Chapter 5
Optimal Estimation

Part 1

5.1 Random Variables, Processes and Transformation

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Outline

- 5.1 Random Variables, Processes and Transformation
  - 5.1.1 Characterizing Uncertainty
  - 5.1.2 Random Variables
  - 5.1.3 Transformation of Uncertainty
  - 5.1.4 Random Processes
  - Summary

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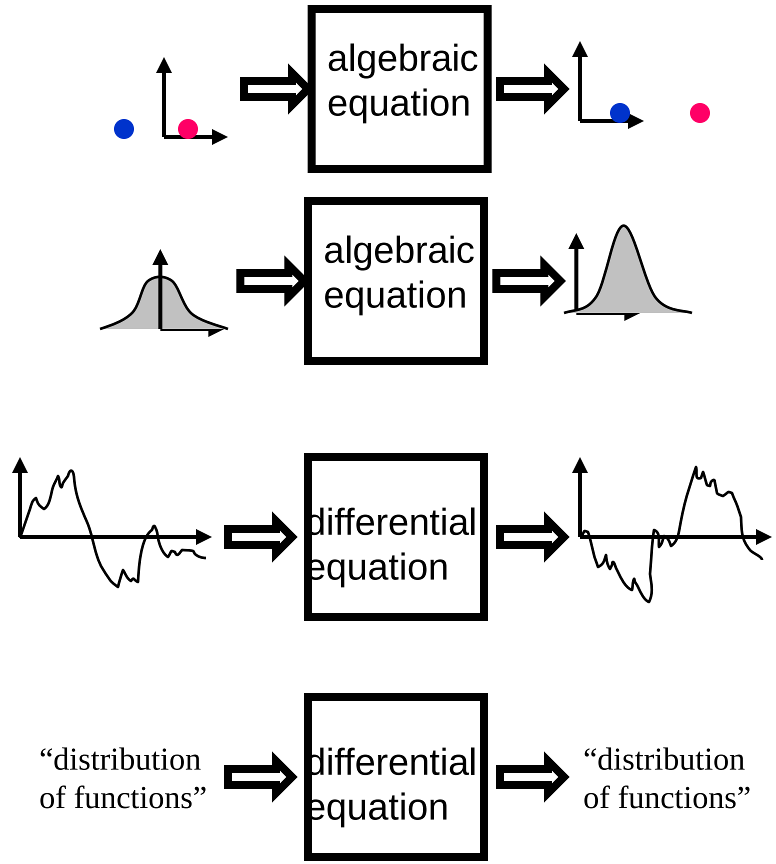
5.1.1 Characterizing Uncertainty

- Uncertainty =
  - Not Known: Bias, scale systematic error
    - E.g. Temperature sensitivity
  - Not Knowable: Noise, randomness, unpredictability
    - E.g. “drift”
- Fact of life:
  - Some randomness is fundamental
  - It can’t be measured.
- Humans do a good job coping…

Modeling Uncertainty

- An oxymoron?
- Distributions are models.
- Algebraic and differential equations are models.
- We can "pass distributions through" equations to get other distributions.
  - one point at a time, or…
  - as a complete distribution



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5.1.1.1 Types of Uncertainty

- We usually consider it to be additive:

$$x_{meas} = x_{true} + \varepsilon \quad \hat{x} = x_{true} + \varepsilon$$

Hat means estimate

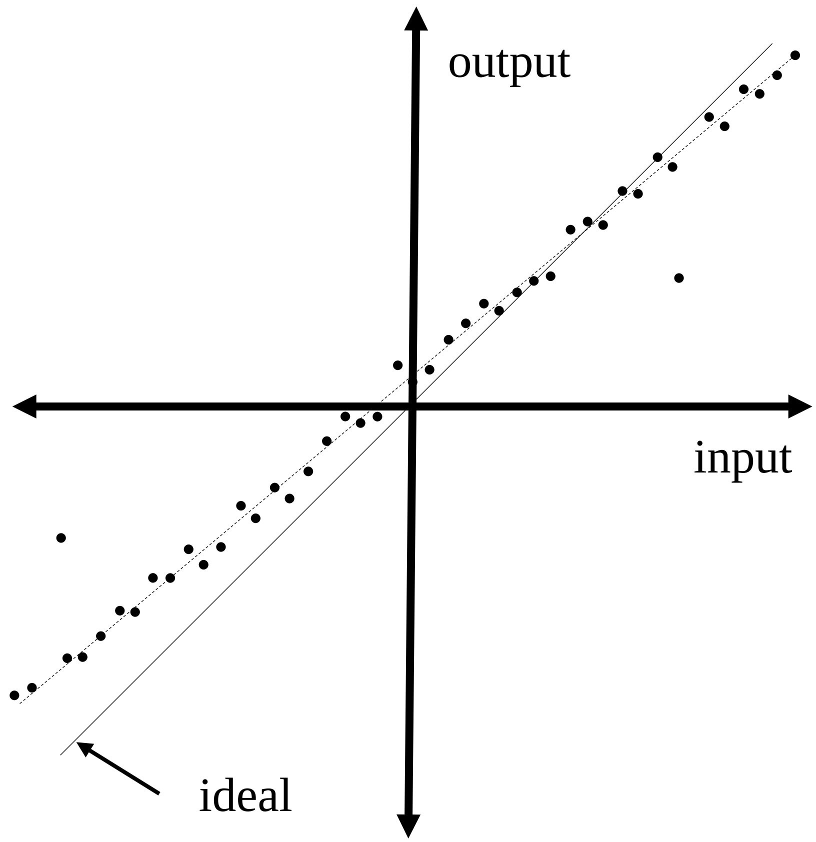
- ε may be zero, a constant, or a function of anything.
- ε may be:
  - Systematic (="deterministic")
  - Random (= "stochastic")
  - a combination of both.

Random error is called **unbiased** of it has a mean of zero.

- Most of all ε is unknown. Otherwise we would take it out.

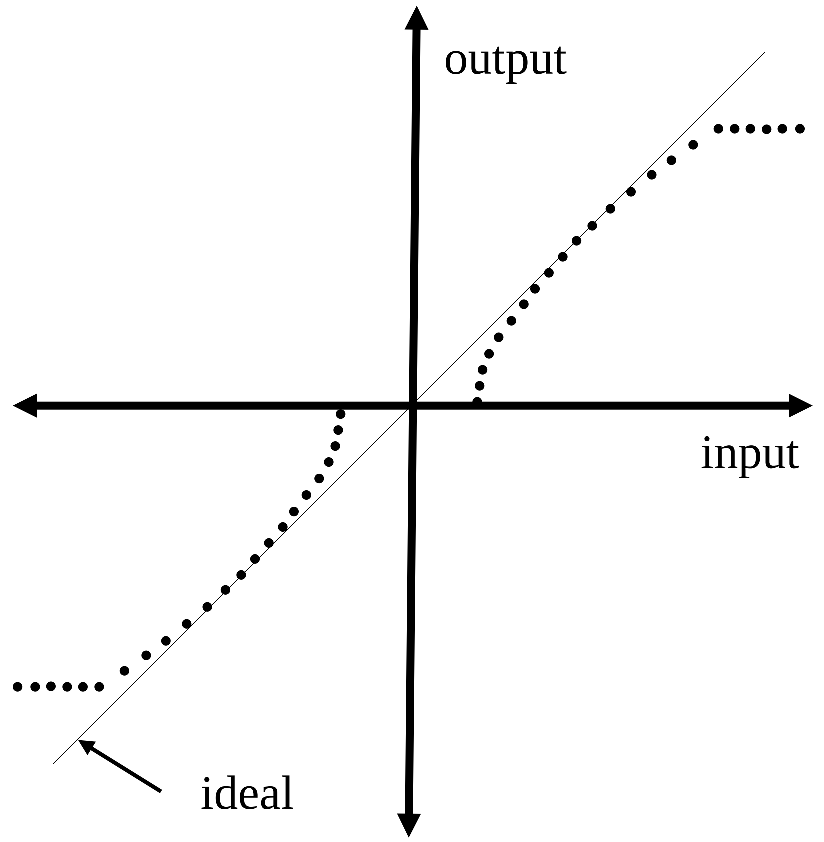
5.1.1.1 Real and Ideal Signals

- Below: bias, scale errors, and two "outliers".



5.1.1.1 Real and Ideal Signals

- Below: Saturation, nonlinearity, deadband

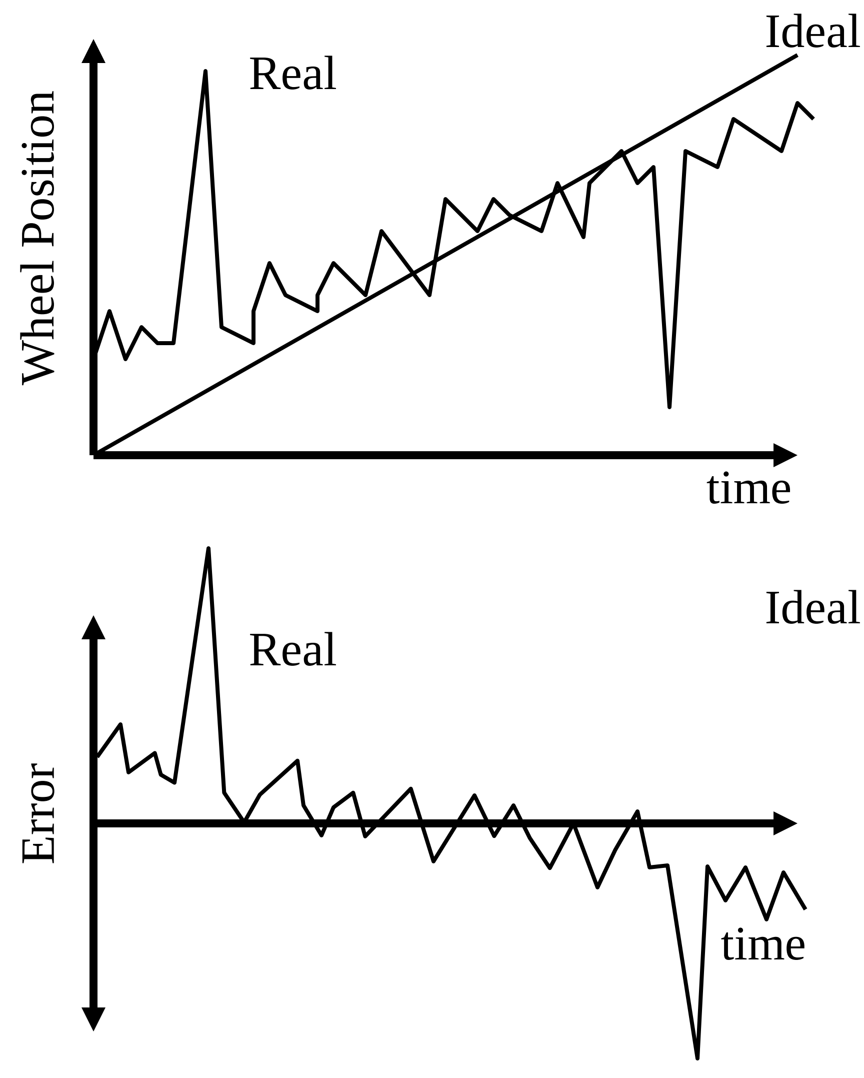


5.1.1.1 Real and Ideal Signals

- Might model the errors like so:

$$\varepsilon = a + b\theta + N(\mu, \sigma)$$

- Note the appearance of model parameters of both kinds:
  - systematic (a,b)
  - stochastic (μ,σ)



Removing errors

- Systematic → calibration:
  - Fit a line to the last graph
- Stochastic → filtering
  - Smooth out the wiggles
- Correlation → differential measurement
  - Reject the common component

Know Your Model

- You can fit a line to anything.

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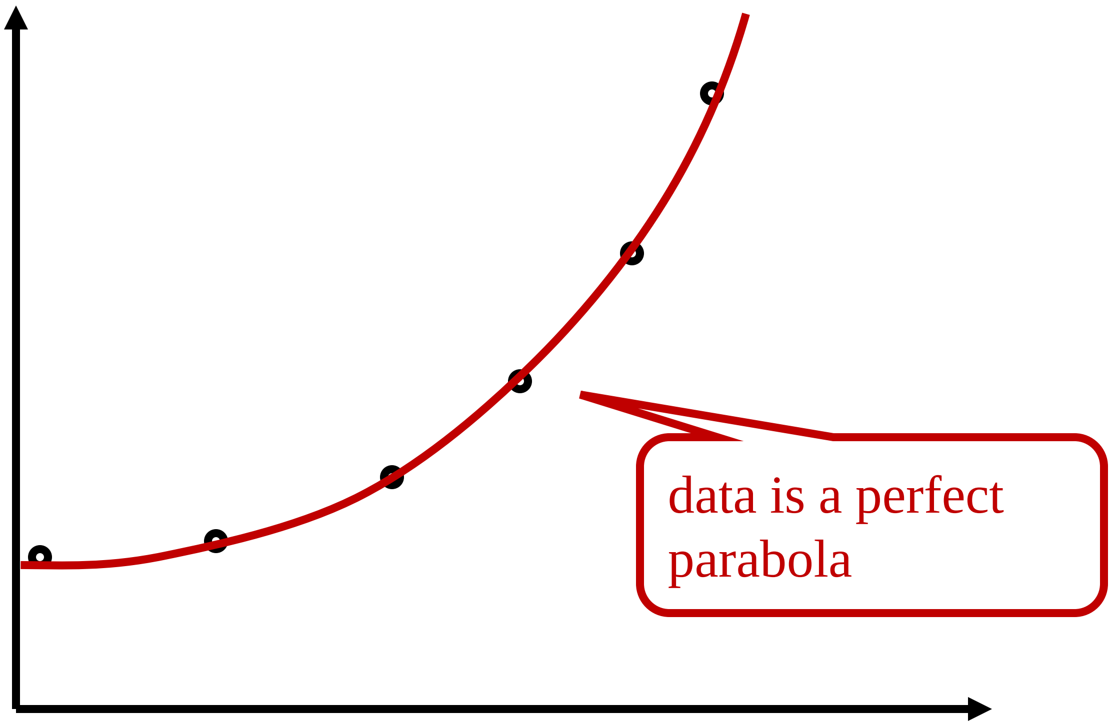
Know Your Model

- You can fit a line to anything.



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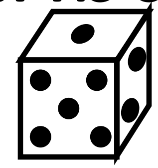
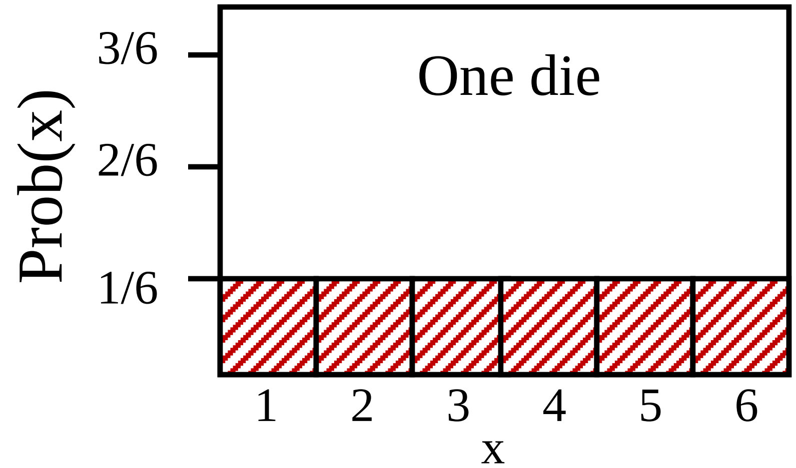
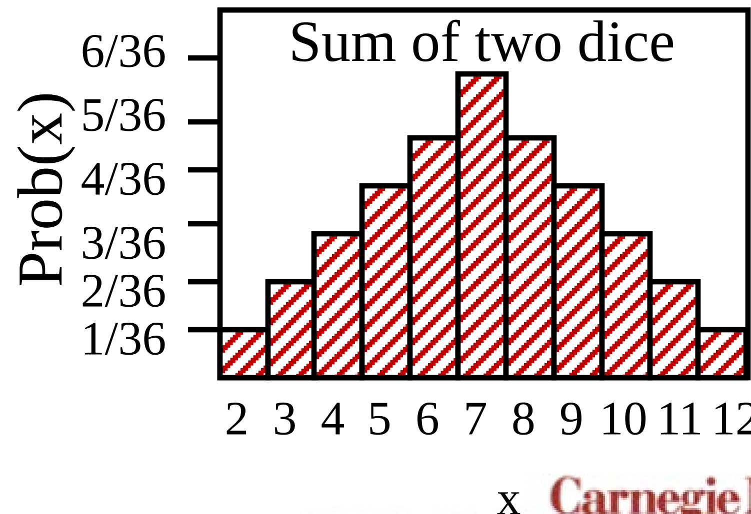
Probability as Frequency Distribution



- Events that occur randomly may nonetheless have a knowable probability distribution.



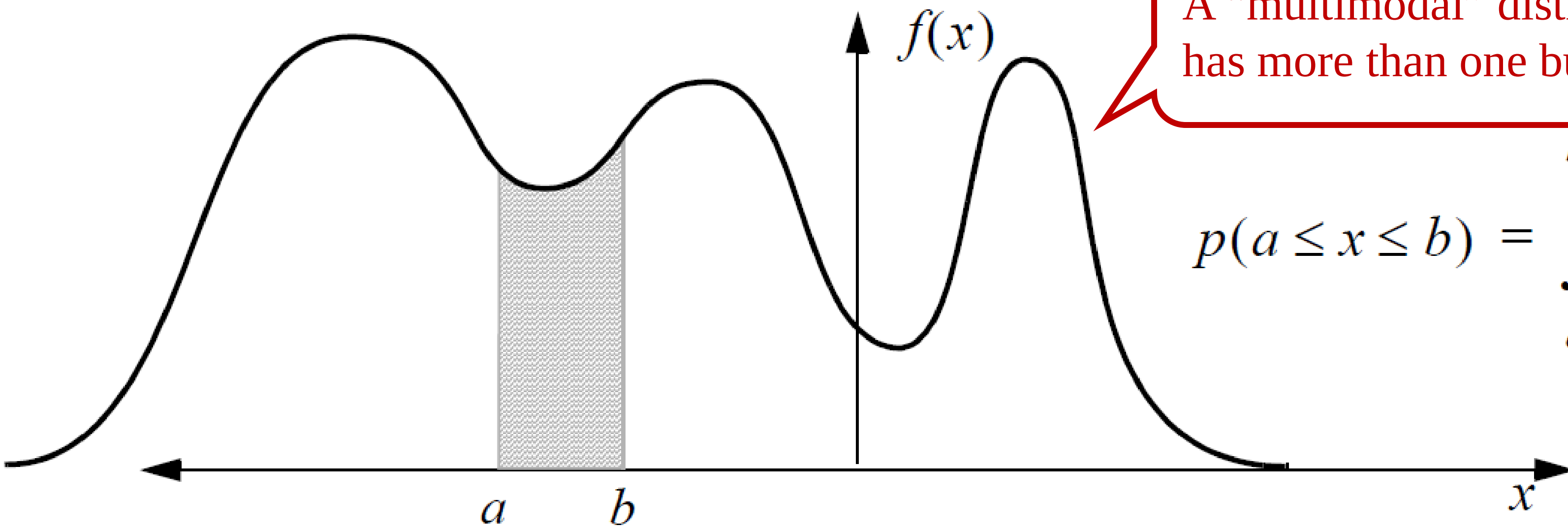
- Its not unusual to know the distribution but never be able to perfectly predict an individual event.



- Knowing one distribution allows you to compute others.
  - Math on distributions is well defined.

Continuous Random Variable

- Pdf – probability density function.



A “multimodal” distribution has more than one bump in it.

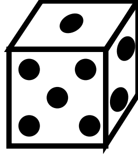
$$p(a \le x \le b) = \int_a^b p(u)\,du$$

Little p

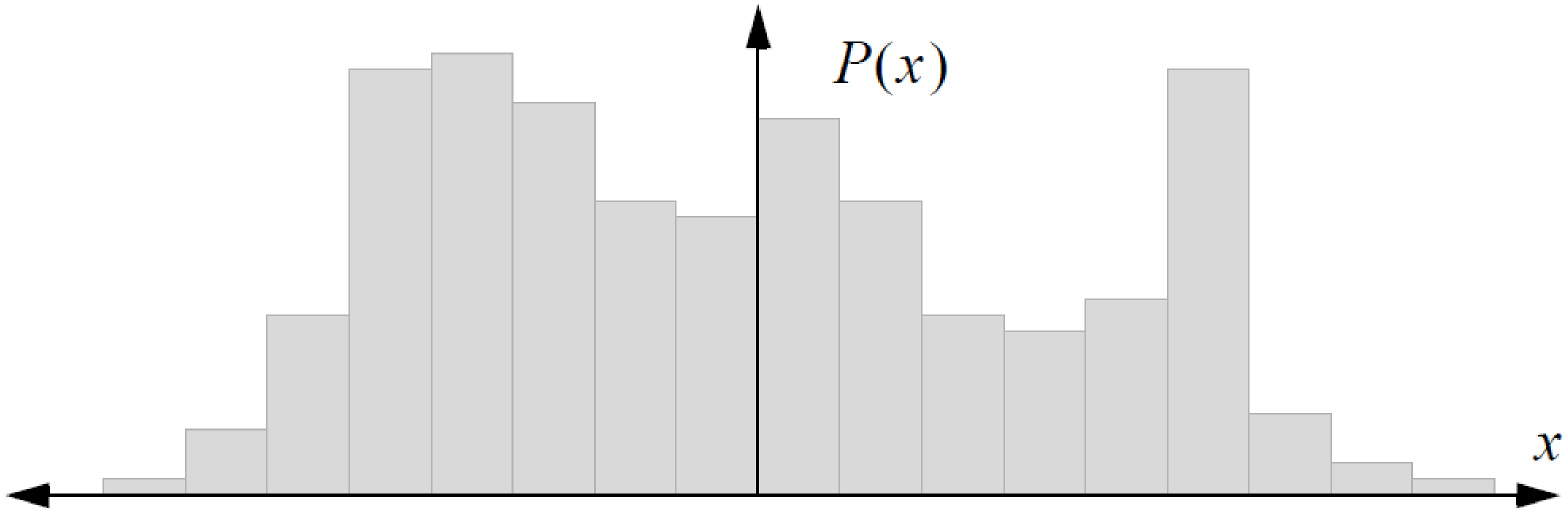
$$p(a \le x \le b) = \int_a^b f(u)\,du$$

- Describes probability of each possible outcome of a single experiment.

Discrete Random Variable



- Pf – probability function.

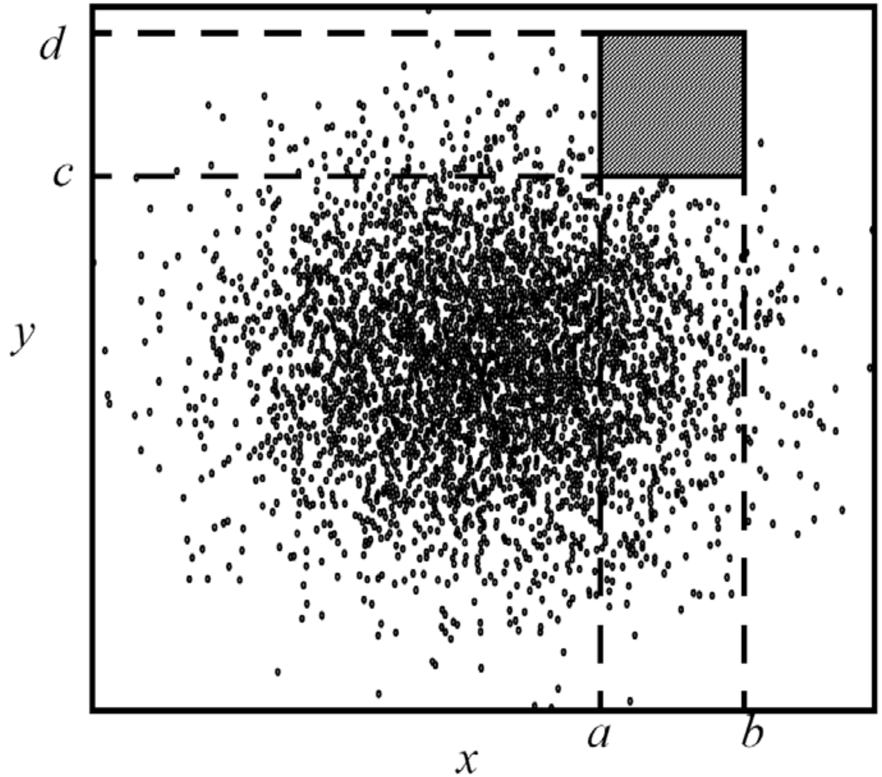


Big p → $$P(a \le x \le b) = \sum_i P(x_i)$$

For the sum to hold, the outcomes must be <u>mutually exclusive</u>.

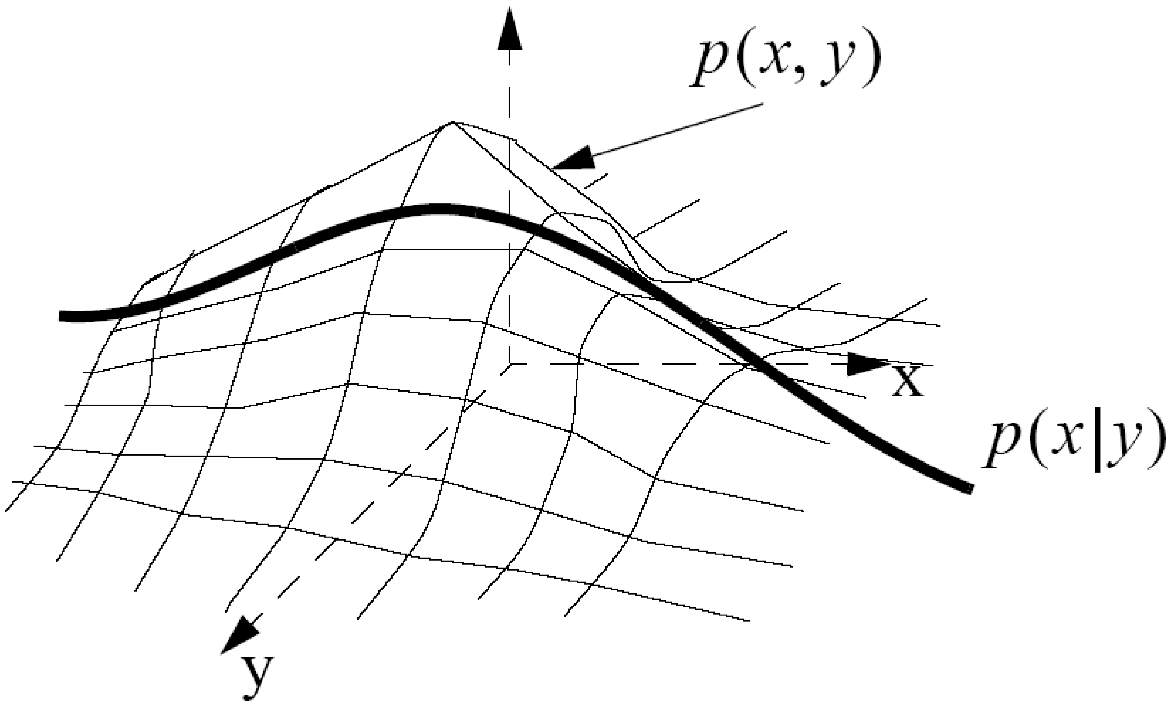
- Describes probability of each possible outcome of a single event.

(Joint) 2D Distributions



$$p(a \le x \le b \wedge c \le y \le d) = \int_{c}\int_{a} f(u, v)\,du\,dv$$

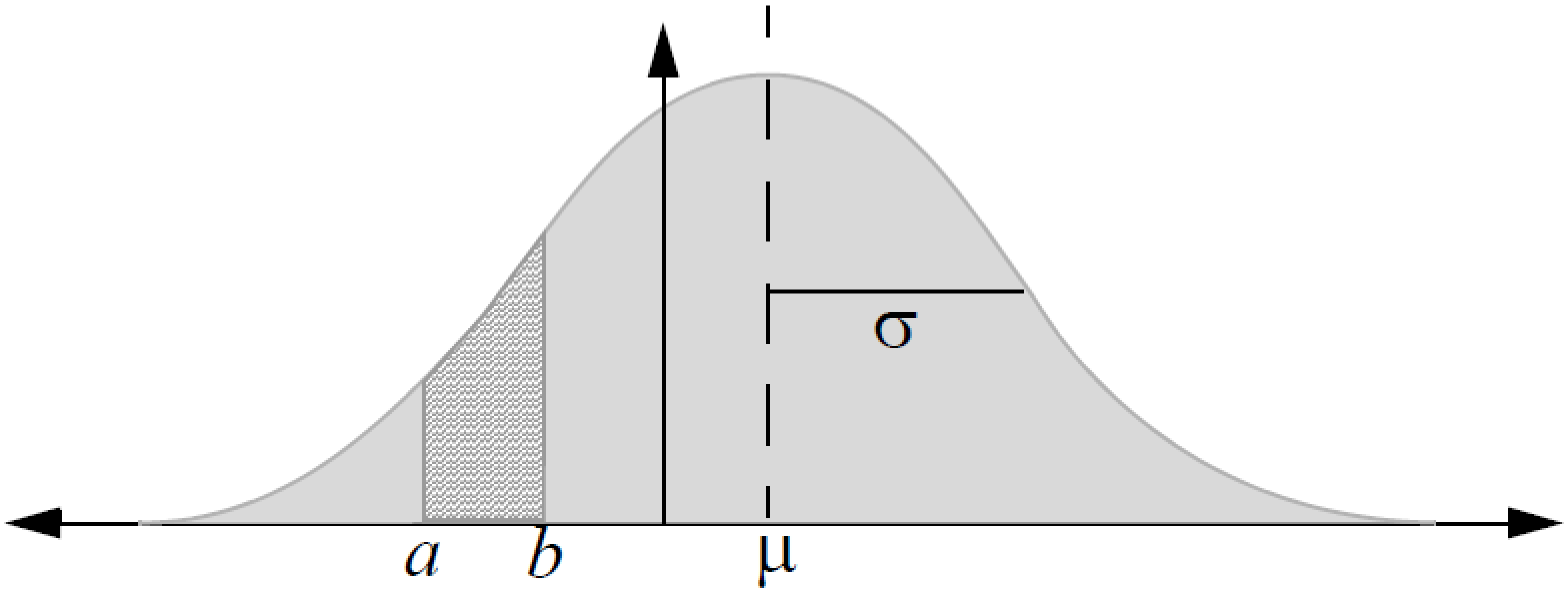
(Conditional) 2D Distributions



$$p(a \le x \le b \mid y) = \int_{a} f(u, y)\,du \Big/ \int_{-\infty} f(u, y)\,du$$

- Take a slice and renormalize.

Gaussian Pdf



$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

N Dimensional Gaussian

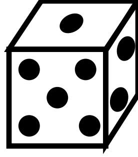
- Formula:

$$p(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{|C|}} exp\left(-\frac{[\underline{x}-\underline{\mu}]^T C^{-1}[\underline{x}-\underline{\mu}]}{2}\right)$$

- Mahalanobis distance:

$$[\underline{x}-\underline{\mu}]^T C^{-1}[\underline{x}-\underline{\mu}]$$

- C is "covariance matrix" defined later.



5.1.2.2 Expectation

- For any function of x, this is just a weighted average where the pdf is the weight.

$$Exp[h(x)] = \int_{-\infty}^{\infty} h(x)p(x)dx \quad \textit{scalar-scalar continuous}$$

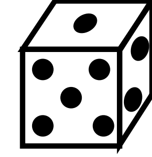
$$Exp[\underline{h}(x)] = \int_{-\infty}^{\infty} \underline{h}(x)p(x)dx \quad \textit{vector-scalar continuous}$$

$$Exp[\underline{h}(\underline{x})] = \int_{-\infty}^{\infty} \underline{h}(\underline{x})p(\underline{x})d\underline{x} \quad \textit{vector-vector continuous}$$

Notation means volume integral

$$Exp[\underline{h}(x)] = \sum_{i=1}^{n} \underline{h}(\underline{u}_i)P(\underline{u}_i) \quad \textit{vector-vector discrete}$$

- This is a functional or moment (with infinite limits of integration) so you need the entire pdf to work it out.



5.1.2.2 Expectation

- Properties inherited from integrals.

$$Exp[k] = k$$

$$Exp[kh(x)] = kExp[h(x)]$$

$$Exp[h(x) + g(x)] = Exp[h(x)] + Exp[g(x)]$$

Expectation is a linear operator over <u>functions</u>.

5.1.2.2 Mean

- Set h(x) → x etc.

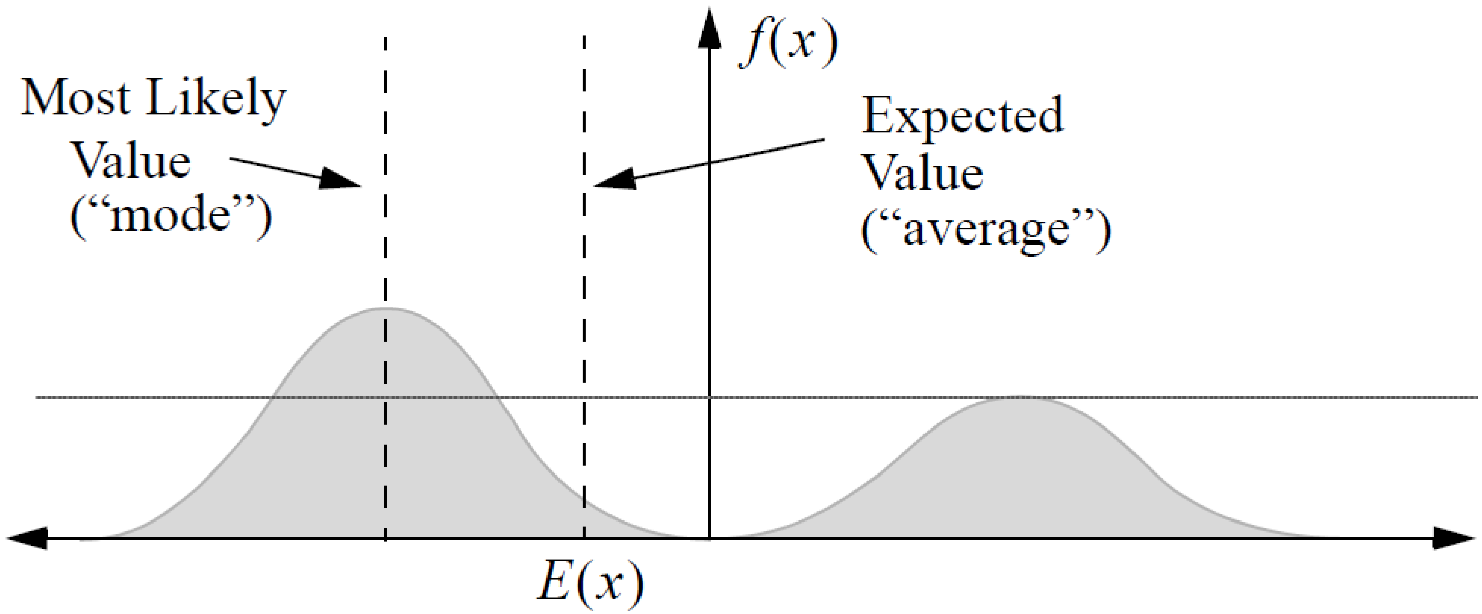
$$\mu = Exp[x] = \int_{-\infty}^{\infty} [xp(x)]dx \qquad \textit{scalar continuous}$$

$$\underline{\mu} = Exp(\underline{x}) = \int_{-\infty}^{\infty} \underline{x}p(\underline{x})d\underline{x} \qquad \textit{vector continuous}$$

$$\underline{\mu} = Exp[\underline{x}] = \sum_{i=1}^{n} \underline{x}_i P(\underline{x}_i) \qquad \textit{vector discrete}$$

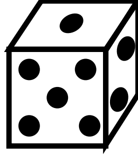
- This is a property of the distribution of the population.

5.1.2.2 Mean and Most Likely Value



- Expected value is a centroid.
- It is not always the most likely value to occur.

Variance of a Random Scalar



- Set h(x) → [x-μ]2.

$$\sigma_{xx} = \int_{-\infty}^{\infty} [(x-\mu)^2 \cdot p(x)]dx$$

- Alternative notation: $\sigma^2(x)$

- Standard deviation defined as:

$$\sigma_x = \sigma(x) = \sqrt{\sigma_{xx}}$$

Recall : “Outer” Product

- Opposite of “inner” or dot product.

$$\underline{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

- Generates a symmetric matrix from a vector.

$$\underline{x}\,\underline{x}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{bmatrix}$$

Co-Variance of a Random Vector

- Continuous and discrete cases.

$$\Sigma = E([\underline{x}-\underline{\mu}][\underline{x}-\underline{\mu}]^T) = \int_{-\infty}^{\infty}[\underline{x}-\underline{\mu}][\underline{x}-\underline{\mu}]^T f(\underline{x})\,d\underline{x}$$

$$\Sigma = E([\underline{x}-\underline{\mu}][\underline{x}-\underline{\mu}]^T) = \sum_{i=1}^{n}[\underline{x}-\underline{\mu}][\underline{x}-\underline{\mu}]^T p(\underline{x})$$

- Integral of a matrix is the matrix of the integrals.

Sample Statistics

- Mean:

$$\bar{\underline{x}} = \frac{1}{n}\sum_{i=1}^{n} \underline{x}$$

- Sample Covariance.

?

$$S = \frac{1}{n}\sum_{i=1}^{n} [\underline{x} - \underline{\mu}][\underline{x} - \underline{\mu}]^T$$

- Elemental variances and co variances

$$s_{ii} = \frac{1}{n}\sum_{i=1}^{n} [x_i - \mu_i][x_i - \mu_i] \quad s_{ij} = \frac{1}{n}\sum_{i=1}^{n} [x_i - \mu_i][x_j - \mu_j]$$

5.1.2.3 Sampling Distributions and Statistics

- "Batch" Methods:

$$\underline{\bar{x}} = \frac{1}{n}\sum_{i=1}^{n} \underline{x}$$

$$S = \frac{1}{n}\sum_{i=1}^{n} [\underline{x} - \underline{\mu}][\underline{x} - \underline{\mu}]^T$$

- Not feasible computationally for continuous update when N is large.

5.1.2.4 Computing Sample Statistics

- "Recursive" Methods:

$$\bar{\underline{x}}_{k+1} = \frac{(k\bar{\underline{x}}_k + \underline{x}_{k+1})}{(k+1)}$$

$$S_{k+1} = \frac{kS_k + [\underline{x}_{k+1} - \underline{\mu}][\underline{x}_{k+1} - \underline{\mu}]^T}{(k+1)}$$

- Related to the Kalman Filter.

Computing Sample Statistics

- "Calculator" Methods use accumulators:

mean $\underline{T}_{k+1} = \underline{T}_k + \underline{x}_{k+1}$ when data arrives

$\bar{\underline{x}}_{k+1} = \frac{T_{k+1}}{(k+1)}$ when answer necessary

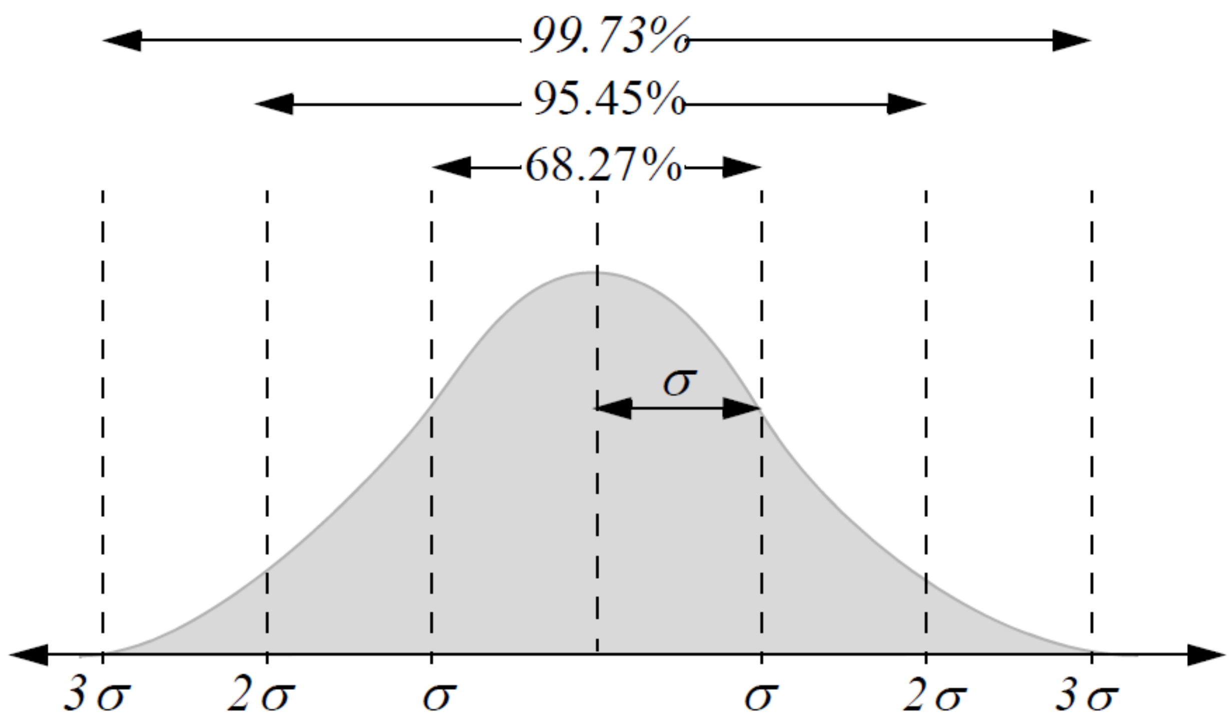
covariance $Q_{k+1} = Q_k + [\underline{x}_{k+1} - \underline{\mu}][\underline{x}_{k+1} - \underline{\mu}]^T$ when data arrives

$S_{k+1} = \frac{Q_k}{(k+1)}$ when answer necessary

- Used in … you guessed it … hand calculators.

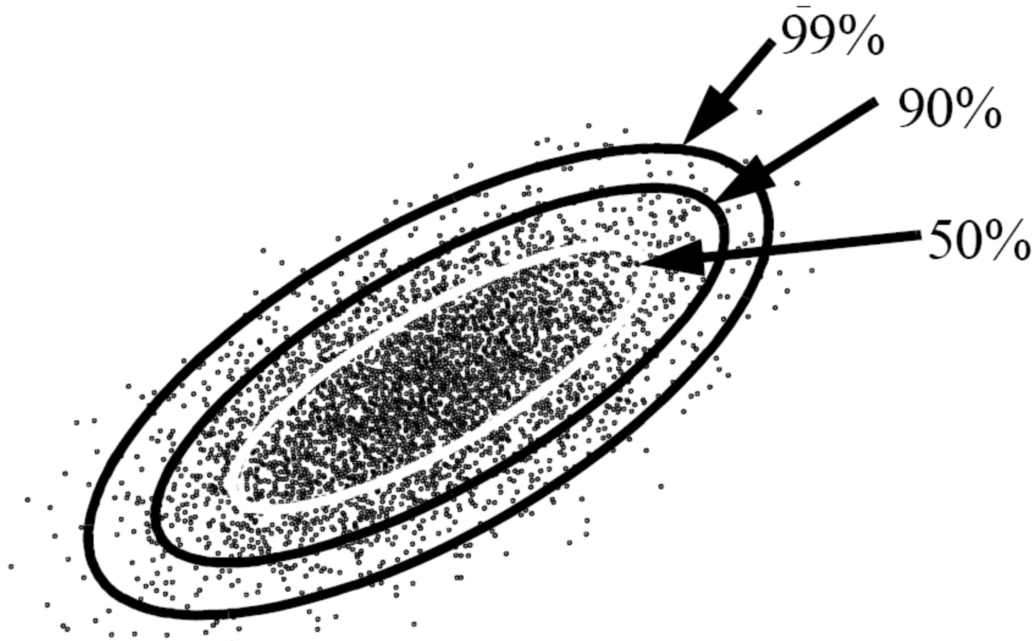
Contours of Constant Probability

- Consider the probability contained within a symmetric interval on the x axis.



5.1.2.6 Contours of Constant Probability

- In 2D, consider contours of constant exponent.



- These are ellipsoids in n dimensions:

$$(\underline{x} - \underline{\mu})^T \Sigma^{-1} (\underline{x} - \mu) = k^2(p)$$

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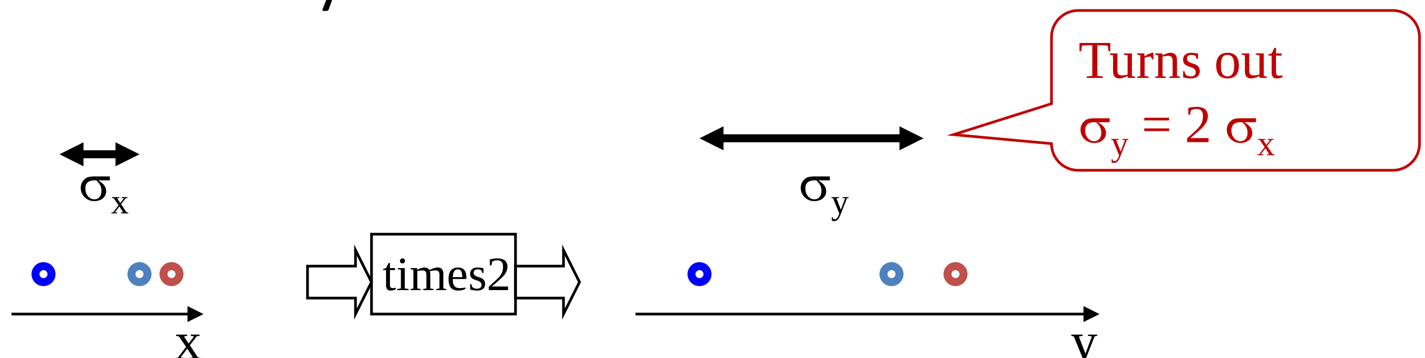
Transformation

- "Pass covariance through a function":
- Suppose $y = 2x$ and x is random.
  - 1st point: if x is random, y must be random – even if "2" is not.
  - 2nd point: if we know cov(x) we can find cov(y). How? Here's the hard way.



- This works even for nonlinear functions $y = f(x)$ but there is a simpler way.

Linearization

“Analytic continuation”

- The Taylor series allows us to extend any function into a neighborhood around a given point if we know the derivatives at that point:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + f''(x)\frac{\Delta x^2}{2} + \ldots$$

- Error involved in truncation is related to magnitude of first neglected term.
- We linearize like so:

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

- Errors involved are “second order”

5.1.3.1 Linear Transformation: Mean

- Suppose we know μ_x and want μ_y where:

$$\underline{y} = F\underline{x}$$

F independent of x

- Because expectation is an integral and hence a linear operator:

$$\mu_{\underline{y}} = Exp(F\underline{x}) = FExp(\underline{x})$$

- In other words

$$\mu_{\underline{y}} = F\mu_{\underline{x}}$$

5.1.3.1 Linear Transformation: Covariance

- Suppose we know σ_x and want σ_y where:

$$\underline{y} = F\underline{x}$$

F independent of x

- Because covariance is an integral and hence a linear operator:

$$\Sigma_{\underline{y}\underline{y}} = Exp(F\underline{x}\underline{x}^T F^T) = F Exp(\underline{x}\underline{x}^T) F^T$$

- In other words

$$\Sigma_{\underline{y}\underline{y}} = F \Sigma_{\underline{x}\underline{x}} F^T$$

5.1.3.2 Variance of a Sum of RVs

- Suppose there are n random variables x_i of same distribution.

$$x_i \sim N(\mu, \sigma) \quad , \quad i = 1, n$$

Variance of x'es known and equal.

- Define a new variable y as the sum of these:

$$y = \sum_{i=1}^{n} x_i$$

- What is the variance of y?

5.1.3.2 Variance of a Sum of RVs

- By our rules for uncertainty transforms:

$$\Sigma_{\underline{y}\underline{y}} = F\Sigma_{\underline{x}\underline{x}}F^T$$

- Where, in this case: $F = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$

- Hence:

$$\sigma_y^2 = F\Sigma_{\underline{x}\underline{x}}F^T = \sum_{i=1}^{n} \sigma_{x_i}^2$$

- IOW:

$$\sigma_y^2 = n\sigma_x^2$$

5.1.3.3 Variance of an Average of RVs

- Suppose there are n random variables x_i of same distribution.

$$x_i \sim N(\mu, \sigma) \quad , \quad i = 1, n$$

Variance of x'es known and equal.

- Define a new variable y as the average of these:

$$y = \frac{1}{n}\sum_{i=1}^{n} x_i$$

- What is the variance of y?

5.1.3.3 Variance of an Average of RVs

- By our rules for uncertainty transforms:

$$\Sigma_{\underline{y}\underline{y}} = F\Sigma_{\underline{x}\underline{x}}F^T$$

- Where, in this case: $F = \frac{1}{n}\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$

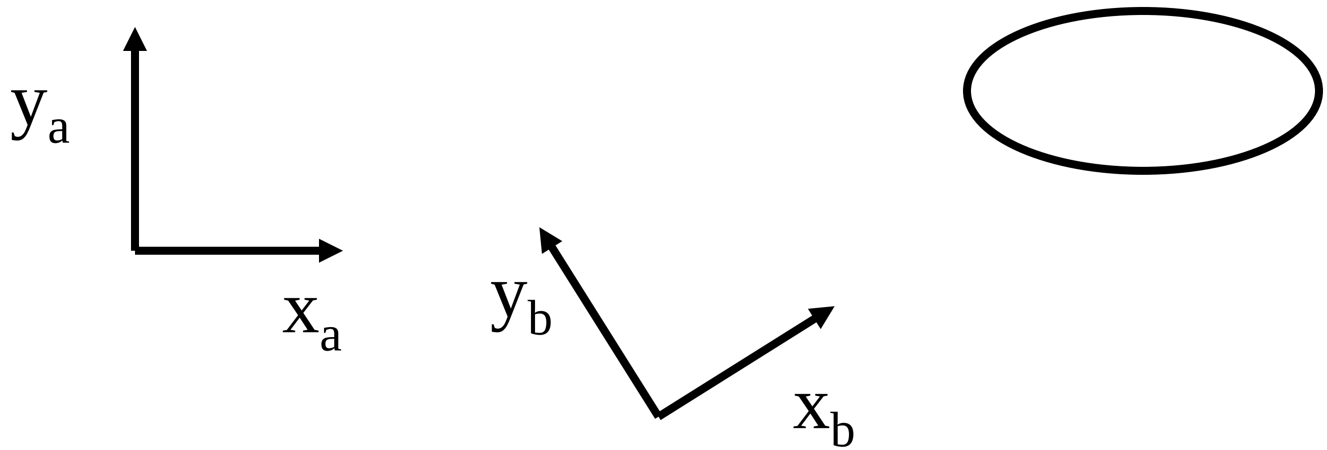
- Hence:

$$\sigma_y^2 = J\Sigma_{\underline{x}\underline{x}}J^T = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_{x_i}^2$$

- IOW:

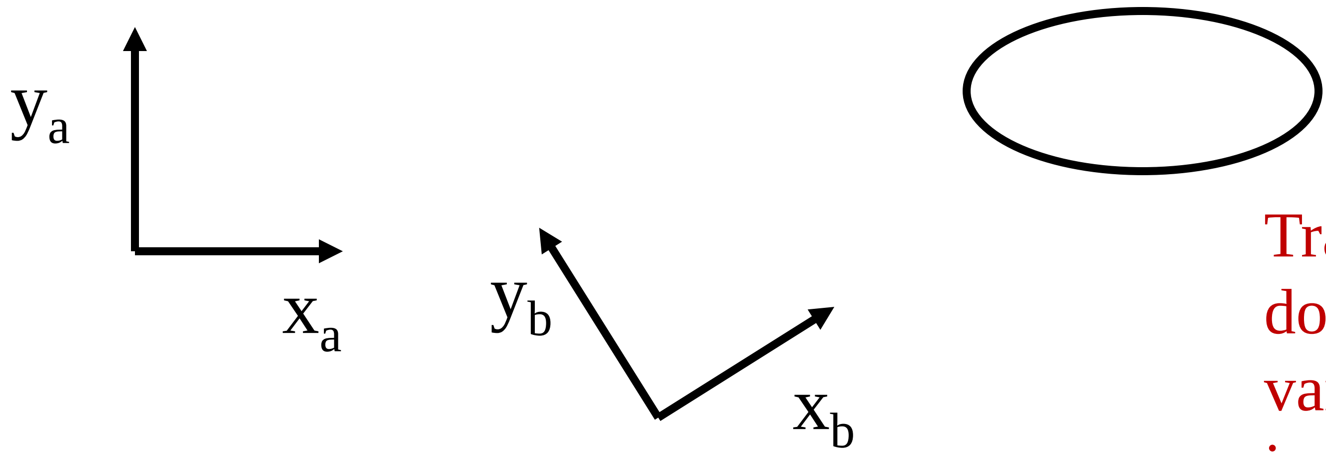
$$\sigma_y^2 = \frac{1}{n}\sigma_x^2$$

5.1.3.4 Coordinate Transformations



- Know covariance in one frame (because its easy to express there).
- Want to know it in another frame.

5.1.3.4 Coordinate Transformations



Translation part does not affect variance so its irrelevant

- If the transform between frames is:

$$ {}^{b}\underline{x} = R\,{}^{a}\underline{x} + \underline{t} $$

- The transformed mean and covariance are:

$$ {}^{b}\underline{\bar{x}} = R\,{}^{a}\underline{\bar{x}} + \underline{t} \qquad {}^{b}\Sigma = R\,{}^{a}\Sigma R^{T} $$

5.1.3.5 Nonlinear Transformation: Mean

- Suppose we know μ_x and want μ_y where: $\underline{y} = f(\underline{x})$
- Write x in terms of a deviation from a reference x': $\underline{x} = \underline{x}' + \underline{\varepsilon}$
- Can use Jacobian to linearize: $y = f(x) = f(x' + \varepsilon) \approx f(x') + J\varepsilon$
- The mean of the distribution of y is.... $\mu_{\underline{y}} = Exp(\underline{y}) \approx Exp[\underline{f}(\underline{x}')] + Exp(J\underline{\varepsilon})$
- x' is not random, so… $Exp[\underline{f}(\underline{x}')] = \underline{f}(\underline{x}')$
- If e is unbiased, then………. $Exp(J\underline{\varepsilon}) = JExp(\underline{\varepsilon}) = 0$
- And if we choose ……… $\underline{x}' = \mu_{\underline{x}}$

$$\mu_{\underline{y}} = Exp(\underline{y}) = \underline{f}(\mu_{\underline{x}})$$

Mean of the f() is the f() of the mean.

"to first order"

"for unbiased error"

5.1.3.5 NonLinear Transformation: Covariance

- Rewriting:

$$\underline{y} = \underline{f}(\underline{x}) = \underline{f}(\underline{x}' + \underline{\varepsilon}) \approx \underline{f}(\underline{x}') + J\underline{\varepsilon}$$

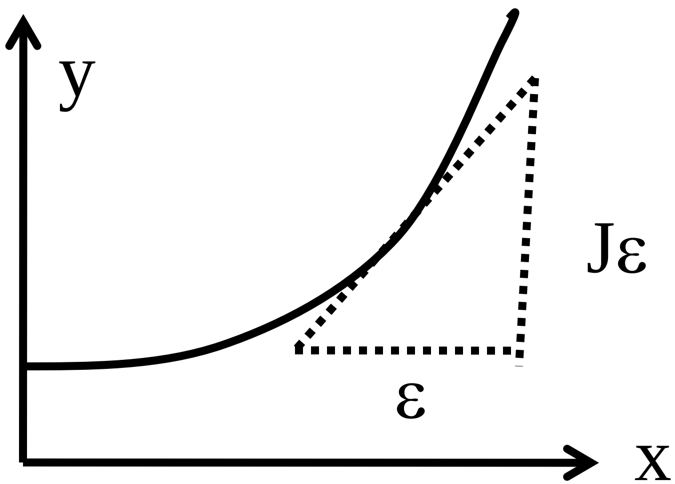
$$\underline{y} - \underline{y}' = J\underline{\varepsilon}$$

- By definition:

$$\Sigma_{\underline{y}\underline{y}} = Exp([\underline{y} - \underline{y}'][\underline{y} - \underline{y}']^T)$$

- Which is:

$$\Sigma_{\underline{y}\underline{y}} = Exp(J\underline{\varepsilon}\underline{\varepsilon}^T J^T)$$



$$\Sigma_{\underline{y}\underline{y}} = J\Sigma_{\underline{x}\underline{x}}J^T$$

Linearization: Again

- Whenever you write:

$$\Sigma_{\underline{y}\underline{y}} = J\Sigma_{\underline{x}\underline{x}}J^T$$

- Unless all derivatives beyond J vanish (i.e unless the mapping from x to y really is linear)
  - You have written an approximation.

5.1.3.6 Covariance with Partitioned Inputs

- Suppose we have: $\underline{y} = f(\underline{x}) \quad \underline{x} = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 \end{bmatrix}^T$

- Partition the Jacobian and the Covariance:

$$J_x = \begin{bmatrix} J_1 & J_2 \end{bmatrix} \qquad \Sigma_{xx} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

- We already know that the covariance of y is:

$$\Sigma_{yy} = J_x \Sigma_{xx} J_x^T \Rightarrow \Sigma_{yy} = \begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} J_1^T \\ J_2^T \end{bmatrix}$$

Uncorrelated Partitioned Inputs

- Suppose we have:

$$\Sigma_{12} = \Sigma_{21} = [0]$$

$$\therefore \quad \Sigma_{yy} = \begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \begin{bmatrix} J_1^T \\ J_2^T \end{bmatrix}$$

- Hence:

$$\Sigma_y = J_1 \Sigma_{11} J_1^T + J_2 \Sigma_{22} J_2^T$$

Uncertainties of uncorrelated inputs add to produce the output uncertainty in $y=f(x_1,x_2)$

Box 5.1 Formulae for Transformation of Uncertainty

For the following nonlinear transformation relating random vector $\underline{x}$ to random vector $\underline{y}$:

$$\underline{y} = \underline{f}(\underline{x})$$

The mean and covariance of $\underline{y}$ are related to those of $\underline{x}$ by:

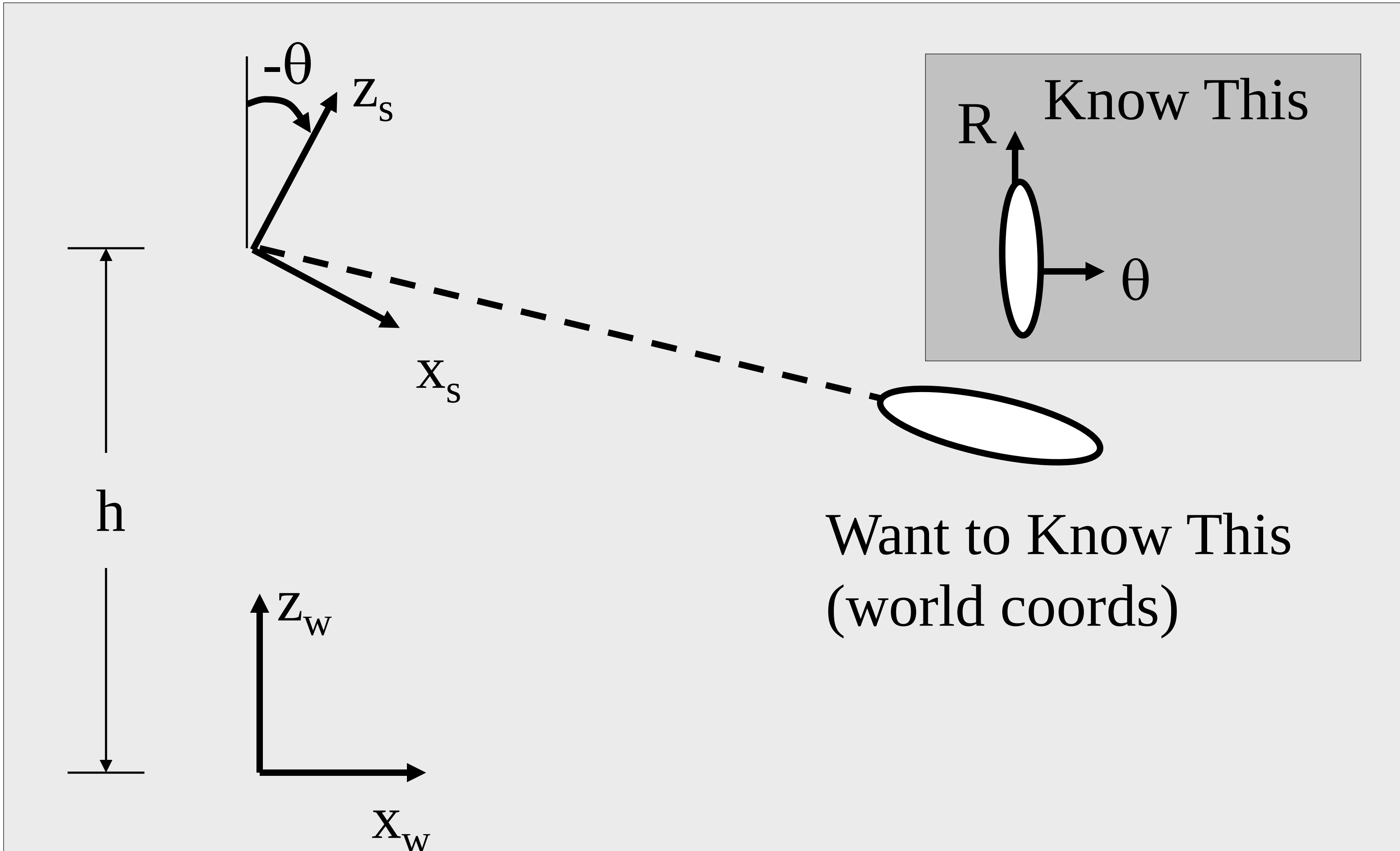
$$\mu_{\underline{y}} = \underline{f}(\mu_{\underline{x}}) \qquad \Sigma_{\underline{y}\underline{y}} = J\Sigma_{\underline{x}\underline{x}}J^T$$

Remember that, when using this result, unless all derivatives beyond J vanish (unless the original mapping really was linear), the result is a linear approximation to the true mean and covariance.

When $\underline{x}$ can be partitioned into two uncorrelated components, then:

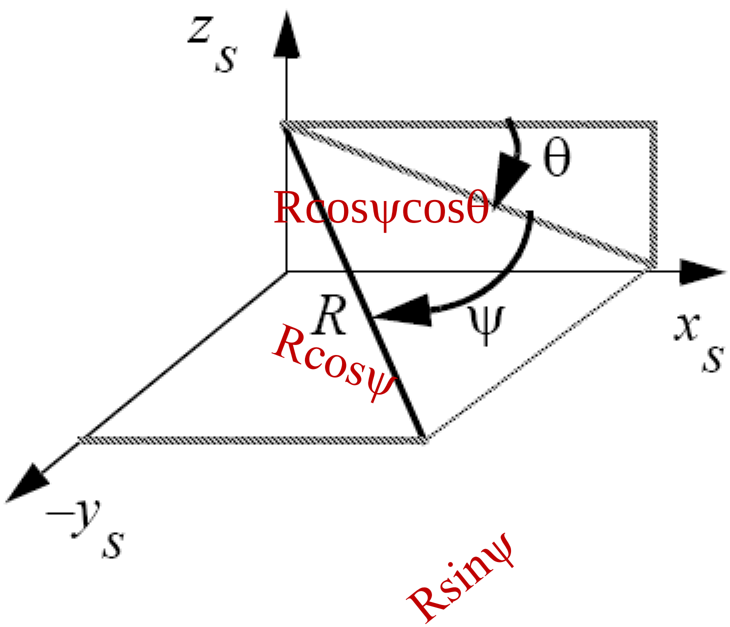
$$\Sigma_{\underline{y}\underline{y}} = J_1\Sigma_{11}J_1^T + J_2\Sigma_{22}J_2^T$$

5.1.3.7 Example : Azimuth Scanner Transforming Uncertainty from 's' to 'w'



Step 1: From i to s

$$\underline{v}_s = \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} = \begin{bmatrix} Rc\psi c\theta \\ -Rs\psi \\ -Rc\psi s\theta \end{bmatrix}$$



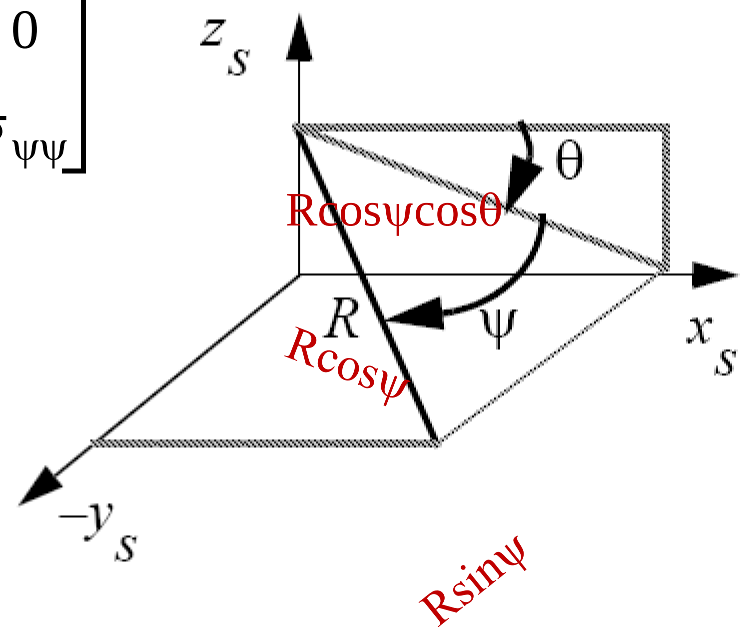
- Differentiate:

$$J_i^s = \begin{array}{c} \\ \mathbf{x} \\ \mathbf{y} \\ \mathbf{z} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{R} & \boldsymbol{\psi} & \boldsymbol{\theta} \end{array} \\ \begin{bmatrix} c\psi c\theta & -Rs\psi c\theta & -Rc\psi s\theta \\ -s\psi & -Rc\psi & 0 \\ -c\psi s\theta & Rs\psi s\theta & -Rc\psi c\theta \end{bmatrix} \end{array}$$

Step 1: Transformation

- Assume we know:

$$\Sigma_i = \begin{bmatrix} \sigma_{RR} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\psi\psi} \end{bmatrix}$$



- Diagonal = "uncorrelated".

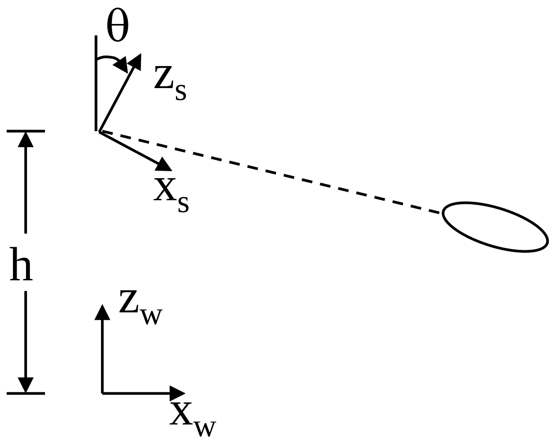
- So……

$$\Sigma_s = J_i^s \Sigma_i (J_i^s)^T$$

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

Step 2: From s to w

- T_s^w matrix relates s to w.
- Translation part is additive and irrelevant, so....



$$\Sigma_w = R_s^w \Sigma_s (R_s^w)^T$$

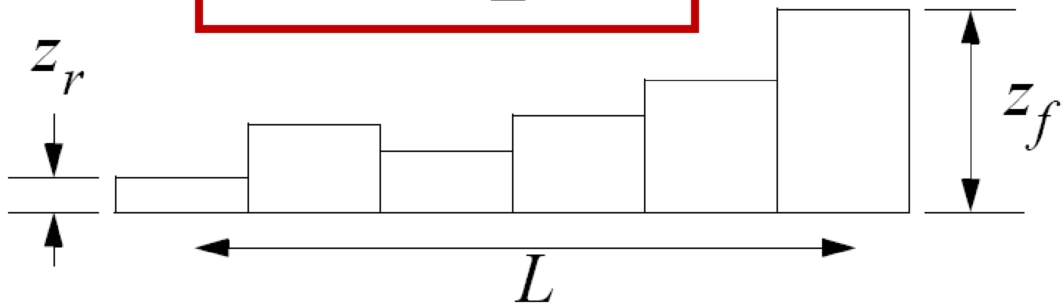
$$T_s^w = Trans(0, 0, h) Roty(\theta)$$

$$T_s^w = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta & 0 \\ 0 & 1 & 0 & 0 \\ -s\theta & 0 & c\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\Sigma_w = R_s^w J_i^s \Sigma_i (J_i^s)^T (R_s^w)^T}$$

5.1.3.8 Example: Attitude from Terrain Map

$$\theta = \frac{(z_f - z_r)}{L}$$



- Find uncertainty in computed pitch angle given uncertainty in terrain
  - Which came from uncertainty in sensor.

5.1.3.8 Example: Attitude from Terrain Map

- Suppose the uncertainty in elevation is:

$$\Sigma_z = \begin{bmatrix} \sigma_f^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

- Variance of computed pitch angle is:

$$\Sigma_\theta = J \Sigma_z J^T$$

- Where the Jacobian in this case is a gradient:

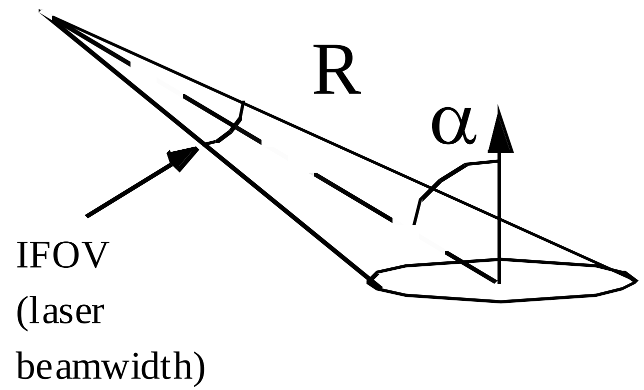
$$J = \begin{bmatrix} \frac{\partial\theta}{\partial z_f} & \frac{\partial\theta}{\partial z_r} \end{bmatrix} = \begin{bmatrix} \frac{1}{L} & -\frac{1}{L} \end{bmatrix}$$

- The result is:

$$\boxed{\sigma_\theta^2 = \frac{1}{L^2}[\sigma_f^2 + \sigma_r^2]}$$

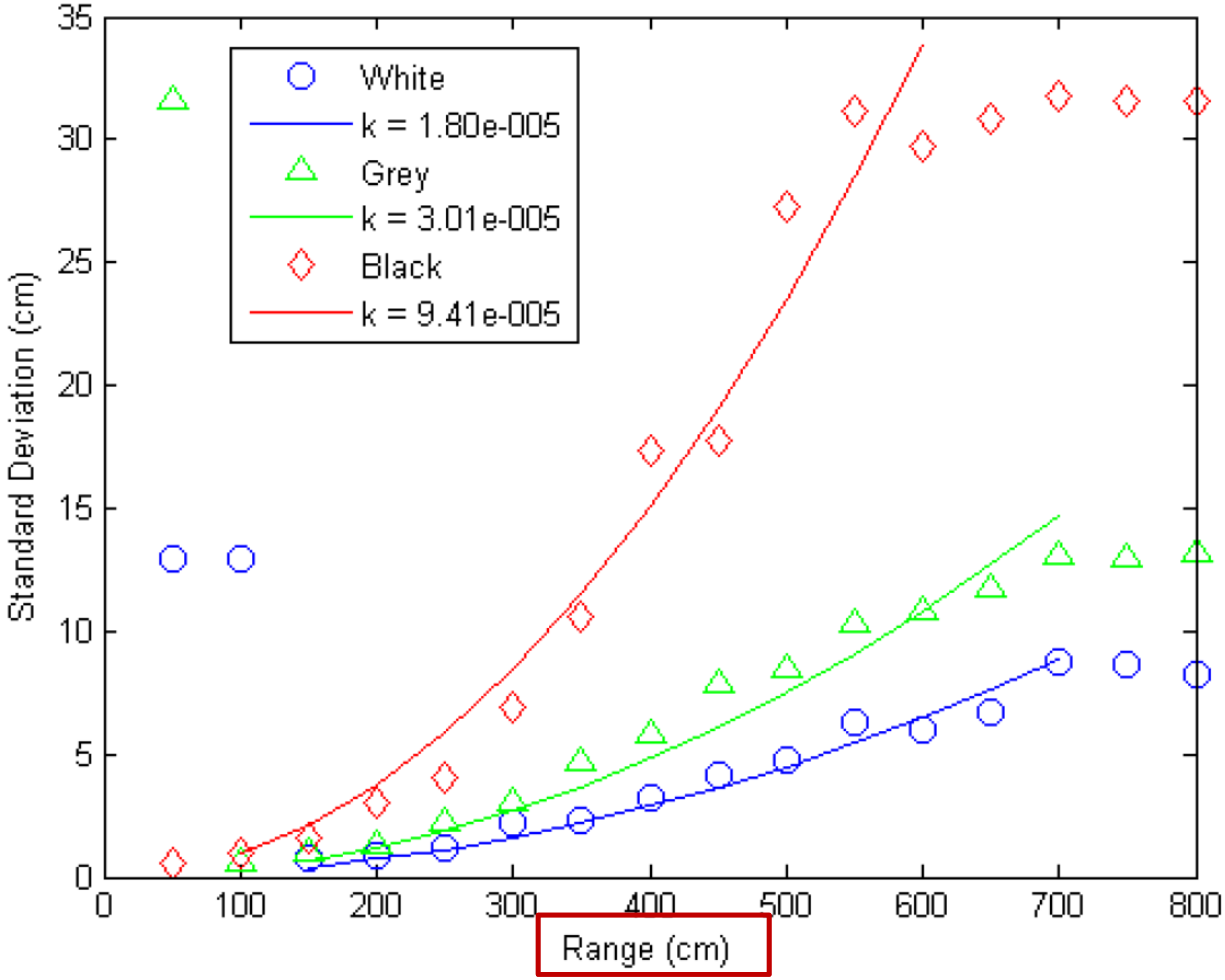
5.1.3.9 Example: Range Error in Rangefinders

- Where do the input variances come from?
- Variance in measured range depends on Range (R) reflectance (r) and incidence angle (a).



$$\sigma_R \propto \left[\frac{\lambda R^2}{\rho \cos \alpha}\right]$$

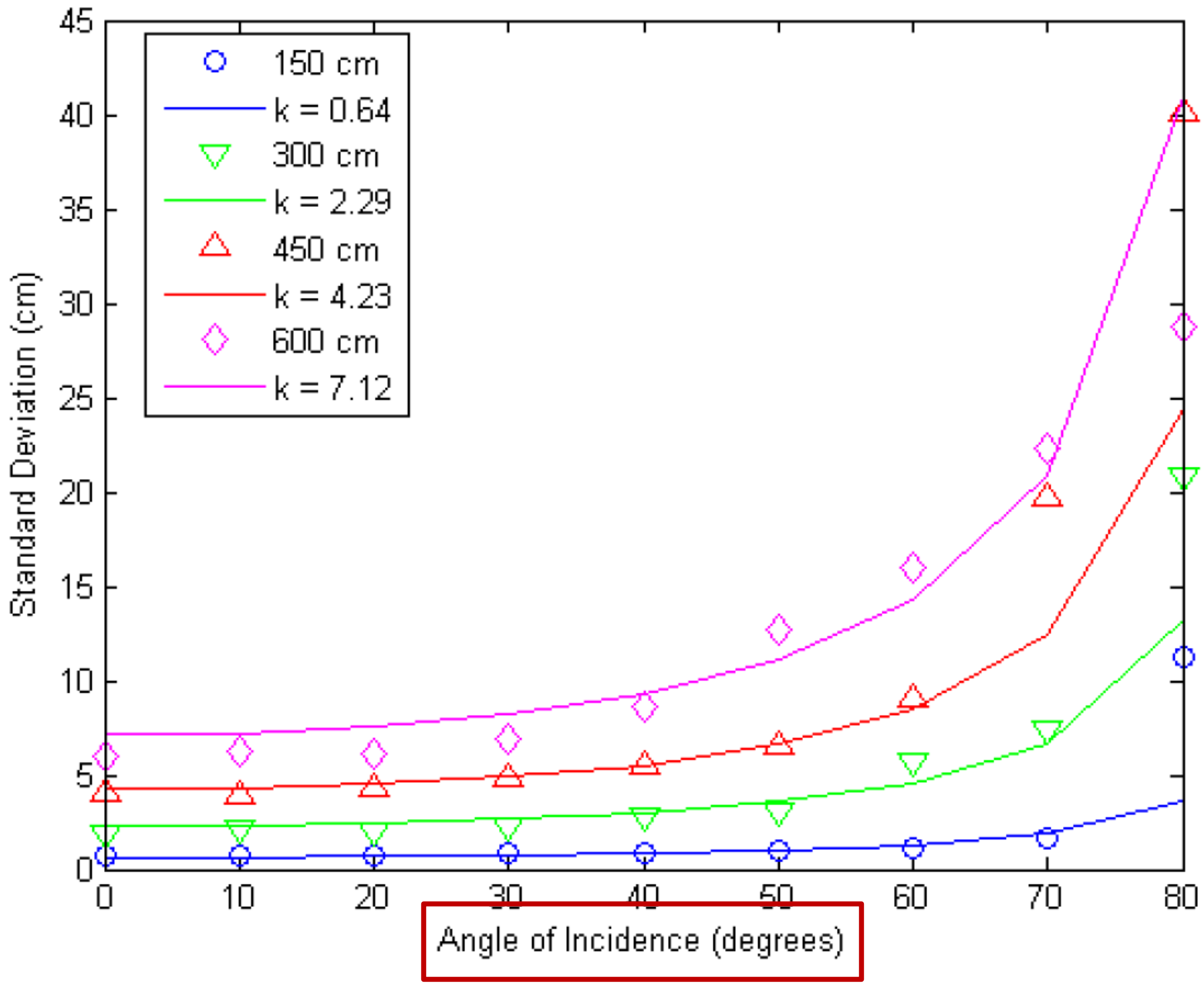
5.1.3.9 Example: Range Error in Rangefinders (Real Data)



- Normal incidence, various reflectance.
- Dark surfaces are 10 X noisier
- Fit lines of the form:

$$\sigma_R = k_1(\rho)R^2$$

5.1.3.9 Example: Range Error in Rangefinders (Real Data)



- White target, various ranges.
- Fit lines of the form:

$$\sigma_R = \frac{k_2(R)}{\cos\alpha}$$

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5.1.3.10 Example: Stereo Vision

- From similar triangles:

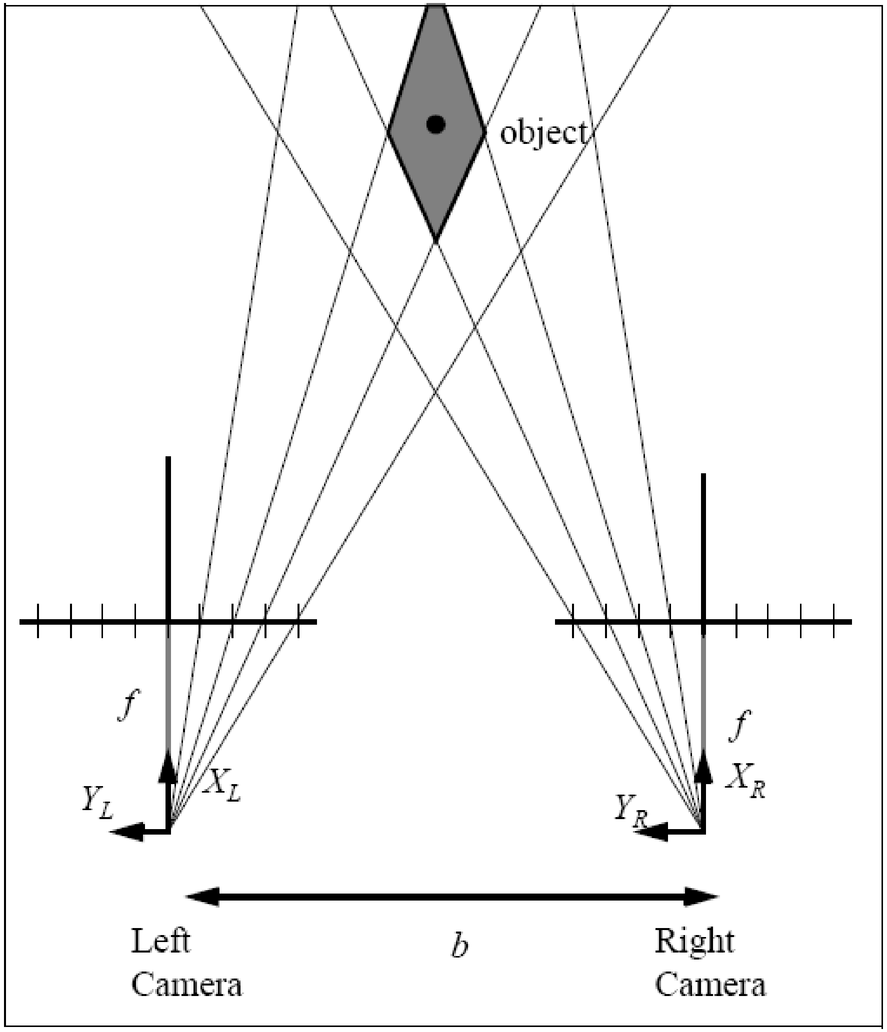
$$\frac{Y_L}{X_L} = \frac{Y_L}{X} = \frac{y_l}{f} \quad \frac{Y_R}{X_R} = \frac{Y_R}{X} = \frac{y_r}{f}$$

- Subtract:

$$Y_L - Y_R = \frac{X[x_l - x_r]}{f}$$

- Hence:

$$X = \frac{bf}{d} \qquad b = \frac{Xd}{f}$$



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5.1.3.11 Example: Stereo Uncertainty

- Define the normalized disparity:

$$\delta = \frac{d}{f} = \frac{b}{X}$$

- Now triangulation looks like:

$$X = \frac{b}{\delta}$$

- Uncertainty transformation:

$$\sigma_{xx} = J\sigma_{\delta\delta}J^T$$

- Jacobian is the scalar:

$$J = \frac{\partial X}{\partial \delta} = \frac{-b}{\delta^2}$$

- Variance goes with 4th power of range:
- Standard deviation with the square of range.
  - Famous result.

$$\sigma_{xx} = \left[\frac{b^2}{\delta^4}\right]\sigma_{\delta\delta} = \left[\frac{X^4}{b^2}\right]\sigma_{\delta\delta}$$

Outline

- 5.1 Random Variables, Processes and Transformation
  - 5.1.1 Characterizing Uncertainty
  - 5.1.2 Random Variables
  - 5.1.3 Transformation of Uncertainty
  - 5.1.4 Random Processes - **SKIP**
  - Summary

Outline

- 5.1 Random Variables, Processes and Transformation
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  - 5.1.3 Transformation of Uncertainty
  - 5.1.4 Random Processes
  - <u>Summary</u>

Summary

- There are many kinds of error.
  - They can be removed with calibration, filtering.
- Covariance measures spread. Level curves are ellipsoids.
- Covariance is transformed with a matrix quadratic form.
- ~~Variance of a random walk process grows linearly with time.~~
- ~~Stochastic Diff Eqs are almost as easy to solve as deterministic ones.~~