{\omega}_m^{\,f} \times \vec{r}_o^{\,m}]$$

- Let “o” = v, “m” = e, and “f” = i:

$$\vec{a}_v^{\,i} = \vec{a}_v^{\,e} + \vec{a}_e^{\,i} + 2\vec{\omega}_e^{\,i} \times \vec{v}_v^{\,e} + \vec{\omega}_e^{\,i} \times [\vec{\omega}_e^{\,i} \times \vec{r}_v^{\,e}]$$

- The i and e origins are coincident. Hence:

$$\vec{a}_e^{\,i} = \vec{0}$$

6.3.2.2 Second Fix: Remove Apparent Forces

- Also, let the earth sidereal rate be given by:

$$\vec{\omega}_e^i = \vec{\Omega}$$

- Now, moving the earth acceleration to the left hand side, we have:

$$\vec{a}_v^e = \vec{a}_v^i - 2\vec{\Omega} \times \vec{v}_v^e - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_v^e)$$

- Substituting for specific force:

"Gravity"

$$\vec{a}_v^e = \vec{t} - 2\vec{\Omega} \times \vec{v}_v^e + \vec{w} - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_v^e)$$

Centripetal

6.3.2.2 Second Fix: Remove Apparent Forces

- The quantity: $\vec{w} - \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_v^e)$
- Is known as "gravity" and denoted $\vec{g}$
- Finally, we have "the" equation of inertial navigation.

$$\vec{a}_v^e = \left(\frac{d\vec{v}_v^e}{dt}\right)_e = \vec{t} - 2\vec{\Omega} \times \vec{v}_v^e + \vec{g}$$

Specific Force

This is the derivative of the velocity relative to e as computed by an earth-fixed observer.

Coriolis

Gravity

6.3.2.2 Second Fix: Remove Apparent Forces

- The computed solution in coordinate system independent form is:

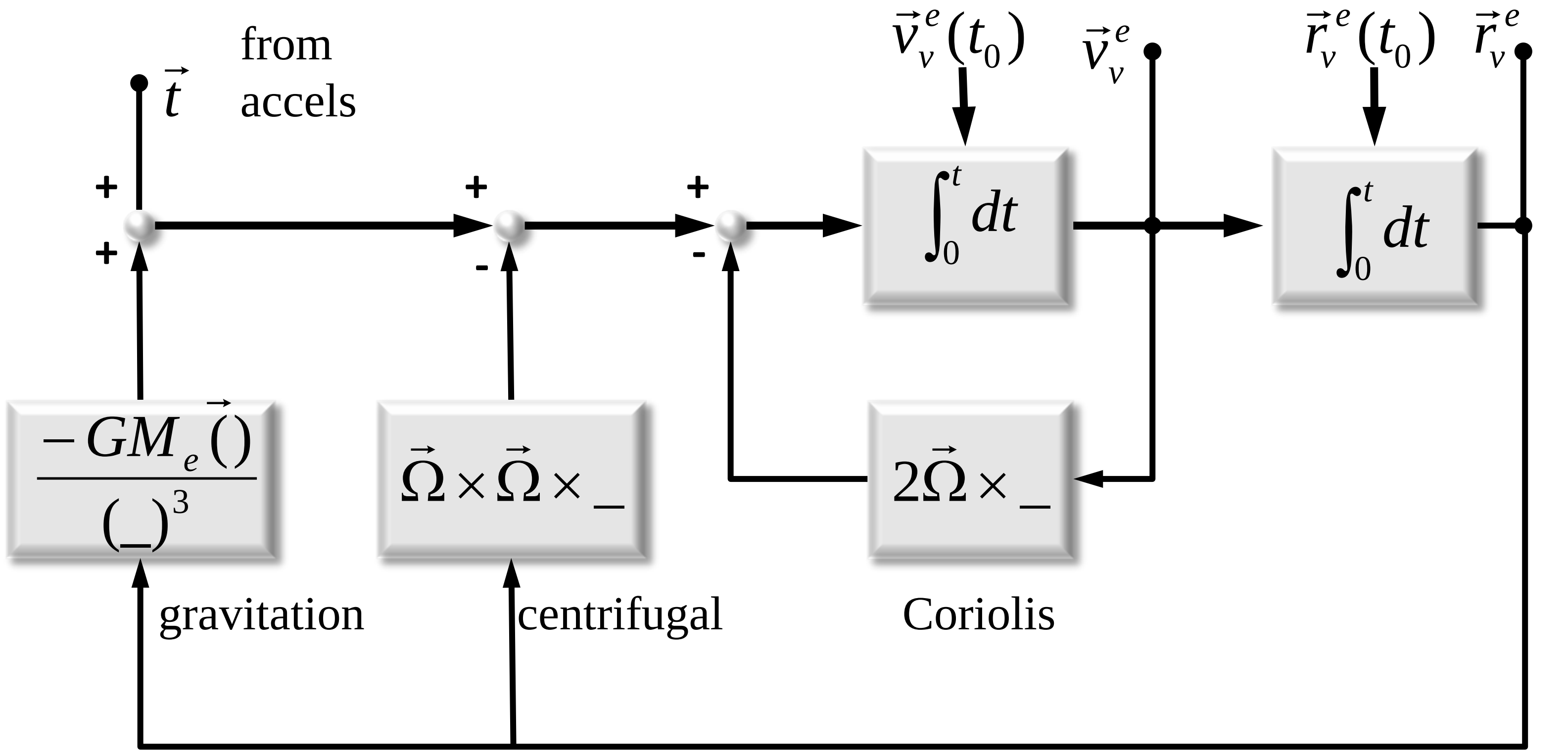
$$\vec{v}_v^{\,e} = \int^{t} [\vec{t} - 2\vec{\Omega} \times \vec{v}_v^{\,e} + \vec{g}]\,dt\Big|_e + \vec{v}_v^{\,e}(t_0)$$

$$\vec{r}_v^{\,e} = \int_0^{t} \vec{v}_v^{\,e}\,dt\Big|_e + \vec{r}_v^{\,e}(t_0)$$

You need to know:
- a model of gravity
- earth sidereal rate
- specific forces
- initial position
- initial velocity
- (gyros don't appear in vector form)

- These are only valid if you integrate in the earth frame (i.e. in earth-fixed coordinates).

6.3.2.2.1 Vector Formulation



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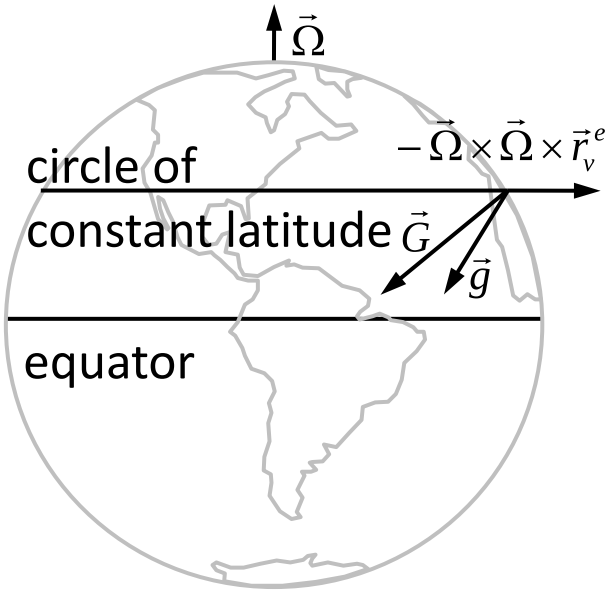
6.3.2.2 Gravity and Gravitation

- Gravity is the force per unit mass required to fix an object wrt the Earth. It includes centrifugal force.
- Gravitation is the force described in Newton's law of gravitation.

$$\vec{W} = \frac{\vec{w}}{m} = -\left[\frac{GM_e \vec{r}_v^{\,e}}{\left|\vec{r}_v^{\,e}\right|^3}\right]$$

- Only at the equator and at the poles does gravity point toward the center of the earth.



6.3.2.3 Third Fix: Adopt a Coordinate System

- The heart of the INS is the inertial measurement unit (IMU) containing 3 accelerometers and 3 gyros.
- The gyros are used to track the orientation of the vehicle wrt the earth.
- You need orientation because:
  - $\vec{g}$ and $\vec{\Omega}$ are known in earth coordinates, whereas....
  - $\vec{t}$ and $\vec{\omega}$ are measured in body coordinates in a modern strapdown system.
- Can't add em up unless they are in the same coordinate system.

6.3.2.3.1 Third Fix: Euler Angles

- Step 1: Integrate the gyros:

$$\begin{bmatrix}\theta\\ \phi\\ \psi\end{bmatrix} = \int_0^t \begin{bmatrix}\dot\theta\\ \dot\phi\\ \dot\psi\end{bmatrix} dt + \begin{bmatrix}\theta\\ \phi\\ \psi\end{bmatrix}_0 = \int_0^t \begin{bmatrix} c\phi & 0 & s\phi \\ t\theta s\phi & 1 & -t\theta c\phi \\ -\frac{s\phi}{c\theta} & 0 & \frac{c\phi}{c\theta}\end{bmatrix} \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} dt + \begin{bmatrix}\theta\\ \phi\\ \psi\end{bmatrix}_0$$

6.3.2.3.2 Third Fix: Direction Cosines

- Step 1: Or, use direction cosine form (better):

$$\delta\underline{\Theta} = \underline{\omega}\delta t \qquad \delta\Theta = \left|\delta\underline{\Theta}\right|$$

$$f_1(\delta\Theta) = \frac{\sin\delta\Theta}{\delta\Theta} \qquad f_2(\delta\Theta) = \frac{(1-\cos\delta\Theta)}{\delta\Theta^2}$$

$$R_{k+1}^{k} = I + f_1(\delta\Theta)[\delta\underline{\Theta}]^{X} + f_1(\delta\Theta)([\delta\underline{\Theta}]^{X})^{2}$$

$$R_{k+1}^{n} = R_k^n R_{k+1}^{k}$$

6.3.2.3.3 Third Fix: Quaternions

- Step 1: Or, use the quaternion form (best):

$$\delta\underline{\Theta} = \underline{\omega}\delta t \qquad \delta\Theta = \left|\delta\underline{\Theta}\right|$$

$$\tilde{q}^{k}_{k+1} = \cos\delta\Theta[I] + \sin\delta\Theta\left[\left({}^{\times}[\tilde{\omega}_b]\right)/\left|\vec{\omega}_b\right|\right]$$

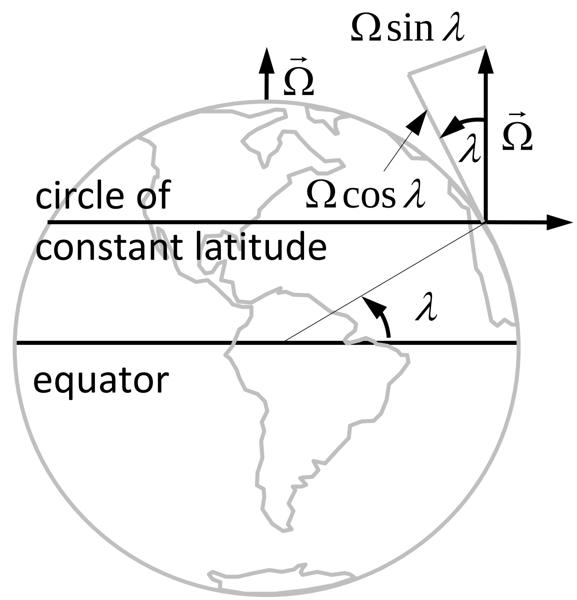
$$\tilde{q}^{n}_{k+1} = \tilde{q}^{k}_{k+1}\tilde{q}^{n}_{k}$$

6.3.2.3.4 Third Fix: Earth Rate Compensation

- When orientation aiding is rare (yaw aiding is typically rare), it may be useful to remove earth rate from the gyros:

$$ {}^{v}\underline{\omega}^{e}_{v} = {}^{v}\underline{\omega}^{i}_{v} - R^{v}_{e} R^{e}_{i} \underline{\Omega}^{i}_{e} $$

- .. Or its projection onto the yaw axis will be integrated.
  - Where is this projection greatest?



$$ {}^{i}\underline{\Omega}^{i}_{n} = \Omega \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{T} $$

$$ {}^{n}\underline{\Omega}^{i}_{n} = \Omega \begin{bmatrix} c\lambda & 0 & -s\lambda \end{bmatrix}^{T} $$

Note: n = e here

6.3.2.3 Third Fix: Adopt a Coordinate System

- Step 2: Integrate the accels:

$$\underline{v}_v^e = \int_0^t [R_v^e \underline{t} + \underline{g} - 2\underline{\Omega} \times \underline{v}_v^e] dt + \underline{v}_v^e(t_0)$$

- Step 3: Integrate the velocity:

$$\underline{r}_v^e = \int_0^t \underline{v}_v^e dt + \underline{r}_v^e(t_0)$$

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6.3.3.1 Sensitivity

For a vehicle at the equator, moving eastward at a velocity of 10 meters per second, and accelerating at 0.1 g

TABLE 3. Term Magnitudes

Term Name	Expression	Nominal Value
Specific Force	$\vec{t}$	0.1 g
Gravitational	$\vec{g}$	1.0 g
Centrifugal	$\vec{\Omega} \times \vec{\Omega} \times \vec{r}_v^e$	$3.5\text{x}10^{-3}$ g
Coriolis	$2\vec{\Omega} \times \vec{v}_v^e$	$1.5\text{x}10^{-4}$ g

- Acceleration is multiplied by the square of time.
  - 1 hour2 = 13 million secs^2.
- After 1 hour, the Coriolis (smallest) term accounts for over 9.5 Km of error.

Error Explosion

- For a 10 m/s vehicle at the equator, the Coriolis term is tiny:
  - 1.5x10-4 g
- Consider an error of this magnitude…
- In one hour:
  - $t^2 = (3600)^2 = 13$ million !!
- Position Error:
  - 9.5 Kilometers!!!

$$\vec{a}_{Coriolis} = 2\vec{\Omega} \times \vec{v}_v^e$$

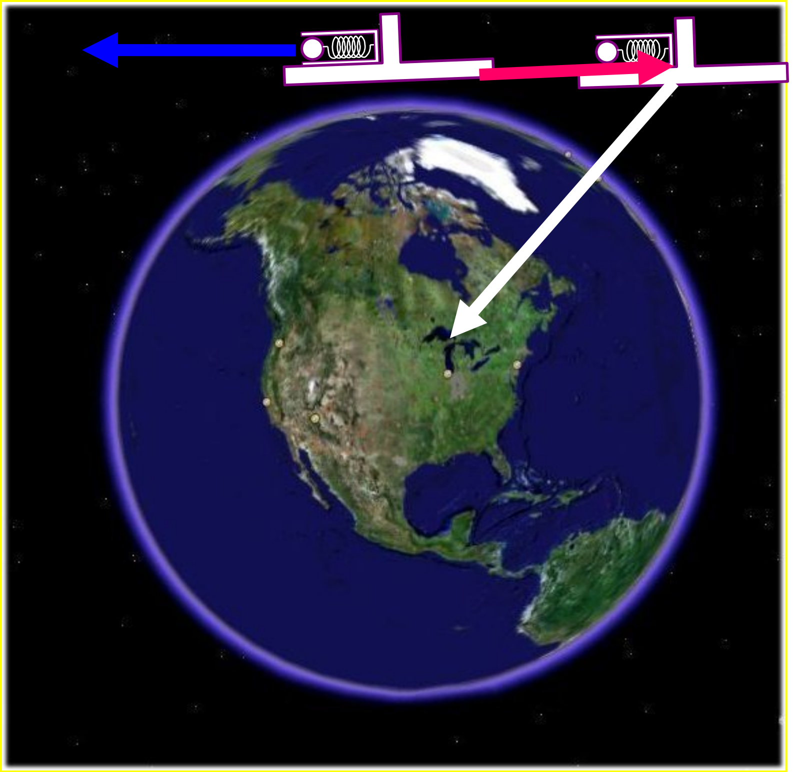
$$\vec{a}_{meas}(t) = \vec{a}_{true}(t) + \vec{a}_{err}(t)$$

$$\vec{r}_{err}(t) = \int_0^t \int_0^t \vec{a}_{err}(t) dt = \frac{\vec{a}_{err} t^2}{2}$$

Error Dynamics: Gravity Feedback

Step 2: Spring is Deflected this way.

Step 1: Orientation error, (system thinks it is level).



Step 3: Interpret as Motion this way.

Step 4: Which rotates gravity prediction until more motion is unnecessary

- Consider predictions of gravity direction based on position.
- This is called a Shuler loop.

Here

Perturbative Analysis

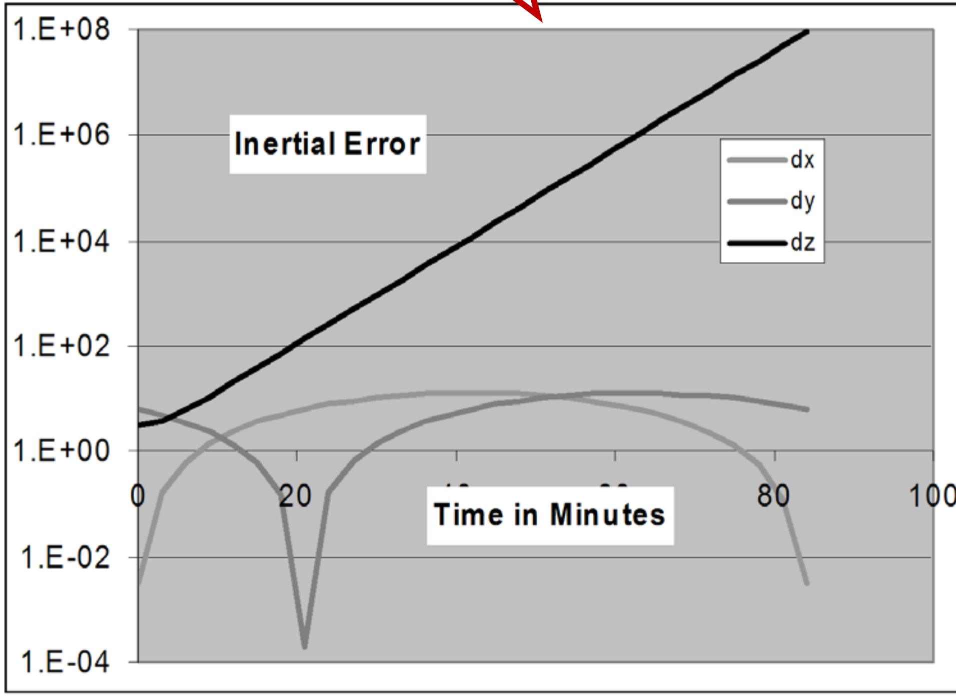
- If the accelerometer biases are constant, the solutions are:

$$\delta x = \frac{\delta t_x}{g_0/R_0}\left[1 - \cos\left(\sqrt{\frac{g_0}{R_0}}t\right)\right]$$

$$\delta y = \frac{\delta t_y}{g_0/R_0}\left[1 - \sin\left(\sqrt{\frac{g_0}{R_0}}t\right)\right]$$

$$\delta z = \frac{\delta t_z}{2g_0/R_0}\left[\cosh\left(\sqrt{\frac{2g_0}{R_0}}t\right)\right]$$

Error magnitudes for 1 micro-g biases.



- Gravity field is a mixed blessing !!

6.3.3.2 Aided Inertial Mode

- Used on mobile robots:
  - Zero velocity update
  - Odometry
  - GPS
  - Landmarks / Map matching
  - Magnetic heading
- Used more generally:
  - Barometric altitude
  - Radar altimeters
  - Doppler radar velocity

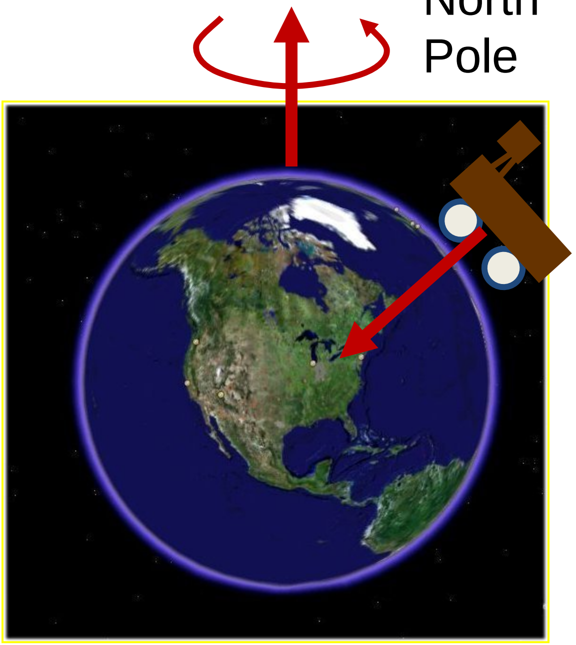
Note: Net effect of **velocity** aiding is to **convert error dynamics from that of free Inertial to that of odometry.**

6.3.3.3 Initialization

- In self alignment, the INS is left stationary and:
  - Accels determine direction of gravity in process called levelling.
  - Gyros determine direction of earth's spin vector in a process called gyrocompassing.
    - Latitude can also be estimated in this way but not longitude.
- Modern GPS aided systems do "moving base alignment" where the difference in GPS readings over time can be used to determine vehicle heading.

Initialization

- Need to measure two non-collinear vectors.
- Earth conveniently has two:
  - Gravity - easy
  - Earth spin – takes time, several minutes
  - Angle between them gives latitude.
- Gives orientation wrt earth and latitude.



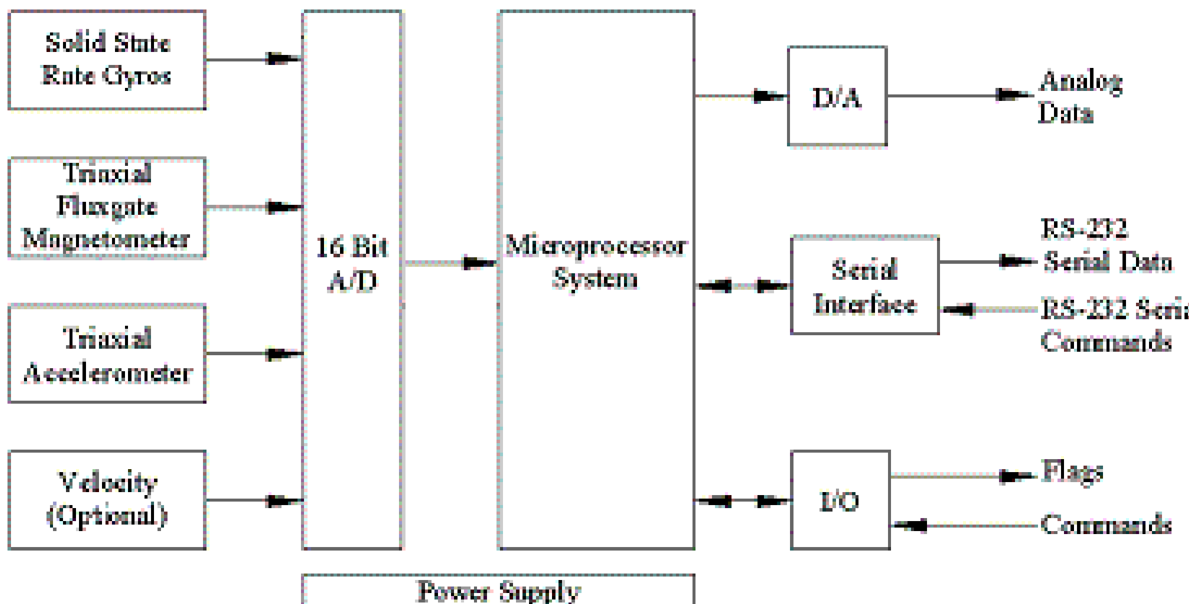
Smiths Industries INS

- Without GPS
  - Static Heading: <0.1 deg. rms
  - Position: <0.35% DT Horizontal
  - Altitude: <0.25% DT Vertical
- With GPS
  - Dynamic Heading: <0.1 deg. rms
  - Position: <10 meters CEP
  - Altitude Accuracy: <10 meters VEP
- Pitch and Roll Outputs: <0.05 deg. rms

- Initialization Time – Static: 3-5 minutes (gyrocompassing)
- Initialization Time – On-the-Move: 1-3 minutes

Watson Industries AHRS E304

- Attitude:
  - 0.25% static, 2% dynamic
- Heading:
  - 1% static, 2% dynamic
- Angular Rate:
  - Scale factor 1%
  - Bias 0.02 deg/sec.
  - Bandwidth 25 Hz
- Acceleration:
  - Scale factor 1%
  - Bias 5 mg
  - Bandwidth 20 Hz



Accuracy

1 nm = 1852 meters = 6078 ft = 1 arc minute on earths surface

- Commercial cruise systems
  - Position: 0.2 nautical miles of error per hour of operation.
    - In some cases, position accuracy along the trajectory (alongtrack) and both normal directions (crosstrack and vertical) are distinguished.
  - Attitude (pitch and roll) : often accurate to 0.05°.
  - Heading: often accurate to 0.5°.
- Land vehicle navigation systems:
  - Position: 0.2% to 2% of distance traveled.
  - Attitude: 0.1°
  - Heading to 0.5°.

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6.3.4 Simple Odometry Aided AHRS

- The AHRS is a degenerate form of inertial navigation system, using much of the same components:
  - indicates orientation only.

Means not stabilized

- Device uses a strapped down IMU today.
  - Accels indicate gravity and acceleration
  - Gyros indicate angular velocity
- Distinguishing acceleration from gravity is still an issue - but less so.

6.4.3.1 Nav Eqns in Body Frame

- Recall the inertial nav equation (Eq 6.46):

$$\vec{a}_v^e = \left(\frac{d\vec{v}_v^e}{dt}\right)_e = \vec{t} - 2\vec{\Omega} \times \vec{v}_v^e + \vec{g}$$

- Lets express this in the body frame so that it becomes unnecessary to known orientation.
- Use the Coriolis theorem:

$$\left(\frac{d\vec{v}_v^e}{dt}\right)_v = \left(\frac{d\vec{v}_v^e}{dt}\right)_e + \vec{\omega}_v^e \times \vec{v}_v^e$$

This adds another Apparent Coriolis Force.

6.4.3.1 Nav Eqns in Body Frame

- Define the strapdown angular velocity:

$$\vec{\omega} = \vec{\omega}_v^i = \vec{\omega}_v^e + \vec{\omega}_e^i = \vec{\omega}_v^e + \vec{\Omega}$$

- Write the inertial navigation equation in the body frame:

$$\left(\frac{d\vec{v}_v^e}{dt}\right)_v = \vec{t} - (\vec{\omega} + \vec{\Omega}) \times \vec{v}_v^e + \vec{g}$$

$$\vec{a}_v^e = \left(\frac{d\vec{v}_v^e}{dt}\right)_e = \vec{t} - 2\vec{\Omega} \times \vec{v}_v^e + \vec{g}$$

Recall Earth Frame

- For this purpose, earth rate can be neglected, so:

$$\left(\frac{d\vec{v}_v^e}{dt}\right)_v = \vec{t} - \vec{\omega} \times \vec{v}_v^e + \vec{g}$$

6.4.3.1 Nav Eqns in Body Frame

- Solve for gravity:

This vanishes on an Ackerman vehicle during periods of constant speed. Otherwise, differentiate numerically.

$$\vec{g} = \left(\frac{d\vec{v}_v^e}{dt}\right)_v - \vec{t} + \vec{\omega} \times \vec{v}_v^e$$

Simply remove Coriolis term from the accel readings.

- Everything on right is known from measurements. g on left is known in world coordinates.

6.4.3.1 Nav Eqns in Body Frame

- Write this in body coordinates:

$$R_w^v \underline{g} = \frac{\mathrm{d}(\underline{{}^v v_v^e})}{\underline{\mathrm{d}t}} - \underline{t} + \underline{{}^v\omega} \times \underline{{}^v v_v^e} = {}^v\underline{g}_{meas}$$

- Can solve this for attitude (not yaw) in the rotation matrix using inverse kinematics.
  - Rotation around g is not observable.

6.4.3.2 Solving for Attitude

- To get the attitude, express in body frame:

$$R_w^v \underline{g} = \frac{d(^v\underline{v}_v^e)}{dt} - \underline{t} + {}^v\underline{\omega} \times {}^v\underline{v}_v^e = {}^v\underline{g}_{meas}$$

- Where:

$$Roty(\theta)Rotx(\phi)\underline{g} = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$$

- The transpose converts from world to body, thus:

$${}^v\underline{g}_{meas} = R_w^v {}^w\underline{g} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ s\theta s\phi & c\phi & c\theta s\phi \\ s\theta c\phi & -s\phi & c\theta c\phi \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = g \begin{bmatrix} -s\theta \\ c\theta s\phi \\ c\theta c\phi \end{bmatrix}$$

6.4.3.2 Solving for Attitude

- The solution is:

$$tan\theta = s\theta / c\theta = -g_x / (\sqrt{g_y^2 + g_z^2})$$

$$tan\phi = s\phi / c\phi = g_y / g_z$$

- To get the yawrate, solve:

$$\dot{\psi} = \frac{s\phi}{c\theta}\omega_y + \frac{c\phi}{c\theta}\omega_z$$

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  - 6.3.2 Mathematics of Inertial Navigation
  - 6.3.3 Errors and Aiding in Inertial Navigation
  - 6.3.4 Example: Simple Odometry Aided AHRS
  - Summary

Summary

- Black magic ?
- Hard to do well.
  – Costs big bucks.
- Most accurate dead reckoning available.
  – Cruise: 0.2 nautical miles of error per hour of operation.
- Indispensable on outdoor mobile robots.
- Complementary technology to GPS.

Summary

- Inertial navigation is based on Newton's laws
  - Works everywhere that gravity is known.
  - It is stealthy and jamproof.
- Modern "strapdown" systems
  - "computationally stabilized".
  - no stabilized platform
- Naive approaches are seriously flawed. Must compensate for
  - Gravity
  - inertial forces
  - body fixed coordinates.

Summary

- Free inertial performs miserably…
  - 1 part in 10,000 acceleration error causes kilometers of position error after 1 hour of operation.
- Interesting Error Dynamics
  - Horizontal errors bounded, oscillate every 84 minutes
  - Vertical position is unstable without damping devices
- An AHRS unit can find attitude from accelerometers and gyros and odometry.