Some Useful Results for Closed-Form Propagation of Error in Vehicle Odometry

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Rev 2.0 is issued to correct many typos and introduce notation consistent with the eventual journal paper. Changes from Rev 1.0 are identifed by change bars in the margin

Abstract

Odometry can be modelled as a nonlinear dynamical system. The linearized error propagation equations for both deterministic and random errors in the odometry process have time varying coefficients and therefore may not be easy to solve. However, the odometry process exibits a property here called "commutable dynamics" which makes the transition matrix easy to compute.

As a result, an essentially closed form solution to both deterministic and random linearized error propagation is available. Examination of the general solution indicates that error expressions depend on a few simple path functionals which are analogous to the moments of mechanics and equal to the first two coefficients of the power and Fourier series of the path followed.

The resulting intuitive understanding of error dynamics is a valuable tool for many problems of mobile robotics. Required sensor performance can be computed from tolerable error, trajectories can be designed to minimize error for operation or to maximize it for calibration and evaluation purposes. Optimal estimation algorithms can be implemented in nearly closed form for small footprint embedded applications, etc.

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1. Introduction

Students of optimal estimation and Kalman filtering know how to propagate random error through a differential equation. The general solution in the absence of measurements of state is given by the linear variance equation[1]:

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + L(t)Q(t)L(t)^T$$

Similarly, students of linear system theory know how to linearize a differential equation to produce its first order behavior. For the general nonlinear differential equation $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$, the first order linearized dynamics are:

$$\delta\dot{\underline{x}}(t) = \frac{\partial}{\partial \underline{x}}\underline{f}\bigg|_{\underline{x}} \delta\underline{x}(t) + \frac{\partial}{\partial \underline{u}}\underline{f}\bigg|_{\underline{x}} \delta\underline{u}(t)$$

So the general solution for the propagation of systematic error is also well known.

Unfortunately, however, the use of numerical techniques to model error propagation tends to rob the user of an intuitive grasp of the behavior of the system. As wonderful as it is to have such general solutions, few of us can integrate matrix differential equations in our heads. Yet, it turns out that alot can be said in general about uncertainty propagation in odometry. This report will investigate error propagation in odometry in order to develop an intuitive grasp of its fundamental behavior.

1.1 Motivation as Calibration Tool

Closed form error propagation expressions can have great practical value in addition to their aid to our understanding. For example, external behavior can be used to determine whether an error source is systematic or random. If systematic, error models can be used to calibrate the error out because the model itself is an extremely sensitive observation of the underlying error.

Generally, if a set of observations $\underline{z}$ are supposed to depend on calibration parameters $\underline{p}$ and state $\underline{x}$ through an observer relationship such as:

$$\underline{z} = f(\underline{x}, \underline{p})$$

then it is possible, given observations and state to determine the value of the parameters which predicts the observations in a best-fit sense. Hence, many of the results of this report also have great value in the calibration of systematic error models and the elimination of the associated error.

1.2 Analytical Results

As a motivation to read the rest of the report, its main results are introduced here. Consider the scenario where a vehicle is travelling on a linear trajectory and computing its position from odometry. Suppose the linear trajectory that is nominally aligned along the x axis called the

1. This becomes nonlinear with the addition of a $-PH^TR^{-1}HP$ term due to the addition of measurements. It is then known as the matrix Ricatti equation - a much studied equation that is central to the theory of the Kalman filter because it is the basis of error prediction.

alongtrack direction and let y be called the **crosstrack** direction. Let s represent distance (for this trajectory also equal to x) and let V represent velocity. In the balance of the report, three cases of odometry implementation are evaluated for this trajectory, the general case, the linear case, and a constant curvature trajectory, but only the linear trajectory case is presented here.

1.2.1 Direct Heading Odometry

This is the case where a compass is used to indicate heading directly and a transmission or equivalent encoder is used to measure differential distance travelled. Let δV_v represent the magnitude of a scale factor error on the encoder expressed as a fraction of distance. Let $\delta\theta_c$ represent the amplitude of the sinusoidal heading error on the compass which is due to the magnetic field generated by the vehicle itself. Let $\sigma_{vv}^{(v)}$ scale velocity onto the noise amplitude on the encoder and let $\sigma_{\theta\theta}$ represent the variance of the noise on the compass. If we ignore initial conditions on the errors, the expected evolution of error in computed position and orientation is as follows:

Table 1: Odometry Error for Linear Trajectory, Direct Heading Odometry

Systematic Error	Expression	Random Error	Expression
$\delta x(s)$	$\delta V_v s$	$\sigma_{xx}(s)$	$\sigma_{vv}^{(v)} s$
$\delta y(s)$	$\delta\theta_c s$	$\sigma_{yy}(s)$	$\sigma_{\theta\theta}\lvert V\rvert s$

Constant velocity was assumed for the crosstrack variance expression. For this case, all systematic errors and variances are linear in distance.

1.2.2 Integrated Heading Odometry

This is the case where a gyro is used to indicate angular velocity and a transmission or equivalent encoder is used to measure differential distance travelled. Let δV_v represent the magnitude of a scale factor error on the encoder expressed as a fraction of distance. Let $\delta\omega$ represent a bias on the gyro output. Let $\sigma_{vv}^{(v)}$ scale velocity onto the noise amplitude on the encoder and let $\sigma_{\omega\omega}$ represent the variance of the gyro noise. If we ignore initial conditions on the errors, the expected evolution of error in computed position and orientation is as follows:

Table 2: Odometry Error for Linear Trajectory, Integrated Heading Odometry

Systematic Error	Expression	Random Error	Expression
$\delta x(t)$	$\delta V_v s(t)$	$\sigma_{xx}(t)$	$\sigma_{vv}^{(v)} s(t)$
$\delta y(t)$	$\delta\omega s t/2$	$\sigma_{yy}(t)$	$\sigma_{\omega\omega} s^2 t/3$

Table 2: Odometry Error for Linear Trajectory, Integrated Heading Odometry

Systematic Error	Expression	Random Error	Expression
$\delta\theta(t)$	$\delta\omega t$	$\sigma_{\theta\theta}(t)$	$\sigma_{\omega\omega} t$

Hence both $\delta x(t)$ and $\sigma_{xx}(t)$ are linear in distance while both $\delta\theta(t)$ and $\sigma_{\theta\theta}(t)$ are linear in time. Systematic crosstrack error $\delta y(t)$ is quadratic in time whereas random crosstrack error variance $\sigma_{yy}(t)$ is cubic in time. Constant velocity was assumed in the solutions for $\delta y(t)$ and $\sigma_{yy}(t)$.

1.2.3 Differential Heading Odometry

This is the case where encoders on two separated wheels are used to indicate both incremental change in heading (through their difference) as well as incremental distance travelled (through their average). Let δl_l and δr_r represent the magnitude of scale factor errors on the left and right wheels respectively - again expressed as a fraction of distance. Let W represent the “tread” or length of the axle joining the wheels. Further define the derived constants:

$$\delta V_v = (\delta l_l + \delta r_r)/2 \qquad \delta\omega_v = (\delta r_r - \delta l_l)/W$$

Let $\sigma_{rr}^{(r)}$ and $\sigma_{ll}^{(l)}$ scale velocity onto the noise amplitude on the encoders for each wheel. Also define for convenience:

$$\sigma_{vv}^{(v)} = \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4} \qquad \sigma_{\omega\omega}^{(v)} = \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{W^2}$$

If we ignore initial conditions on the errors, the expected evolution of error in computed position and orientation is as follows:

Table 3: Odometry Error for Linear Trajectory, Differential Heading Odometry

Systematic Error	Expression	Random Error	Expression
$\delta x(s)$	$\delta V_v s$	$\sigma_{xx}(s)$	$\sigma_{vv}^{(v)} s$
$\delta y(s)$	$\delta\omega_v s^2/2$	$\sigma_{yy}(s)$	$\sigma_{\omega\omega}^{(v)} s^3/3$
$\delta\theta(s)$	$\delta\omega_v s$	$\sigma_{\theta\theta}(s)$	$\sigma_{\omega\omega}^{(v)} s$

This result is similar to the previous one except that time dependence, where it existed, has been replaced with distance dependence. Now both $\delta x(s)$ and $\sigma_{xx}(s)$ as well as both $\delta\theta(s)$ and

$\sigma_{\theta\theta}(s)$ are linear in distance. Systematic crosstrack error $\delta y(s)$ is quadratic in distance whereas random crosstrack error variance $\sigma_{yy}(s)$ is cubic in distance.

1.3 Summary

As a motivation to read the balance of the report, some of its theoretical content is repeated here. Section 2 discusses the dynamical system from the perspective of its utility as a model of odometry. From this perspective, certain properties become important. The conditions under which errors are erased by driving backwards, stop accumulating when motion stops, and cancel out on closed trajectories, among others, are discussed.

Section 3 defines the three cases of odometry which will be studied. These are chosen so as to expose the most issues in least time but virtually any type of odometry can be treated with the results of the report. The general behavior of the odometry equations is cast in the context of Section 2 and special test trajectories are defined for use in the balance of the report. A short analysis of nonlinear odometry error propagation reveals why it is so hard to solve. Instantaneously equivalent relationships between error sources in different types of odometry are also developed.

Section 4 is a tutorial on the aspects of linear systems theory, perturbation theory, and stochastic calculus which will be relevant to achieving our goals. Essentially, the vector convolution integral is the solution to the linear perturbation equation and the matrix convolution integral is the solution to the linear variance equation. A particular condition called commutable dynamics happens to apply to the linearized odometry error propagation equations and because of this fact, the general solutions developed in the rest of the report are possible.

Section 5 develops the main results of the report. These include the general solutions for error propagation for the three cases studied, for any error model, and any trajectory. These results could be used as the basis of numerical simulation but they are most useful when particular errors and trajectories are substituted analytically. The results can be understood in graphical terms. Also, important behaviors in terms of response to initial conditions and inputs, as well as path dependent and path independent error propagation become obvious.

Section 6 develops a set of functionals known as moments which are as central to odometry error propagation as the Laplace transforms are to control theory. These moments can be tabulated once and then used to convert error propagation problems expressed as differential equations into much easier equivalent problems expressed as algebraic and transcendental equations. Moments exhibit interesting properties of zeros, extrema, monotonicity, and conservation which are the underlying sources of the behavior of odometry systems. Moments for linear and arc trajectories are tabulated.

Sections 7 and 8 use the moments developed earlier to immediately write the solutions for particular cases of error models on two test trajectories. Significantly, these results can be written by inspection using moments whereas an appendix demonstrates that it takes many pages to derive just one of the six results from scratch.

Section 9 uses all of the previous results to generate interesting analyses and graphs which demonstrate the utility of the results in practical applications.

2. Dynamical Systems

The nonlinear dynamical system is perhaps the closest approximation available to an engineering "theory of everything". It applies to processes as diverse as chemical reactions, growth of bacteria, financial markets, and motion of the planets. For our purposes, it is the governing theory of the process of odometry. This section introduces some notation that will be used throughout the document and it mentions some properties of dynamical systems which will be important to us later on.

2.1 The Continuous-Time Nonlinear System

2.1.1 Nonlinear State Equations

A fairly general expression for a nonlinear dynamical system takes the form:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), \underline{w}(t), t)$$
$$\underline{z}(t) = \underline{h}(\underline{x}(t), \underline{u}(t), \underline{v}(t), t) \tag{1}$$

These equations will be called the nonlinear "state equations". The first equation is sometimes called the process model or system dynamics. The process or system is described by the time varying "state" $\underline{x}(t)$ of the system. The controllable input vector $\underline{u}(t)$ captures any controllable forcing functions. The uncontrollable input vector $\underline{w}(t)$ captures any uncontrollable forcing functions (deterministic or random). The general nonlinear function $f(\ )$ captures the manner in which the state derivative depends on the state itself and both forms of input.

The second equation is often called the observer or measurement equation. It comes into play because we will be interested also in the process by which various sensory devices can be used to observe the system state. The measurement vector $\underline{z}(t)$ captures the information that is obtained from the sensors. The uncontrollable input vector $\underline{v}(t)$ captures any uncontrollable forcing functions (deterministic or random). The general nonlinear function $h(\ )$ models the indirect process by which the measurements are related to the underlying state $\underline{x}(t)$ and both forms of input.

2.1.2 Solution to the Nonlinear State Equations

While alot is known about the solution to linear systems - particularly in the constant coefficient case - general results for the nonlinear case are not available. Indeed, unlike in the linear case, there is no guarantee that an explicit solution to such nonlinear differential equations exists at all. Practical numerical solutions are available, however, by direct integration of the first equation[1]:

$$\underline{x}(t) = \underline{x}(0) + \int_0^t \underline{f}(\underline{x}(\tau), \underline{u}(\tau), \underline{w}(\tau), \tau)d\tau \tag{2}$$

1. Throughout the document we will use the standard loose notation of where integrals and (formally nonexistent) derivatives of random processes will be written in lieu of more formal methods which generate the same results.

Also, the technique of generating an approximate linear solution is well known - but we must be careful not to assume that the conclusions for the linearized system apply to the original nonlinear one. While little can be said about the general nonlinear case, we will see that odometry is nonlinear, so it will be important to try to learn as much as possible about the nonlinear case. However, since the nonlinear case subsumes the linear one, **any conclusions that apply to the above system will apply to linear systems**.

2.2 Relevant Derivative Properties of the Nonlinear Dynamical System

Certain properties of dynamical systems that will be very important to us later are discussed here. Later, it will be shown that the controllable and uncontrollable inputs can be treated separately. For this section, the uncontrollable input $\underline{w}(t)$ will be suppressed in the discussion to avoid confusion.

2.2.1 Special Cases of Controllability

Generally, controllability is the property of being able to drive the system state to a particular place in state space by adjusting the inputs. We will be interested in several special cases of how the system responds to a change in its inputs. Suppose that at a particular point in time, the input to the system is $\underline{u}(t)$. Suppose further that we change this input to a new input by operating on the input in a manner symbolized by the function $g(\ )$:

$$\underline{u}(t) \rightarrow g[\underline{u}(t)]$$

We are interested in how the state derivative changes in response to this change in the input. The following special cases will be defined.

2.2.1.1 Nullability

A system is nullable if zeroing the input zeros the state derivative:

$$\underline{f}(\underline{x}(t), 0, t) = \underline{0} \tag{3}$$

Such systems do not remember what they were doing when the input is removed, so we will also call them memoryless[1]. A memoryless system must have no zeroth order term in its Taylor Series over its inputs. Systems with nullable error dynamics are easier to analyze.

2.2.1.2 Reversibility

A system is reversible if switching the input sign switches the sign of the state derivative:

$$\underline{f}(\underline{x}(t), -\underline{u}(t), t) = -\underline{f}(\underline{x}(t), \underline{u}(t), t) \tag{4}$$

A sufficient condition for reversibility is the condition that the system dynamics $f(\ )$ be an odd function of the input. Equivalently, $f(\ )$ should have no even powers in its Taylor series over the input. By continuity, reversible systems are also nullable because $f(\ )$ must be reversible even when it has vanishingly small magnitude. Systems with reversible error dynamics can have their error erased by replaying the opposite inputs backwards.

1. The equivalent designation “drift-free” is used in control systems.

2.2.1.3 Scalability

A system will be called scalable if it is possible to scale the derivative by scaling only the controllable input:

$$\underline{f}(\underline{x}(t), k \times \underline{u}(t), t) = k \times \underline{f}(\underline{x}(t), \underline{u}(t), t) \tag{5}$$

Clearly, such a system is also reversible and therefore is also nullable. A sufficient condition for scalability is the condition that the system dynamics $f(\ )$ be linear in the input. Systems with error dynamics scalable in the coordinates of the reference trajectory have errors which are independent of the path followed.

2.2.1.4 Factorability

A system will be called factorable if it can be written in the form:

$$\underline{f}(\underline{x}(t), \underline{u}(t), t) = \underline{\phi}(\underline{x}(t), \underline{u}(t), t) \times g[\underline{u}(t)] \tag{6}$$

where a factor dependent only on the input can be isolated. Depending on the form of $g[\ ]$, this system can be scalable or only reversible, or only nullable.

2.2.2 Motion Dependence

Let s represent a general position variable such as distance, cartesian coordinates x,y,z or angles such as pitch, yaw, and roll. A change of variable from time t to s is possible if the system dynamics can be divided by the time derivative of s without creating a singularity:

$$\frac{\dot{\underline{x}}(t)}{ds/dt} = \frac{\underline{f}(\underline{x}(t), \underline{u}(t), t)}{ds/dt}$$

$$\frac{d}{ds}\underline{x}(s) = \underline{\phi}(\underline{x}(s), \underline{u}(s), s)$$

Clearly, a sufficient condition for motion dependence is that it be possible to factor a motion derivative out of the original system:

$$\underline{f}(\underline{x}(t), \underline{u}(t), t) = \underline{\phi}(\underline{x}(s), \underline{u}(s), s)\left(\frac{ds}{dt}\right) \tag{7}$$

An equivalent integral condition can be generated by substituting this into equation (2).

This property can be defined for each scalar element of the state vector and each need not have the same motion derivative as a factor. Motion dependence is important because, when the system dynamics are a model of error propagation, this property indicates that error accumulation halts whenever motion stops. When the position derivative chosen can be identified with an input, the motion dependent system is also memoryless because setting $ds/dt = 0$ sets the state derivative to zero.

2.3 Relevant Integral Properties of the Nonlinear Dynamical System

Other important properties can be expressed in terms of a solution integral for the state. Assuming that the "initial" conditions are known at time zero, a general solution for the system state will take the form of an integral which depends on the input thus:

$$\underline{x}(t) = \underline{x}(0) + \int_0^t \underline{F}(\underline{u}(\tau), \tau)d\tau \tag{8}$$

2.3.1 Closure

Certain inputs have the property that they return the state to its initial value after some time T has elapsed. Substituting $\underline{x}(T) = \underline{x}(0)$ into the above gives the trajectory closure condition:

$$\int_0^T \underline{F}(\underline{u}(\tau), \tau)d\tau = \underline{0} \tag{9}$$

Closure is a necessary condition for repetition and periodicity of the state because a continuous system can only be in the state where it started the first cycle and where it ended the first cycle at the same time if they are the same place.

This property can be defined for each scalar element of the state vector. Closure is important because, when the system dynamics are a model of error propagation and the input represents a particular error model, this property indicates that the cumulative effect of the input errors vanishes under certain conditions.

2.3.2 Symmetry

One of the ways in which the trajectory closure condition can be satisfied is when the integrand is symmetric on the interval $[0, T]$. Suppose for example that the interval $[0, T]$ can be partitioned into two sets of times where each time t_1 in the first set can be paired with a unique time t_2 in the second set and the following condition holds:

$$\underline{F}(\underline{u}(t_1), t_1)dt = -\underline{F}(\underline{u}(t_2), t_2)dt$$

Under these conditions, the trajectory closure condition will be satisfied by the associated input. One way to guarantee that the integral vanishes on $(0, T)$ is to make the part after $T/2$ equal to the negative of the half before $T/2$. The second half can be repeated in the same order or the opposite order to the first, so there are two cases. The two cases are functions that are "odd" and functions that are "aperiodic" about the center of the interval.

An odd function about the center of $(0, T)$ and its integral is illustrated below.

$$f(T/2 + t) = -f(T/2 - t) \qquad \forall t \in [0, T/2]$$

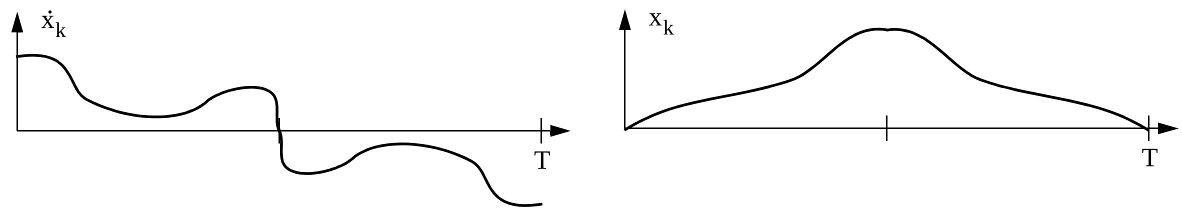

Figure 1 Odd Symmetry

Note that continuity requires that such a function pass through zero at $T/2$. Also, the integral of such a function (and only of such a function) is even about the point $T/2$ so the second half of the integral is just the first half played back in reverse.

An aperiodic function about the center of $(0, T)$ and its integral is illustrated below. Note that continuity requires that such a function achieve the negative of its initial value at $T/2$. Also, if the area under the function happens to vanish at $T/2$, the integral of the function is also aperiodic as illustrated.

$$f(t + T/2) = -f(t) \qquad \forall t \in [0, T/2]$$

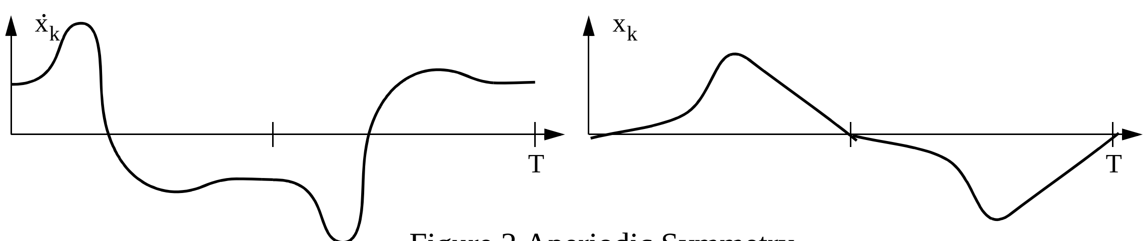

Figure 2 Aperiodic Symmetry

Of course, if the interval $(0, T)$ can be covered by any number of functions exhibiting one of these two symmetries on their respective interval, the associated trajectory closure integral still vanishes.

This property can be defined for each scalar element of the state vector. Symmetry is an important sufficient condition for closure.

2.3.3 Path Independence

There is also a case where the trajectory closure condition will be satisfied for **any** input. When the integrand can be written as:

$$\underline{F}(\underline{u}(\tau), \tau)dt = \frac{\partial}{\partial \underline{u}}\underline{p}(\underline{u}) \bullet d\underline{u} = d\underline{p}(\underline{u}) \qquad \textbf{(10)}$$

the middle expression is a Jacobian matrix (gradient of a vector) of some potential over the input. The right had side is a total differential which is integrable immediately. Under these conditions, the solution is of the form:

$$\underline{x}(t) = p(\underline{u}(t)) \qquad \textbf{(11)}$$

Here, the state is a closed-form function of the input. In this case, the state is totally dependent on the present value of the input, not its history, so such systems are called path independent[1]. Whenever the input returns to its initial value, the state will as well, so **any closed** input trajectory will have an associated closed state trajectory.

This property can be defined for each scalar element of the state vector. Path independence is an important sufficient condition for closure where the only condition imposed on the input is that it also be closed.

1. In vector field theory, all of the following notions are provably equivalent:
- A line integral is path independent if the value of the integral does not depend on the path from the start to the endpoint (it depends only on the position of the endpoint).
- The line integral over an arbitrary closed path is zero.
- The integrand is a total differential.
- The integrand can be derived from an underlying potential field.
- The curl of the associated vector is zero everywhere within a closed path.

3. Odometry Processes and Error Models

This section develops the equations of odometry with reference to the nonlinear dynamical system model presented in the last section. The resulting equations are solved for two simple trajectories. Models of error are presented for typical sensor modalities and closed form solutions for error propagation are computed for these simple cases.

3.1 Odometry Models

Odometry is the process of computing vehicle position and orientation by integrating differential indications of translational and rotational motion. In certain cases, the integral takes place over time; in others over space. Various sensors can be used and there are also various degrees of sophistication of the odometry process.

3.1.1 Coordinate Conventions

The coordinate system that will be used throughout the analysis is presented in Figure 1. The vehicle will be assumed to move in the plane and the zero heading direction will be defined to be along the x axis.

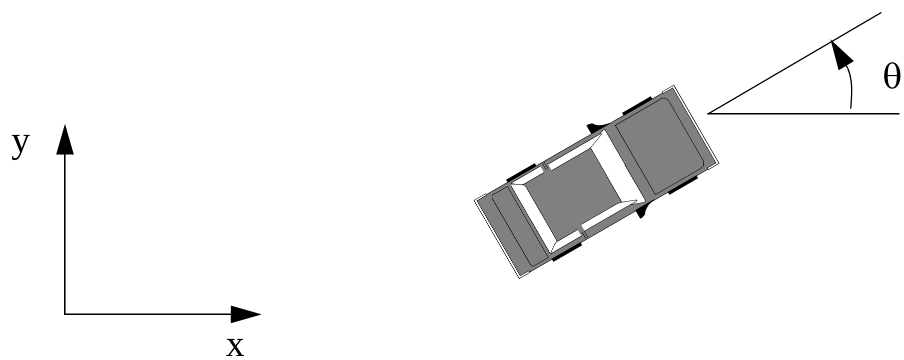

Figure 3 Coordinates for Odometry

In two dimensions, the typical state vector includes two position and one orientation component. For a continuous-time representation, this is:

$$\underline{x}(t) = \begin{bmatrix} x(t) & y(t) & \theta(t) \end{bmatrix}^T \tag{12}$$

For a discrete-time representation, the state vector is:

$$x_k = \begin{bmatrix} x_k & y_k & \theta_k \end{bmatrix}^T \tag{13}$$

There are at least two ways that the former nonlinear dynamical system of equation (1) can be used to model this process:

- Forced Dynamics: We can use the input vector to represent the measurements and compute the associated response. We have the option of including velocity states in the state vector in order to model the system dynamics.
- Observer: We can associate the measurements with the z vector in the observer equation.

We will use a combination of these two methods. The forced dynamics formulation will be used when the inputs are the linear and angular velocity. The observer formulation will be used when inputs other than these are to be modeled. To save space, the continuous time formulation will be used throughout but an equivalent discrete-time version of every result in the document can be generated as well.

3.1.2 Direct Heading Odometry

This scenario is the simplest because the heading is an input rather than a state. Measurements of linear velocity and heading are assumed to be available. The state equations are:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \end{bmatrix} \tag{14}$$

and there are no observer equations. The state and input vectors are:

$$\underline{x}(t) = \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T \qquad \underline{u}(t) = \begin{bmatrix} V(t) & \theta(t) \end{bmatrix}^T$$

where the input vector consists of the linear velocity and heading of the vehicle.

3.1.3 Integrated Heading Odometry

This is the case where measurements of linear and angular velocity are presumed to be available. The state equations are:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix} \tag{15}$$

and there are no observer equations. The state and input vectors are:

$$\underline{x}(t) = \begin{bmatrix} x(t) & y(t) & \theta(t) \end{bmatrix}^T \qquad \underline{u}(t) = \begin{bmatrix} V(t) & \omega(t) \end{bmatrix}^T$$

where the input vector consists of the linear and angular velocity of the vehicle and the state vector now includes heading.

3.1.4 Differential Heading Odometry

This is the case where direct linear and angular velocity are derived from the linear velocities of two or more points on the vehicle. For example, let there be a left wheel and a right wheel on either side of the vehicle reference point. This common case of differential heading odometry, based on two separated wheels is indicated below:

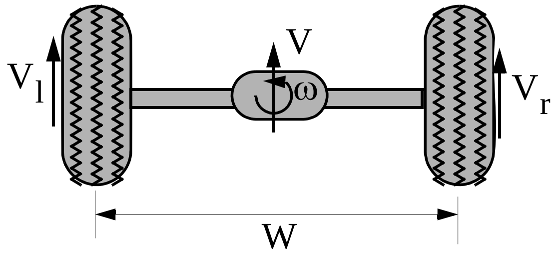

Figure 4 Differential Heading

W is the wheel separation or "tread". Encoders can be considered velocity measurement devices if we suppose that velocity is computed by dividing incremental distance over a small time interval by the length of the interval.

While it is possible in this simple case to solve for the equivalent integrated heading input $\underline{u}(t)$ and use the previous result, or to consider the wheel velocities to be the inputs, we will formulate this form of odometry with an observer to illustrate the more general case where the measurements $\underline{z}(t)$ may depend nonlinearly on both the state and the input, and may even overdetermine the input. As before the state and input vectors are:

$$\underline{x}(t) = \begin{bmatrix} x(t) & y(t) & \theta(t) \end{bmatrix}^T \qquad \underline{u}(t) = \begin{bmatrix} V(t) & \omega(t) \end{bmatrix}^T$$

Let the measurement vector be the velocities of the two wheels:

$$\underline{z}(t) = \begin{bmatrix} V_r(t) & V_l(t) \end{bmatrix}^T$$

The relationship between these and the equivalent integrated heading inputs is:

$$\underline{z}(t) = M\underline{u}(t)$$

$$\begin{bmatrix} V_r(t) \\ V_l(t) \end{bmatrix} = \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} \tag{16}$$

This is a particularly simple version of the more general form of the observer in equation (1). The inverse relationship is also immediate:

$$\underline{u}(t) = M^{-1}\underline{z}(t)$$

$$\begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} V_r(t) \\ V_l(t) \end{bmatrix} \tag{17}$$

3.2 Properties of Odometry

Odometry as formulated here possesses many of the properties described earlier.

3.2.1 Underactuation, Coupling, and Nonlinearity

The integrated and differential models are **underactuated** since two inputs control three outputs. Note also that heading is not independent of the direction of velocity. Such a system has to move in the direction it is pointing:

$$V(t) = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$tan\theta = \dot{y}/\dot{x}$$

The fact that the equations are differential equations also means that when the equations are integrated it is likely that position will depend on the entire time history of heading state as well as the input speed. The heading state appears inside a trigonometric function so the equations are also **nonlinear**.

Since the position derivatives depend on the heading state, the equations are **coupled**. The coupling is also asymmetric. While position derivatives depend on heading, heading does not depend on position. This difference between position and heading error dynamics will show up many times later.

3.2.2 Echelon Form

Fortunately, the equation dependencies of odometry are in echelon form, meaning that each state variable derivative depends only on others “below” it in the ordering of the equations as listed here. As a result, a solution can be generated by solving for the heading first and then substituting the result into the position equations before solving them.

3.2.3 Factorable

All forms of odometry mentioned above possess no terms involving state variables which are not multiplied by an input. If we make the substitution:

$$\omega(t) = \kappa(t)V(t)$$

where $\kappa(t)$ is the instantaneous curvature, then it becomes clear that all forms of odometry above are both input factorable and linear in the velocity. Hence, they are memoryless, reversible, and scalable.

3.2.4 Motion Dependent

It also turns out that the input is velocity in this case, so these equations are motion dependent.

3.3 Solutions for Simple Trajectories

Although the odometry equations are nonlinear, their echelon form makes it possible to solve them easily. We can solve for heading and then substitute the solution into the position equations and solve those. This process is illustrated below for two important simple trajectories.

3.3.1 Straight Line Trajectory

Consider the simple case of straight line motion $\omega(t) = 0$ at any speed $V(t)$ along the x axis starting from the origin. The trajectory is given by the solution to equation (15) which can be written by inspection:

$$\begin{aligned} x(t) &= s(t) \\ y(t) &= 0 \\ \theta(t) &= 0 \end{aligned} \tag{18}$$

where $s(t)$ is the total distance travelled.

3.3.2 Constant Curvature Trajectory

A constant curvature trajectory can be described by:

$$\omega(t) = \kappa V(t) \qquad \frac{V(t)}{\omega(t)} = \frac{1}{\kappa} = R$$

for some constant curvature κ or radius of curvature R. This type of trajectory is a circular arc. Note that the two velocities need not be constant in time - the trajectory is circular so long as their ratio is constant. Let the vehicle start out from the origin pointing along the x axis.

The solution to equation (15) is obtained as follows:

$$\begin{aligned} \theta(s) &= \int_0^t \omega(t)dt = \int_0^t \kappa V(t)dt = \int_0^s \kappa ds = \kappa s(t) = \frac{s}{R} \\ x(s) &= \int_0^t V\cos(\omega t)dt = \int_0^s \cos\left(\frac{s}{R}\right)ds = R\sin\left(\frac{s}{R}\right) = R\sin(\kappa s) \\ y(s) &= \int_0^t V\sin(\omega t)dt = \int_0^s \sin\left(\frac{s}{R}\right)ds = R\left[1-\cos\left(\frac{s}{R}\right)\right] = R[1-\cos(\kappa s)] \end{aligned} \tag{19}$$

It was not necessary to assume constant linear or angular velocity above. In the specific case when the linear and angular velocities[1] are known to be constant we also have:

$$\theta(t) = \int_0^t \omega(t)dt = \omega t$$

$$x(t) = \int_0^t V\cos(\omega t)dt = \left(\frac{V}{\omega}\right)\sin(\omega t) = R\sin(\omega t) \tag{20}$$

$$y(t) = \int_0^t V\sin(\omega t)dt = \left(\frac{V}{\omega}\right)[1 - \cos(\omega t)] = R[1 - \cos(\omega t)]$$

Clearly the analogy between distance and time is perfect when curvature is substituted for angular velocity.

3.4 Nonlinear Error Dynamics

It will be important for our purposes to be able to model the behavior of a small "perturbation" about a known solution to the state equations because doing so models the behavior of errors as they propagate through the system dynamics. The notation δ is standard for perturbations and necessary because it will often be necessary to write expressions involving both δx which is "small" and dx which is "infinitesimal".

3.4.1 Nonlinear Perturbation

Let us therefore assume that any of the earlier nonlinear odometry models applies. We will neglect the uncontrollable input $\underline{w}(t)$ and deal with its impact separately later. Assume that a nominal input $\underline{u}(t)$ and the associated nominal solution $\underline{x}(t)$ are known. That is, they satisfy:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t) \tag{21}$$

Suppose now that solution is desired for a slightly different input.

$$\underline{u}'(t) = \underline{u}(t) + \delta\underline{u}(t)$$

We will call $\underline{u}'(t)$ the "perturbed input", and $\delta\underline{u}(t)$ the "input perturbation". Let us designate the solution associated with this input as follows:

$$\underline{x}'(t) = \underline{x}(t) + \delta\underline{x}(t)$$

1. If either one is constant, the assumption of constant curvature requires the other to be.

This expression amounts to a definition of the state perturbation $\delta\underline{x}(t)$ as the difference between the perturbed and the nominal state. The time derivative of this solution is clearly the sum of the time derivatives of the nominal solution and the perturbation. This slightly different solution also, by definition, satisfies the original state equation, so we can write:

$$\dot{\underline{x}}'(t) = \dot{\underline{x}}(t) + \delta\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t) + \delta\underline{x}(t), \underline{u}(t) + \delta\underline{u}(t), t)$$

The state perturbation can be obtained by the difference between the perturbed and unperturbed responses:

$$\delta\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t) + \delta\underline{x}(t), \underline{u}(t) + \delta\underline{u}(t), t) - \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

In other words, we can assess the impact of errors on dynamical systems by solving the system differential equations with the error, then solving them without the error and taking the difference.

Of course, numerical techniques can be always be used to integrate the equations and compute perturbed trajectories, nominal trajectories, and the difference between them. However, while the numerical technique is completely general, it is not very illuminating.

3.4.2 Exact Error Dynamics

We have seen that odometry, though nonlinear, can often be solved in closed form for specific inputs because the equations are in echelon form. Hence, it seems like it might be possible to solve for the exact error dynamics under certain conditions. This turns out not to be so except in trivial cases.

For example, recall again the equations of integrated heading odometry:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\,cos\,\theta(t) \\ V(t)\,sin\,\theta(t) \\ \omega(t) \end{bmatrix}$$

Let the inputs be corrupted by errors as follows:

$$V'(t) = V(t) + \delta V(t)$$

$$\omega'(t) = \omega(t) + \delta\omega(t)$$

The perturbed heading is:

$$\theta'(t) = \theta'(0) + \int_0^t \omega'(t)dt = \theta'(0) + \int_0^t [\omega(t) + \delta\omega(t)]dt$$
$$\theta'(t) = \theta(0) + \int_0^t \omega(t)dt + \delta\theta(0) + \int_0^t \delta\omega(t)dt = \theta(t) + \delta\theta(t) \quad \textbf{(22)}$$

Hence, the resulting heading error is just the integral of the input angular velocity error:

$$\delta\theta(t) = \delta\theta(0) + \int_0^t \delta\omega(t)dt \quad \textbf{(23)}$$

The perturbed x coordinate can be expressed as:

$$x'(t) = x'(0) + \int_0^t V'(t)\cos\theta'(t)dt = x'(0) + \int_0^t [V(t) + \delta V(t)]\cos[\theta(t) + \delta\theta(t)]dt$$
$$x'(t) = x'(0) + \int_0^t [V(t) + \delta V(t)]\{\cos\theta(t)\cos\delta\theta(t) - \sin\theta(t)\sin\delta\theta(t)\}dt \quad \textbf{(24)}$$

Similarly, the perturbed y coordinate can be expressed as:

$$y'(t) = y'(0) + \int_0^t V'(t)\sin\theta'(t)dt = y'(0) + \int_0^t [V(t) + \delta V(t)]\sin[\theta(t) + \delta\theta(t)]dt$$
$$y'(t) = y'(0) + \int_0^t [V(t) + \delta V(t)]\{\sin\theta(t)\cos\delta\theta(t) + \cos\theta(t)\sin\delta\theta(t)\}dt \quad \textbf{(25)}$$

Discovering closed-form solutions to the exact position perturbation equations above can be difficult. When one or both of the perturbative inputs are constant or null functions of time, the above integrals can become straightforward. In most cases, the exact solutions are integrals of trigonometric functions, so they are also trigonometric functions. We will investigate these equations for specific trajectories and error models once the error models are specified next.

A general closed-form solution for any trajectory and error model is not available in the nonlinear case, but later, we will linearize the equations and then it will be possible to generate a closed form general solution. A second reason for exploring linearization is that stochastic error propagation results are often available only for linearized equations.

3.5 Error Models for Odometry Sensors

This section will present deterministic and random error models for typical sensors used in odometry.

3.6 Error Gradient Notation

Error models such as constant biases and scale errors in the systematic case and time or distance dependent random walks in the random case will be used in examples. These assumptions will generate a large number of constants for which a consistent notation will be adviseable. Most commonly, gradients of error with time, distance or angle will occur.

For systematic error, the partial derivative will be indicated by a right subscript. Thus:

$$\delta V_v = \frac{\partial}{\partial V}\{\delta V\}$$

On the assumption that velocity sensor error is proporation to velocity, this error gradient is constant and:

$$\delta V = \delta V_v \times V$$

For random error, right subscripts are already used to indicate the variable operated upon by the variance operator. In this case the right bracketed superscript will be used. Thus:

$$\sigma_{v\omega}^{(v)} = \frac{\partial}{\partial V}\{\sigma_{v\omega}\}$$

When this gradient is constant, the indicated covariance varies linearly with (usually unsigned) velocity:

$$\sigma_{v\omega} = \sigma_{v\omega}^{(v)} \times |V|$$

3.6.1 Direct Heading Odometry

Suppose that a compass is used to measure heading directly and that a transmission encoder is used to measure the linear velocity of the vehicle reference point - the point whose pose is considered to be that of the vehicle.

3.6.1.1 Systematic Error

One of the more common types of error for a compass is the response of the compass to the field generated by the vehicle itself. Without loss of generality, let the earth magnetic field point along the x axis:

$$\vec{E}_e = C\hat{i}$$

The field generated by the vehicle will be constant in the vehicle frame:

$$\vec{E}_v = R_b \cos\Theta \hat{i}_b + R_b \sin\Theta \hat{j}_b$$

When the vehicle is at orientation θ, this field can be expressed in the earth frame as:

$$\vec{E}_v = R_b cos(\Theta + \theta)\hat{i} + R_b sin(\Theta + \theta)\hat{j}$$

The compass will respond to the total field:

$$\vec{E}_T = \vec{E}_e + \vec{E}_v$$

The angle of the total field is:

$$\theta' = \theta + atan\left(\frac{R_b sin(\Theta + \theta)}{C + R_b cos(\Theta + \theta)}\right)$$

If we assume that the vehicle field is small, then the argument is small and if $R_b \ll C$ we have:

$$\delta\theta \approx \frac{R_b sin(\Theta + \theta)}{C} = \delta\theta_c cos\theta + \delta\theta_s sin\theta$$

so the error is a sinusoid of constant amplitude and phase. We will use a scale error for the encoder so the total error model is:

$$\begin{aligned} \delta V &= \delta V_v V \text{ where } (\delta V_v \ll 1) \\ \delta\theta &= \delta\theta_c cos\theta + \delta\theta_s sin\theta \end{aligned} \quad \textbf{(26)}$$

3.6.1.2 Random Error

For encoders, we will have a process linear in distance rather than time because encoder error accumulation stops when motion stops. A constant spectral probability density for the compass leads to a random walk contribution to the position coordinates. These considerations lead to the following models[1]:

$$\begin{aligned} \sigma_{vv} &= \sigma_{vv}^{(v)}|V| \\ \sigma_{\theta\theta} &= \text{const} \end{aligned} \quad \textbf{(27)}$$

3.6.2 Integrated Heading Odometry

Suppose that a gyro is used to measure angular velocity and that a transmission encoder is used to measure the linear velocity of the center of the rear axle of the vehicle.

3.6.2.1 Systematic Error

Let there be a constant bias error on the gyro and a scale error on the encoder. These errors can be

1. A negative variance is impossible so we must interpret all velocity dependent expressions in terms of the (unsigned) magnitude of the velocity vector.

represented as follows:

$$\delta V = \delta V_v V \text{ where } (\delta V_v \ll 1) \qquad \textbf{(28)}$$
$$\delta\omega = \delta\omega$$

3.6.2.2 Random Error

If the spectral probability density of the encoder error is constant, it results in a random walk process. However, because we are considering an encoder, it makes more sense to have a process linear in distance rather than time because encoder error accumulation will stop when motion stops. Constant variance of the gyro signal (constant bias stability) is a typical model, and it results in a "random walk" heading process whose variance grows linearly with time. These errors will be assumed to be uncorrelated. These considerations lead to the following models:

$$\sigma_{vv} = \sigma_{vv}^{(v)}|V| \qquad \textbf{(29)}$$
$$\sigma_{\omega\omega} = \sigma_{\omega\omega}$$
$$\sigma_{v\omega} = 0$$

3.6.3 Differential Heading Odometry

Let there be a wheel encoder mounted on each of the left and right wheels.

3.6.3.1 Systematic Error

One of the most commonly occurring errors in this type of sensor is a scale error due to a poor estimate of effective wheel radius caused by tire wear, uneven floors, or some other source. We will model errors in the two encoders for this case as scale errors[1]:

$$\delta V_r = \delta Vr_v V_r \text{ where } (\delta Vr_v \ll 1) \qquad \textbf{(30)}$$
$$\delta V_l = \delta Vl_v V_l \text{ where } (\delta Vl_v \ll 1)$$

3.6.3.2 Random Error

If the spectral probability densities of the encoder errors are constant, the measurement noise processes are equivalent to a random walk since a single integration renders them linear in time. We would like to model random walk sensors but, because we are considering encoders, it again makes more sense to have a process linear in distance rather than time because encoder error accumulation stops when motion stops. These errors will be assumed to be uncorrelated. These

1. Whereas scale errors make sense for encoders, constant errors is probably a better model of visual odometry.

considerations lead to the following models:

$$\begin{aligned} \sigma_{rr} &= \sigma_{rr}^{(v)}|V_r| \\ \sigma_{ll} &= \sigma_{ll}^{(v)}|V_l| \\ \sigma_{rl} &= 0 \end{aligned} \quad (31)$$

Hence, it will be understood that error accumulation stops when $V = 0$ and that the variance of the integrated encoder readings increases linearly with distance whether travelling forward or backward.

3.7 Treatment of Arithmetic Sign

It is clear from that last section that it is often convenient to model the fact that sensor error variance should be monotone in some velocity by requiring the use of the magnitude of the velocity. When these expressions occur inside integrals later we will be required to consider writing expressions like:

$$\begin{aligned} |ds| &= |V|dt \\ |d\theta| &= |\omega|dt \end{aligned}$$

in order to capture the fact that, if velocity is to be unsigned, the associated differential position coordinates must be unsigned. Rather than invent a special notation, we will refer to these cases as integrals over unsigned variables. In practice, such integrals can be evaluated by separating them into regions of constant sign of velocity and reversing the signs of those for which velocity is negative.

A more subtle case occurs when we encounter integrals like:

$$I(t) = \int_0^t V^2 dt$$

and desire a change of coordinates from time to distance. In such a case, we can write:

$$I(s) = \int_0^s |V| ds$$

where the differential ds is also required to be unsigned.

3.8 Error Propagation for Simple Cases

This section will derive closed form error propagation solutions for some cases where it is simple enough to derive them. Such cases will be limited here to deterministic errors with no initial conditions.

3.8.1 Direct Heading

This case will be investigated on a straight trajectory. The perturbed input is given by:

$$\begin{aligned} V'(t) &= (1+\delta V_v)V(t) \\ \theta'(t) &= A\cos\theta + B\sin\theta \end{aligned} \quad \textbf{(32)}$$

Substituting into equations (23) through (25), the perturbed trajectory is given by:

$$\begin{aligned} x'(t) &= \int_0^t [V(t) + \delta V_v V(t)]\cos\theta'(t)dt \\ y'(t) &= \int_0^t [V(t) + \delta V_v V(t)]\sin\theta'(t)dt \end{aligned} \quad \textbf{(33)}$$

This is already too complicated to integrate because we have expressions like:

$$\int_0^t \cos(A\cos\theta + B\sin\theta)dt$$

3.8.2 Integrated Heading

First, consider a linear trajectory. The perturbed input is given by:

$$\begin{aligned} V'(t) &= (1+\delta V_v)V(t) \\ \omega'(t) &= \delta\omega \end{aligned} \quad \textbf{(34)}$$

When the velocity is constant, the perturbed trajectory is clearly a constant curvature arc. Its curvature is given by:

$$\kappa'(t) = \omega'(t)/V'(t) = \frac{\delta\omega}{(1+\delta V_v)V(t)} = \frac{(\delta\omega)/V(t)}{(1+\delta V_v)}$$

We would expect the state perturbation to reflect the differences in position of two points - one moving on a line and one moving on a constant curvature arcs. The state perturbations are immediate:

$$\begin{aligned}\delta\theta(t) &= \theta'(t) - \theta(t) = [\omega'(t) - \omega(t)]t = \delta\omega t \\ \delta x(t) &= x'(t) - x(t) = R' sin\,\omega'(t) - Vt \\ \delta y(t) &= y'(t) - y(t) = R'[1 - cos\,\omega'(t)]\end{aligned} \quad (35)$$

Next, consider a constant curvature trajectory. The perturbed input is given by:

$$\begin{aligned} V'(t) &= (1 + \delta V_v)V(t) \\ \omega'(t) &= \omega(t) + \delta\omega \end{aligned} \quad (36)$$

When the velocity is constant, the perturbed trajectory is clearly another constant curvature arc. Its curvature is given by:

$$\kappa'(t) = \omega'(t) / V'(t) = \frac{\omega(t) + \delta\omega}{(1 + \delta V_v)V(t)} = \frac{\kappa(t) + \delta\omega / V(t)}{(1 + \delta V_v)}$$

We would expect the state perturbation to reflect the differences in position of two points moving on two different constant curvature arcs. The state perturbations are therefore immediate:

$$\begin{aligned}\delta\theta(t) &= \theta'(t) - \theta(t) = [\omega'(t) - \omega(t)]t = \delta\omega t \\ \delta x(t) &= x'(t) - x(t) = R' sin\,\omega'(t) - R\, sin\,\omega(t) \\ \delta y(t) &= y'(t) - y(t) = R'[1 - cos\,\omega'(t)] - R[1 - cos\,\omega(t)]\end{aligned} \quad (37)$$

3.8.3 Differential Heading

Rather than solve this case from first principles, the next section will reveal a method to turn any constant curvature differential heading error propagation problem into an equivalent integrated heading problem.

3.9 Instantaneous Differential Error Equivalent of Integrated Heading

Equations (16) and (17) outline how the inputs in differential heading are related to those of integrated heading. Given this relationship, an equivalent relationship between errors seems likely.

3.9.1 Systematic Error

Differentiating equation (17) leads to:

$$\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \delta r(t) \\ \delta l(t) \end{bmatrix} \quad \textbf{(38)}$$

Substituting our error models into this gives:

$$\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \delta r_r r(t) \\ \delta l_l l(t) \end{bmatrix}$$

Moving the scale factors into the matrix isolates the velocities:

$$\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix} = \begin{bmatrix} \frac{\delta r_r}{2} & \frac{\delta l_l}{2} \\ \frac{\delta r_r}{W} & -\frac{\delta l_l}{W} \end{bmatrix} \begin{bmatrix} r(t) \\ l(t) \end{bmatrix} \quad \textbf{(39)}$$

Substituting equation (16) into this gives:

$$\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix} = \begin{bmatrix} \frac{\delta r_r}{2} & \frac{\delta l_l}{2} \\ \frac{\delta r_r}{W} & -\frac{\delta l_l}{W} \end{bmatrix} \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix}$$

Multiplying the matrices:

$$\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix} = \begin{bmatrix} \frac{\delta r_r + \delta l_l}{2} & \frac{(\delta r_r - \delta l_l)W}{4} \\ \frac{\delta r_r - \delta l_l}{W} & \frac{\delta r_r + \delta l_l}{2} \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix}$$

Written out and substituting for angular velocity in terms of curvature gives:

$$\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix} = \begin{bmatrix} \left\{\left(\frac{\delta r_r + \delta l_l}{2}\right) + \frac{(\delta r_r - \delta l_l)}{4} W\kappa(t)\right\} V(t) \\ \left\{\frac{(\delta r_r - \delta l_l)}{W} + \left(\frac{\delta r_r + \delta l_l}{2}\right)\kappa(t)\right\} V(t) \end{bmatrix} \quad (40)$$

Substituting our integrated heading error model in the left hand side gives:

$$\begin{aligned} \delta V_v V(t) &= \left\{\left(\frac{\delta r_r + \delta l_l}{2}\right) + \frac{(\delta r_r - \delta l_l)}{4} W\kappa(t)\right\} V(t) \\ \delta\omega_v V(t) &= \left\{\frac{(\delta r_r - \delta l_l)}{W} + \left(\frac{\delta r_r + \delta l_l}{2}\right)\kappa(t)\right\} V(t) \end{aligned} \quad (41)$$

Equating coefficients of $V(t)$, the equivalent error parameters are:

$$\boxed{\begin{aligned} \delta V_v\big|_{equiv} &= \left(\frac{\delta r_r + \delta l_l}{2}\right) + \frac{(\delta r_r - \delta l_l)}{4R(t)} W \\ \delta\omega\big|_{equiv} &= \delta\omega_v V(t) = \left\{\frac{(\delta r_r - \delta l_l)}{W} + \left(\frac{\delta r_r + \delta l_l}{2R(t)}\right)\right\} V(t) \end{aligned}} \quad (42)$$

Where the motion dependent equivalent of gyro bias (that is, gradient of error with distance) is denoted $\delta\omega_v$ and $R(t)$ is the instantaneous radius of curvature. Clearly these are time dependent expressions but when curvature is constant, δV_v and $\delta\omega_v$ are also constant. Under this assumption, there exits a pair of constants $(\delta V_v, \delta\omega)$ derivable from $(\delta r_r, \delta l_l)$ and $R(t)$ which are the motion dependent equivalent of the integrated heading error parameters $(\delta V_v, \delta\omega)$.

3.9.2 Random Error

Equation (38) leads to:

$$\begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \sigma_{rr} & \sigma_{rl} \\ \sigma_{rl} & \sigma_{ll} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix}^T \quad (43)$$

This gives three scalar equations:

$$\begin{bmatrix} \sigma_{vv} \\ \sigma_{v\omega} \\ \sigma_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2W} & -\frac{1}{2W} & 0 \\ \frac{1}{W^2} & \frac{1}{W^2} & -\frac{2}{W^2} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{ll} \\ \sigma_{rl} \end{bmatrix}$$

Substituting our error model:

$$\begin{bmatrix} \sigma_{vv} \\ \sigma_{v\omega} \\ \sigma_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2W} & -\frac{1}{2W} & 0 \\ \frac{1}{W^2} & \frac{1}{W^2} & -\frac{2}{W^2} \end{bmatrix} \begin{bmatrix} \sigma_{rr}^{(r)}|r(t)| \\ \sigma_{ll}^{(l)}|l(t)| \\ 0 \end{bmatrix}$$

Moving the scale factors into a matrix isolates the velocities:

$$\begin{bmatrix} \sigma_{vv} \\ \sigma_{v\omega} \\ \sigma_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2W} & -\frac{1}{2W} & 0 \\ \frac{1}{W^2} & \frac{1}{W^2} & -\frac{2}{W^2} \end{bmatrix} \begin{bmatrix} \sigma_{rr}^{(r)} & 0 \\ 0 & \sigma_{ll}^{(l)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ l(t) \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{rr}^{(r)}}{4} & \frac{\sigma_{ll}^{(l)}}{4} \\ \frac{\sigma_{rr}^{(r)}}{2W} & -\frac{\sigma_{ll}^{(l)}}{2W} \\ \frac{\sigma_{rr}^{(r)}}{W^2} & \frac{\sigma_{ll}^{(l)}}{W^2} \end{bmatrix} \begin{bmatrix} |r(t)| \\ |l(t)| \end{bmatrix}$$

Now if the wheel velocities are unsigned, then the “equivalent” linear and angular velocities are

$$\left.\begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix}\right|_{DH} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} |r(t)| \\ |l(t)| \end{bmatrix} = \begin{bmatrix} \frac{|r(t)| + |l(t)|}{2} \\ \frac{|r(t)| - |l(t)|}{W} \end{bmatrix}$$

This device accounts for the requirement to use the absolute value of the left and right wheel velocities in order to keep the associated variances positive regardless of the direction of motion.

Inverting this relationship gives:

$$\begin{bmatrix} |r(t)| \\ |l(t)| \end{bmatrix} = \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} \Bigg|_{DH}$$

Substituting this gives:

$$\begin{bmatrix} \sigma_{vv} \\ \sigma_{v\omega} \\ \sigma_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{rr}^{(r)}}{4} & \frac{\sigma_{ll}^{(l)}}{4} \\ \frac{\sigma_{rr}^{(r)}}{2W} & -\frac{\sigma_{ll}^{(l)}}{2W} \\ \frac{\sigma_{rr}^{(r)}}{W^2} & \frac{\sigma_{ll}^{(l)}}{W^2} \end{bmatrix} \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} \Bigg|_{DH}$$

Multiplying the matrices

$$\begin{bmatrix} \sigma_{vv} \\ \sigma_{v\omega} \\ \sigma_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4} & \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{4}\frac{W}{2} \\ \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{2W} & \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4} \\ \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{W^2} & \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{W^2}\frac{W}{2} \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} \Bigg|_{DH} \qquad \textbf{(44)}$$

Let us define:

$$\kappa(t)|_{DH} = \omega(t)|_{DH} / V(t)|_{DH} \qquad\qquad R(t)|_{DH} = V(t)|_{DH} / \omega(t)|_{DH}$$

Written out and substituting for angular velocity in terms of curvature gives:

$$\begin{bmatrix} \sigma_{vv} \\ \sigma_{v\omega} \\ \sigma_{\omega\omega} \end{bmatrix} = \begin{bmatrix} \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{4} \frac{W}{2R(t)|_{DH}} \\ \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{2W} + \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4R(t)|_{DH}} \\ \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{W^2} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{W^2} \frac{W}{2R(t)|_{DH}} \end{bmatrix} V(t)|_{DH}$$

Substituting our integrated heading error model in the left hand side gives:

$$\begin{bmatrix} \sigma_{vv}^{(v)} \\ \sigma_{v\omega}^{(v)} \\ \sigma_{\omega\omega}^{(v)} \end{bmatrix} V(t)|_{DH} = \begin{bmatrix} \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{4} \frac{W}{2R(t)|_{DH}} \\ \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{2W} + \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4R(t)|_{DH}} \\ \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{W^2} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{W^2} \frac{W}{2R(t)|_{DH}} \end{bmatrix} V(t)|_{DH}$$

Equating coefficients of $V(t)$, the equivalent error parameters are:

$$\begin{aligned} \sigma_{vv}|_{equiv} &= \sigma_{vv}^{(v)} V(t)|_{DH} = \left\{ \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{4} \frac{W}{2R(t)|_{DH}} \right\} V(t)|_{DH} \\ \sigma_{v\omega}|_{equiv} &= \sigma_{v\omega}^{(v)} V(t)|_{DH} = \left\{ \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{2W} + \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4R(t)|_{DH}} \right\} V(t)|_{DH} \\ \sigma_{\omega\omega}|_{equiv} &= \sigma_{\omega\omega}^{(v)} V(t)|_{DH} = \left\{ \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{W^2} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{W^2} \frac{W}{2R(t)|_{DH}} \right\} V(t)|_{DH} \end{aligned} \qquad \textbf{(45)}$$

Clearly these are time dependent expressions but when curvature is constant, $\sigma_{vv}^{(v)}$, $\sigma_{v\omega}^{(v)}$ and $\sigma_{\omega\omega}^{(v)}$ are also constant. Under this assumption, there exits a set of constants $(\sigma_{vv}^{(v)}, \sigma_{v\omega}^{(v)}, \sigma_{\omega\omega}^{(v)})$ derivable from $(\sigma_{rr}^{(r)}, \sigma_{ll}^{(l)})$ and $R(t)$ which are the motion dependent equivalent of the integrated heading error parameters $(\sigma_{vv}^{(v)}, \sigma_{v\omega}^{(v)}, \sigma_{\omega\omega}^{(v)})$.

4. Relevant Ideas from Linear Systems Theory

The last section has made it clear that exact nonlinear solutions for odometry error propagation are not generally available. By contrast, this section will develop linearized methods which will be applicable to both deterministic and random errors and these methods will also provide a route to general solutions for any trajectory and error model.

Scalar Linear System

Before addressing the case of vector differential equations, the solution to a linear scalar differential equation based on an integrating factor is rewritten here in order to motivate the sequel. The general form of the linear scalar differential equation is:

$$\dot{x}(t) = f(t)x(t) + g(t)u(t) \tag{46}$$

Define the function:

$$\psi(t, t_0) = exp\left[\int_{t_0}^{t} f(\tau)d\tau\right] \tag{47}$$

The solution to the original scalar differential equation is then of the form:

$$x(t) = \psi(t, t_0)x(t_0) + \int_{t_0}^{t} \psi(t, \tau)g(\tau)u(\tau)d\tau \tag{48}$$

As we shall see, the linear vector differential equation has a solution of analgous form.

4.1 The Continuous-Time Linear System

The continuous-time linear lumped-parameter dynamical system is one of the most generally useful representations of applied mathematics. While not as expressive as the nonlinear case, the linear system has the advantage that general solutions exist. It is used to represent some real or imagined process whose behavior can be captured in a time-varying vector.

4.1.1 Linear State Equations

Many variations on the theme are used, but for our purposes, we will be interested in the form with no "output" but with an added "observer". This form is just complicated enough for our purposes. In continuous time, this representation is of the form:

$$\begin{aligned}\dot{\underline{x}}(t) &= F(t)\underline{x}(t) + G(t)\underline{u}(t) + L(t)\underline{w}(t)\\ \underline{z}(t) &= H(t)\underline{x}(t) + M(t)\underline{u}(t) + N(t)\underline{v}(t)\end{aligned} \tag{49}$$

These equations will be called the linear "state equations". The process or system is described by the time varying “state” $\underline{x}(t)$ of the system. The matrix $F(t)$, the system dynamics matrix, determines the unforced response of the system. The controllable input vector $\underline{u}(t)$ captures any controllable forcing functions which are mapped onto the state vector through the input transformation matrix $G(t)$. The uncontrollable input vector $\underline{w}(t)$ captures any uncontrollable forcing functions (deterministic or random) which are mapped onto the state vector through the transformation matrix $L(t)$.

The second equation comes into play because we will be interested also in the process by which various sensory devices can be used to observe the system state. The measurement vector $\underline{z}(t)$ captures the information that is obtained from the sensors and the measurement matrix $H(t)$ models the process by which the measurements are related to the underlying state $\underline{x}(t)$. On occasion, it will be necessary to model the manner which the input $\underline{u}(t)$ is observed indirectly through the transformation $M(t)$. The measurement noise $\underline{v}(t)$ is used to model the manner in which observations are corrupted by random noise.

The special case where the coefficient matrices are not time dependent is the theoretically important “constant coefficient” case. Note that the equations are still linear even in the time dependent case given above since the matrices do not depend on the state.

4.1.2 Solution to the Linear State Equations

We will for the moment, restrict attention to the case where there is no uncontrollable input $\underline{w}(t)$. It is well known that under such conditions the above linear time-dependent system has a general solution known as the vector convolution integral:

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)\underline{u}(\tau)d\tau$$

Vector Convolution Integral. General solution to the linear dynamical system **(50)**

where the matrix $\Phi(t, \tau)$, called the “transition matrix” is clearly the only unknown quantity. One can verify by substitution that the above expression satisfies the state equations if the transition matrix has two properties:

The first property is that the matrix is a direct mapping (or “transition”) of states from one time to any other time:

$$\underline{x}(t) = \Phi(t, \tau)\underline{x}(\tau)$$

The second property, often considered the definition, is that it satisfies the unforced state equation:

$$\frac{d}{dt}\Phi(t, \tau) = F(t)\Phi(t, \tau) \qquad \Phi(t, t) = I \qquad \textbf{(51)}$$

In the constant coefficient case, the transition matrix takes the particularly elegant form of the “matrix exponential” of the constant dynamics matrix:

$$\Phi(t, \tau) = e^{F(t-\tau)} \tag{52}$$

The matrix exponential is defined for any matrix and can be interpreted as an infinite matrix series:

$$e^{A} = exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots \tag{53}$$

In our case, the time dependent coefficient case, $\Phi(t, \tau)$ is known to exist but it may not be easy to find.

4.1.3 Solution For Commutable Dynamics

Consider the following generalization of the solution to the constant coefficient case:

$$\Psi(t, \tau) = exp\left(\int_{\tau}^{t} F(\zeta)d\zeta\right) \tag{54}$$

Define:

$$R(t, \tau) = \int_{\tau}^{t} F(\zeta)d\zeta$$

Notice immediately that:

$$\dot{R}(t, \tau) = \frac{d}{dt}R(t, \tau) = F(t)$$

By definition of the matrix exponential:

$$\Psi(t, \tau) = exp\left(\int_{\tau}^{t} F(\zeta)d\zeta\right) = exp[R(t, \tau)] = I + R + \frac{R^2}{2!} + \frac{R^3}{3!} + \frac{R^4}{4!} + \dots$$

Differentiating with respect to time:

$$\frac{d}{dt}\Psi(t, \tau) = 0 + \dot{R} + \frac{1}{2!}[R\dot{R} + \dot{R}R] + \frac{1}{3!}[RR\dot{R} + R\dot{R}R + \dot{R}RR] + \dots$$

$$\frac{d}{dt}\Psi(t, \tau) = 0 + F + \frac{1}{2!}[RF + FR] + \frac{1}{3!}[RRF + RFR + FRR] + \dots$$

Now in the particular case where R commutes with its own time derivative F, that is when $RF = FR$, we have:

$$\frac{d}{dt}\Psi(t, \tau) = F(t)\left\{I + R(t, \tau) + \frac{R(t, \tau)^2}{2!} + \frac{R(t, \tau)^3}{3!} + \ldots\right\} = F(t)\Psi(t, \tau)$$

Since $\Psi(t, \tau)$ satisfies the condition in equation (51), it must be the transition matrix for equation (49). Similiarly, we have:

$$\frac{d}{dt}\Psi(t, \tau) = \left\{I + R(t, \tau) + \frac{R(t, \tau)^2}{2!} + \frac{R(t, \tau)^3}{3!} + \ldots\right\}F(t) = \Psi(t, \tau)F(t)$$

4.1.4 An Equivalent Condition for Commutable Dynamics

All of this was based on the assumption that $RF = FR$. We know this implies that

$$F(t)\Psi(t, \tau) = \Psi(t, \tau)F(t)$$

Lets see if the reverse is true. Rewriting this:

$$F(t)\left\{I + R(t, \tau) + \frac{R(t, \tau)^2}{2!} + \frac{R(t, \tau)^3}{3!} + \ldots\right\} = \left\{I + R(t, \tau) + \frac{R(t, \tau)^2}{2!} + \frac{R(t, \tau)^3}{3!} + \ldots\right\}F(t)$$

$$F\left\{I + R + \frac{R^2}{2!} + \frac{R^3}{3!} + \ldots\right\} = \left\{I + R + \frac{R^2}{2!} + \frac{R^3}{3!} + \ldots\right\}F$$

Equating terms leads to:

$$F = F$$

$$FR = RF$$

$$FR^2 = R^2F$$

$$\ldots$$

$$FR^k = R^kF$$

The second line establishes that $F\Psi = \Psi F$ implies $FR = RF$. Hence, the former is an equivalent condition.

Let us therefore define the "commutable dynamics condition". When:

$$\boxed{\Psi(t, \tau)F(t) = F(t)\Psi(t, \tau)} \tag{55}$$

Commutable Dynamics Condition. Necessary condition for equation (54) to define the transition matrix for given system dynamics $F(t)$.

the matrix $\Psi(t, \tau)$ satisfies the conditions for a transition matrix. We will need a name for this property, so let it be called the property of "commutable dynamics". Some cases where the product commutes are when $F(t)$ is constant or diagonal, but one particular case concerns us here.

4.1.5 Special Case for Commutable Dynamics

Suppose the dynamics matrix can be partitioned as follows:

$$F(t) = \left[\begin{array}{c|c} \underset{[n \times n]}{0} & \underset{[n \times m]}{M(t)} \\ \hline \underset{[m \times n]}{0} & \underset{[m \times m]}{0} \end{array}\right] \tag{56}$$

For such matrices, it is easy to show that all powers of $F(t)$ vanish. In particular:

$$F(t)F(t) = 0$$

The product of any two such matrices of conformable structure also vanishes. Also, because the integrals involved are definite integrals, the associated exponential argument for a transition matrix obeying equation (54) is:

$$R(t, \tau) = \int_{\tau}^{t} F(\zeta)d\zeta = \begin{bmatrix} 0 & \int_{\tau}^{t} M(\zeta)d\zeta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & N(t, \tau) \\ 0 & 0 \end{bmatrix} \tag{57}$$

Recalling the infinite series for the matrix exponential and the vanishing higher power property, the exponential is:

$$\Psi(t, \tau) = exp[R(t, \tau)] = I + R(t, \tau) \tag{58}$$

and its easy to show that because the product $F(t)R(t, \tau)$ also vanishes, equation (55) also holds:

$$\Psi(t, \tau)F(t) = [I + R(t, \tau)]F(t) = F(t) = F(t)[I + R(t, \tau)] = F(t)\Psi(t, \tau)$$

Thus, systems satisfying equation (55) have commutable dynamics and their transition matrix is given by equation (58).

This appears to be a very stringent condition. Some special cases which satisfy the criterion are when F has the block structure mentioned above, when F is constant, and when it is symmetric (includes diagonal as a special case).

4.2 Perturbation Theory

It is generally possible to produce a "linearized" version of a nonlinear system which applies to small perturbations about a reference trajectory. This technique can be used as a substitute for the exact nonlinear perturbations discussed in the last chapter. Let us consider again the nonlinear system:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t) \tag{59}$$

We have again neglected the uncontrollable input $\underline{w}(t)$. Assume that a nominal input $\underline{u}(t)$ and the associated nominal solution $\underline{x}(t)$ are known. That is, they satisfy equation (59). As we did in the last chapter, suppose now that solution is desired for a slightly different input.

$$\underline{u}'(t) = \underline{u}(t) + \delta\underline{u}(t)$$

We will call $\underline{u}'(t)$ the "perturbed input", and $\delta\underline{u}(t)$ the "input perturbation". Let us designate the solution associated with this input as follows:

$$\underline{x}'(t) = \underline{x}(t) + \delta\underline{x}(t)$$

This expression amounts to a definition of the state perturbation $\delta\underline{x}(t)$ as the difference between the perturbed and the nominal state. The time derivative of this solution is clearly the sum of the time derivatives of the nominal solution and the perturbation, and this slightly different solution also, by definition, satisfies the original state equation, so we can write:

$$\dot{\underline{x}}'(t) = \dot{\underline{x}}(t) + \delta\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t) + \delta\underline{x}(t), \underline{u}(t) + \delta\underline{u}(t), t)$$

An approximation for $\delta\underline{x}(t)$ will generate an approximation for $\underline{x}'(t)$. We can get this approximation from the Taylor series expansion as follows:

$$\underline{f}(\underline{x}(t) + \delta\underline{x}(t), \underline{u}(t) + \delta\underline{u}(t), t) \approx \underline{f}(\underline{x}(t), \underline{u}(t), t) + F(t)\delta\underline{x}(t) + G(t)\delta\underline{u}(t)$$

where the two new matrices are the Jacobians of $\underline{f}$ with respect to the state and input - evaluated on the nominal trajectory:

$$F(t) = \left.\frac{\partial}{\partial \underline{x}}\underline{f}\right|_{\underline{x}} \qquad G(t) = \left.\frac{\partial}{\partial \underline{u}}\underline{f}\right|_{\underline{x}}$$

At this point, we have:

$$\dot{\underline{x}}(t) + \delta\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t) + F(t)\delta\underline{x}(t) + G(t)\delta\underline{u}(t)$$

Finally, by cancelling out the original state equation (59), there results a linear system which approximates the behavior of the perturbation.

$$\boxed{\delta\dot{\underline{x}}(t) = F(t)\delta\underline{x}(t) + G(t)\delta\underline{u}(t)}$$

Linear Perturbation Equation. Expresses first order dynamic behavior of systematic error for linear and nonlinear dynamical systems. **(60)**

All of the solution techniques for linear systems can now be applied to determine the behavior of this perturbation. In particular, a transition matrix for the above equation must exist because it is linear, so the first order propagation of systematic error in a dynamical system is a solved problem once the transition matrix is computed.

4.2.1 Perturbation with an Observer

Consider now a situation where an observer is present. Perturbing the observer in equation (1) we have by analogy:

$$\delta\underline{z}(t) = H(t)\delta\underline{x}(t) + M(t)\delta\underline{u}(t) \tag{61}$$

Where the observer Jacobians are:

$$H(t) = \left.\frac{\partial}{\partial\underline{x}}\underline{h}\right|_{\underline{x}} \qquad M(t) = \left.\frac{\partial}{\partial\underline{u}}\underline{h}\right|_{\underline{x}} \tag{62}$$

If $\delta\underline{u}(t)$ is not known directly but $\delta\underline{z}(t)$ is, we can solve the above equation for $\delta\underline{u}(t)$ by first writing:

$$M(t)\delta\underline{u}(t) = \delta\underline{z}(t) - H(t)\delta\underline{x}(t)$$

In the event that the measurements determine or overdetermine the inputs, the left pseudoinverse applies:

$$M^{LI} = [M^T(t)M(t)]^{-1}M^T(t) \tag{63}$$

and we can write:

$$\delta\underline{u}(t) = M^{LI}[\delta\underline{z}(t) - H(t)\delta\underline{x}(t)] \tag{64}$$

This is the input which minimizes the residual:.

$$\delta \underline{r}(t) = \delta \underline{z}(t) - (H(t)\delta \underline{x}(t) + M(t)\delta \underline{u}(t))$$

Substituting this back into the state perturbation equation we have:

$$\delta \dot{\underline{x}}(t) = F(t)\delta \underline{x}(t) + G(t)\{M^{LI}[\delta \underline{z}(t) - H(t)\delta \underline{x}(t)]\}$$

Which reduces to:

$$\delta \dot{\underline{x}}(t) = \{F(t) - G(t)M^{LI}H(t)\}\delta \underline{x}(t) + G(t)M^{LI}\delta \underline{z}(t) \tag{65}$$

This is of the same form as the original perturbation equation with modified matrices and the measurements acting as the input:

$$\delta \dot{\underline{x}}(t) = \tilde{F}(t)\delta \underline{x}(t) + \tilde{G}(t)\delta \underline{z}(t) \tag{66}$$

4.3 The Error Augmented Dynamical System

When our error descriptions result from linearization of some associated dynamical system, they can always be written in the form:

$$\delta \dot{\underline{x}}(t) = F(\underline{x}, \underline{u})\delta \underline{x}(t) + G(\underline{x}, \underline{u})\delta \underline{u}(t) \tag{67}$$

It is natural to consider that the input to this new set of equations is the input perturbations $\delta \underline{u}(t)$. For the purpose of modeling error dynamics, we can often assume and even specify the manner in which the input perturbation depends in some way on the input and the state:

$$\delta \underline{u}(t) = \underline{q}(\underline{x}, \underline{u}) \tag{68}$$

In this case, we can view the original equations and the perturbation equations as an augmented system where both $\delta \underline{x}(t)$ and $\underline{x}(t)$ are controlled through the input $\underline{u}(t)$:

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \delta \dot{\underline{x}}(t) \end{bmatrix} = \tilde{\underline{f}}(\underline{x}, \underline{u}) = \begin{bmatrix} \underline{f}(\underline{x}, \underline{u}) \\ F(\underline{x}, \underline{u})\delta \underline{x}(t) + G(\underline{x}, \underline{u})\underline{q}(\underline{x}, \underline{u}) \end{bmatrix} \tag{69}$$

This is a nonlinear dynamical system of the same form as the original system - but with twice as many states. The important conceptual point is that once the input $\underline{u}(t)$ to the system is specified, **both the state trajectory and its error** are determined by the above equation.

It is also possible under certain circumstances to express the state perturbations entirely in terms of the states by effectively substituting the first set of equations into the second. Here the view is the natural expectation that the error should depend only on the trajectory followed. In summary, the input to the perturbation equations can be considered to be the input perturbations, the original inputs, or the states depending on which dependencies between them are introduced and which are emphasized through substitution.

The error states in this new set of equations can be evaluated in terms of the integral and derivative properties of dynamical systems mentioned earlier.

4.3.1 Memoryless

The error equations are memoryless when the system Jacobian $F(\underline{x}, \underline{u})$ is zero.

4.3.2 Motion Dependence

The error equations are motion dependent when they can be divided by a position variable derivative without creating a singular point at zero. A singularity means intuitively that there is error accumulation happening when the system stops which is "hidden" by the singularity. Motion dependence can also be considered a property of each term. The state Jacobian of odometry will turn out to be motion dependent. Encoder errors, for example, are motion dependent whereas gyro bias is not.

4.3.3 Input Reversibility

When the system is not memoryless, the state perturbation derivative $\delta\dot{\underline{x}}(t)$ is not a reversible function of the input perturbation $\delta\underline{u}(t)$ because its memory prevents it from being odd in $\delta\underline{u}(t)$.

However, this point is not so important because sensory errors are largely outside of our control. A more important question is under what conditions is the state perturbation derivative $\delta\dot{\underline{x}}(t)$ a reversible function of the input $\underline{u}(t)$.

Note that if the original nonlinear dynamics are odd in the input, then the Taylor series of $f(\ )$ involves only the odd terms in $\underline{u}(t)$[1]:

$$\underline{f}(\underline{x}, \underline{u}) = \underline{f}'(\underline{x}, \underline{0})\underline{u} + \frac{\underline{f}'''(\underline{x}, \underline{0})\underline{u}^3}{3!} + \ldots$$

Under this assumption, it is easy to show that:

- the state Jacobian $F(\ )$ must also be odd in $\underline{u}(t)$
- the input Jacobian $G(\ )$ must be even in $\underline{u}(t)$

So, if the original dynamics were input reversible, the first term in equation (67) is odd in the input and the second is even. For such a system, we can achieve input reversibility if the input errors are

1. The notation f'''() etc. is used avoids explicit tensor notation for the cubic and higher order terms. Also, oddness of f is a sufficient, not a necessary condition.

an odd function of the input (because the product of an odd and even function is odd) as follows:

$$\delta\underline{u}(t) = \underline{q}(\underline{x}, \underline{u}) \quad \text{and} \quad \underline{q}(\underline{x}, -\underline{u}) = -\underline{q}(\underline{x}, \underline{u}) \tag{70}$$

So, if the original dynamics are input reversible, the perturbation dynamics are input reversible provided equation (70) is also satisfied.

4.3.4 Input Scalability

This property is of interest in the context of a perturbation input. If we again assume that the perturbation depends itself on the input, then the question rests on the response of equation (67) to a rescaling of the input perturbation.

4.3.5 Closure

Closure of the state perturbation is achieved when the vector convolution integral (in equation (50)) for the perturbed dynamics vanishes on an interval of interest:

$$\int_0^T \Phi(t, \tau)G(\tau)\delta\underline{u}(t)d\tau = 0$$

As always, symmetry of the integrand on the interval is one way to achieve closure.

4.3.6 Path Independence

Path independence can be expressed as an integral or differential property. When the solution integral can be expressed in either of the following forms:

$$\delta\underline{x}(t) = \int_0^t \frac{\partial}{\partial \underline{u}}\underline{p}(\underline{u})d\underline{u} = \int_0^t d\underline{p}(\underline{u}) \qquad \delta\underline{x}(t) = \int_0^t \frac{\partial}{\partial \underline{x}}\underline{p}(\underline{x})d\underline{x} = \int_0^t d\underline{p}(\underline{x})$$

The errors in question are path independent and the above integrals vanish on any closed trajectory of $\underline{u}(t)$ or $\underline{x}(t)$ respectively.[1]

4.4 Stochastic Error Propagation for the Continuous-Time Nonlinear System

In much the same way that deterministic error propagation satisfies a differential equation, stochastic error propagation also follows a differential equation.

4.4.1 Nonlinear State Equations

As before, a continuous time system can be described by:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), t)$$

4.4.2 Random Perturbations

Suppose now that a solution to this equation is desired for a slightly different input.

$$\underline{u}'(t) = \underline{u}(t) + \delta\underline{u}(t) + \delta\underline{w}(t)$$

where $\delta\underline{w}(t)$ is a random vector, often called the process noise. We will call $\underline{u}'(t)$ the "perturbed input", and $\delta\underline{u}(t)$ the "input perturbation". Let us designate the solution associated with this input as follows:

$$\underline{x}'(t) = \underline{x}(t) + \delta\underline{x}(t)$$

The linearized state equations are given by equation (60) with an additional random input $\delta\underline{w}(t)$:

$$\delta\dot{\underline{x}}(t) = F(t)\delta\underline{x}(t) + G(t)\delta\underline{u}(t) + L(t)\delta\underline{w}(t) \tag{71}$$

Where the Jacobians are:

$$F(t) = \left.\frac{\partial}{\partial\underline{x}}\underline{f}\right|_{\underline{x}} \qquad G(t) = L(t) = \left.\frac{\partial}{\partial\underline{u}}\underline{f}\right|_{\underline{x}}$$

It can be shown by taking expected values that the average solution to this equation is unchanged if the process noise is unbiased. Similarly, the deterministic input $\delta\underline{u}(t)$ has no effect on the resulting variance of the state. Hence, deterministic and random inputs can be treated separately. For the purpose of evaluating stochastic error propagation, we can consider the case where the only error input to the system is a random one:

$$\delta\dot{\underline{x}}(t) = F(t)\delta\underline{x}(t) + L(t)\delta\underline{w}(t) \tag{72}$$

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1. It is interesting to note that formulas for position coordinates are intrinsically path independent - and they should be because the endpoint coordinates of a path should not depend on the path used to get to the endpoint. Consider the x coordinate in the integrated heading case:

$$x = \int_0^t V\cos\theta dt = \int_0^t dx$$

This expression satisfies the conditions for path independence intrinsically. Indeed, other than the null integrand, the constants (the integrand above is unity) are the simplest integrands that are path independent.

4.4.3 Linear Variance Equation

Let us therefore reinterpret the state perturbation $\delta\underline{x}(t)$ as a random variable and define the covariances:

$$P = Exp(\delta\underline{x}(t)\delta\underline{x}(t)^T) \qquad Q = Exp(\delta\underline{w}(t)\delta\underline{w}(\tau)^T)\delta(t-\tau)$$

The delta function $\delta(t-\tau)$ is used to indicate that the process noise is uncorrelated in time, or "white". For small values of Δt and for slowly varying $L(t)$ and $Q(t)$, the relationship between the spectral density matrix Q and the equivalent discrete time covariance matrix is:

$$Q_{k-1} \cong L(t_k)Q(t_k)L(t_k)^T\Delta t$$

Which is to say that former is the derivative of the latter[1].

It is tempting to square equation (72) and take its expectation to get:

$$\dot{P} = Exp(\delta\dot{\underline{x}}(t)\delta\dot{\underline{x}}(t)^T) = F(t)PF^T(t) + L(t)QL^T(t)$$

While the second term is a correct expression of the effect of the input noise, the first is not. In general, the square of the derivative is not equal to the derivative of the square:

$$\dot{P} = \frac{d}{dt}Exp(\delta\underline{x}(t)\delta\underline{x}(t)^T) \neq Exp(\delta\dot{\underline{x}}(t)\delta\dot{\underline{x}}(t)^T)$$

We must instead write an expression for P and take its derivative, or write an expression for $\delta\underline{x}(t)$ and take its expectation and derivative. This is derived in several texts including [4] and the result is known as the linear variance equation:

$$\boxed{\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + L(t)Q(t)L(t)^T}$$

Linear Variance Equation. Expresses first order dynamic behavior of random error for linear and nonlinear dynamical systems. **(73)**

Note that while individual components of the state covariance may increase or decrease, the overall character of the equation is such that the third term requires the norm of P to increase over time.

4.4.4 Vectorization

The covariance matrix is symmetric by construction. It will be convenient to recast the linear variance equation in terms of a vector of variables instead of a matrix. This can be easily done by

1. Assuming that Q_{k-1} is independent of Δt can lead to some interesting dilemmas. For example, if 1 count of noise exists on an encoder, the total variance over a time period t is proportional to $\sqrt{t/\Delta t}$ - so reading the encoder faster actually increases the error for a fixed interval t. This behavior is a reality. The only limit on the error is any practical limits on how small Δt can be made. Similar comments apply to spatially dependent errors. For example, in the absence of image distortion, computed distance errors are reduced in visual odometry by waiting until subsequent images barely overlap.

reorganizing the equations. Clearly, there is a scalar equation generated for each element of P. All but the diagonal elements will be repeated twice, so we need only write out the diagonal and all of the elements either above or below it. For an $n \times n$ matrix, there are $n + n(n-1)/2$ unique equations.

We can represent this equivalent system as follows:

$$\dot{\sigma}_{\underline{x}}^2(t) = \tilde{F}(t)\sigma_{\underline{x}}^2(t) + \tilde{L}(t)\sigma_{\underline{w}}^2(t) \tag{74}$$

The new Jacobians are based on the earlier ones but are now larger and they contain more zeros.

4.4.5 Stochastic Error Propagation with a Process Noise Observer

Consider now a situation where an observer is present. Perturbing the observer in equation (1) we have by analogy[1]:

$$\delta\underline{z}(t) = H(t)\delta\underline{x}(t) + N(t)\delta\underline{w}(t) \tag{75}$$

Where the observer Jacobians are:

$$H(t) = \left.\frac{\partial}{\partial\underline{x}}\underline{h}\right|_{\underline{x}} \qquad N(t) = \left.\frac{\partial}{\partial\underline{u}}\underline{h}\right|_{\underline{x}} \tag{76}$$

If the statistics of $\delta\underline{w}(t)$ are not known directly but those of $\delta\underline{z}(t)$ are, we can solve for the measurement covariance Q in terms of the covariance in the observations R and the state P.

First, let us define:

$$R = Exp(\delta\underline{z}(t)\delta\underline{z}(t)^T)$$

And then take the second moment of equation (75) suppressing the time dependence notation temporarily:

$$R = Exp\{\delta\underline{z}\delta\underline{z}^T\} = Exp\{[H\delta\underline{x} + N\delta\underline{w}][H\delta\underline{x} + N\delta\underline{w}]^T\}$$

Writing out the right hand side:

1. Note that this formulation is not the standard noise corrupted observer of the Kalman filter. Here we are assuming that the process noise is observable. In fact, because we are using the input u to model the sensors, it is perhaps more appropriate to consider $\delta\underline{w}(t)$ to be the measurement noise instead of the process noise.

$$R = HExp\{\delta\underline{x}\delta\underline{x}^T\}H^T + HExp\{\delta\underline{x}\delta\underline{w}^T\}N^T + NExp\{\delta\underline{w}\delta\underline{x}^T\}H^T + NExp\{\delta\underline{w}\delta\underline{w}^T\}N^T$$

If the process noises are uncorrelated with the state noise:

$$R = HPH^T + NQN^T$$

This matrix equation can be solved for Q as follows:

$$\begin{aligned}
NQN^T &= R - HPH^T \\
NQN^T[R - HPH^T]^{-1} &= I \\
NQN^T[R - HPH^T]^{-1}N &= N \\
QN^T[R - HPH^T]^{-1}N &= I \\
Q &= \left[N^T[R - HPH^T]^{-1}N\right]^{-1}
\end{aligned}$$

Substituting this into the linear variance equation we have:

$$\dot{P} = FP + PF^T + L\left[N^T[R - HPH^T]^{-1}N\right]^{-1}L^T \qquad \textbf{(77)}$$

When N is square this becomes:

$$\begin{aligned}
\dot{P} &= FP + PF^T + L[N^{-1}[R - HPH^T]N^{-T}]L^T \\
\dot{P} &= FP + PF^T + LN^{-1}RN^{-T}L^T - LN^{-1}HPH^TN^{-T}L^T
\end{aligned} \qquad \textbf{(78)}$$

And when H is the zero matrix we have:

$$\dot{P} = FP + PF^T + LN^{-1}RN^{-T}L^T \qquad \textbf{(79)}$$

This is of same form as the original variance equation with modified matrices and the measurements acting as the input:

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + L(t)\tilde{Q}(t)L(t)^T \qquad \textbf{(80)}$$

where:

$$\tilde{Q}(t) = N^{-1}RN^{-T} \quad \textbf{(81)}$$

By sequentially writing out only the unique equations in each matrix, this can also be written in vector form as:

$$\sigma_{\underline{w}}^2 = \tilde{N}(t)\sigma_{\underline{z}}^2 \quad \textbf{(82)}$$

4.4.6 Solution to the Linear Variance Equation

While it is possible to vectorize the linear variance equation, compute the vectorized Jacobians, the transition matrix, and ultimately the solution, a more direct route is available [6]. If $\Phi(t, t_0)$ is the transition matrix for the original deterministic system dynamics, then the stochastic equivalent of equation (50) is:

$$P(t) = \Phi(t, t_0)P(t_0)\Phi^T(t, t_0) + \int_{t_0}^{t} \Phi(t, \tau)L(\tau)Q(\tau)L^T(\tau)\Phi^T(t, \tau)d\tau \quad \textbf{(83)}$$

Matrix Convolution Integral. General solution to the linear variance equation.

The heuristic derivation proceeds by taking the expected value of the second moment of the original deterministic solution where the state is considered a random variable as a result of the random forcing function $\underline{w}(t)$. A rigorous derivation requires the use of stochastic calculus but the result is an ordinary matrix integral which expresses the time evolution of the system covariance matrix as a function of the initial conditions and the inputs.

<u>4.5 The Discrete-Time Linear System</u>

Often, a system needs to be expressed in a discrete-time form in order to represent it in a computer. Sometimes the state equations are given in discrete form and other times they are generated by discretizing a continuous system.

4.5.1 Linear State Equations

If we are interested in a discrete-time representation, then the values of the vectors and matrices are known only at discrete times and the state equations take the form.

$$\underline{x}_{k+1} = F_k\underline{x}_k + G_k\underline{u}_k$$

$$\underline{z}_k = H_k\underline{x}_k + M_k\underline{u}_k$$

Here, the equations have similar form and similar meaning to the continuous case - with one exception. Note that $F(t)$ maps a state onto a state derivative while F_k maps a state onto a state. Also, whereas the continuous-time equations are differential equations, the discrete-time equations are recurrence equations.

4.5.2 Solution to the Linear State Equations

The solution to the state recurrence equations can be easily discovered by inspection by writing out the terms as k increases from 0 to some general value n and noticing the pattern. The result of this tedious but straightforward exercise is:

$$x_n = \left(\prod_{k=0}^{n-1} F_k\right) x_0 + \sum_{k=0}^{n} \left[\prod_{p=k+1}^{n-1} F_p\right] G_k u_k$$

By analogy to the earlier continuous-time result, the discrete-time transition matrix is:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} F_p$$

So the solution can be written as:

$$x_n = \Phi_{n,0} x_0 + \sum_{k=0}^{n} \Phi_{n,k+1} G_k u_k$$

4.5.3 Solution for Commutable Dynamics

It is always possible to rewrite the system dynamics matrix as follows:

$$F_k = I + R_k$$

by simply solving for R_k. If we suppose that R_k is of the same structure as equation (56), then all powers and cross-products for any two values of k will vanish. Note, in particular that under these conditions:

$$\Phi_{n,k} = \prod_{p=k}^{n-1} (I + R_p) = (I + R_k)(I + R_{k+1})\ldots = I + \sum_{p=k}^{n-1} R_p$$

and we have converted a product into a sum as a result. Let this special transition matrix and sum be denoted as follows:

$$T_{n,k} = I + R_{n,k} = I + \sum_{p=k}^{n-1} R_p$$

4.6 The Discrete-Time Nonlinear System and its Linear Perturbation

Nonlinear discrete-time systems are similar to their continuous-time counterparts.

4.6.1 Nonlinear State Equations

The nonlinear form of the state equations is:

$$\underline{x}_{k+1} = \underline{f}(\underline{x}_k, \underline{u}_k, k)$$
$$\underline{z}_k = \underline{h}(\underline{x}_k, k)$$

Even though a closed-form result for the nonlinear case is not available, numerical solutions are available by direct recurrence on the first equation:.

$$\underline{x}_1 = \underline{f}(\underline{x}_0, \underline{u}_0, 0)$$
$$\underline{x}_2 = \underline{f}(\underline{x}_1, \underline{u}_1, 1)$$
$$\underline{x}_3 = \underline{f}(\underline{x}_2, \underline{u}_2, 2)$$
$$\dots$$

4.6.2 Perturbation Theory

We can also model the behavior of a small "perturbation" about a known solution to the discrete-time state equations. Assume that a nominal input $\underline{u}_k$ and the associated nominal solution $\underline{x}_k$ are known. That is, they satisfy:

$$\underline{x}_{k+1} = \underline{f}(\underline{x}_k, \underline{u}_k, k) \quad \textbf{(84)}$$

Suppose now that solution is desired for a slightly different input.

$$\underline{u}'_k = \underline{u}_k + \delta\underline{u}_k$$

Designate the solution associated with this input as follows:

$$\underline{x}'_k = \underline{x}_k + \delta\underline{x}_k$$

The state perturbation is again the difference between the perturbed and nominal state. This slightly different solution, by definition, also satisfies the original state equation, so we can write:

$$\underline{x}'_{k+1} = \underline{x}_{k+1} + \delta\underline{x}_{k+1} = \underline{f}(\underline{x}_k + \delta\underline{x}_k, \underline{u}_k + \delta\underline{u}_k, k)$$

An approximation for $\delta\underline{x}_k$ will generate an approximation for $\underline{x}'_k$. We can get this approximation from the Taylor series expansion as follows:

$$\underline{f}(\underline{x}_k + \delta\underline{x}_k, \underline{u}_k + \delta\underline{u}_k, k) \approx \underline{f}(\underline{x}_k, \underline{u}_k, k) + F_k\delta\underline{x}_k + G_k\delta\underline{u}_k$$

where the two new matrices are the Jacobians of $\underline{f}$ with respect to the state and input - evaluated on the nominal trajectory:

$$F_k = \frac{\partial}{\partial\underline{x}}\underline{f}\Big|_{\underline{x}} \qquad\qquad G_k = \frac{\partial}{\partial\underline{u}}\underline{f}\Big|_{\underline{x}}$$

At this point, we have:

$$\underline{x}_{k+1} + \delta\underline{x}_{k+1} = \underline{f}(\underline{x}_k, \underline{u}_k, k) + F_k\delta\underline{x}_k + G_k\delta\underline{u}_k$$

Finally, by cancelling out the original state equation (84), there results a linear system which approximates the behavior of the perturbation.

$$\delta\underline{x}_{k+1} = F_k\delta\underline{x}_k + G_k\delta\underline{u}_k$$

All of the solution techniques for linear systems can now be applied to determine the behavior of this perturbation.

4.7 Stochastic Error Propagation for the Discrete-Time Nonlinear System

Discrete time systems can be modeled deterministically as:

$$x_k = \Phi_{k-1}x_{k-1} + \Gamma_{k-1}u_{k-1} + \Lambda_{k-1}w_{k-1}$$

The corresponding covariance propagation equation when w is a white noise sequence is:

$$P_k = \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + \Lambda_{k-1}Q_{k-1}\Lambda_{k-1}^T$$

where:

$$P_k = E[x_k x_k^T] \qquad\qquad Q_k = E[w_k w_k^T]$$

5. Approximate Error Dynamics in Odometry Processes

Systems that have “dynamics” are described by differential equations. It is fundamental that such systems behave in a manner that is described by the solutions to those equations - by integrals. As a result, such systems are also said to have “memory” because the state at any given time depends on the entire time history of inputs presented to it. Similarly, we have seen that errors influence such systems in a manner that is also described by a differential equation and so errors themselves possess dynamics.

Odometry is a process for which errors behave in this manner. Error propagates in such a way that the current error depends not only on the current input errors, but also on the entire time history of errors since the point at which the state was last known directly (the initial conditions). Once an error is injected into the system, its effects may be felt forever unless some other future error happens to cancel its effects. This section determines the linearized error dynamics of odometry for the most general case.

5.1 Odometry Error Propagation

We can use perturbation theory to model the propagation of systematic error in linear systems and even in nonlinear systems such as odometry. Unlike the exact solution discussed in the previous section, however, perturbative techniques provide only first order behavior. So long as the errors remain “small” however, the linearized solution is an excellent one.

In order to reduce the complexity of expressions, the following notation will be used throughout:

$$\begin{aligned} \Delta x(t, \tau) &= [x(t) - x(\tau)] \\ \Delta y(t, \tau) &= [y(t) - y(\tau)] \end{aligned} \tag{85}$$

The process to generation of solutions is as follows. First, the nonlinear state equations are linearized to produce the relevant Jacobians to appear in both the linear perturbation equation and the linear variance equation. The differential equations are repeated here for reference:

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), \underline{u}(t), \underline{w}(t), t) \qquad \text{Nonlinear Process Dynamics}$$

$$\underline{z}(t) = \underline{h}(\underline{x}(t), \underline{u}(t), \underline{v}(t), t) \qquad \text{Nonlinear Observer}$$

$$\delta\dot{\underline{x}}(t) = F(t)\delta\underline{x}(t) + G(t)\delta\underline{u}(t) + L(t)\delta\underline{w}(t) \qquad \text{Linearized Process Dynamics}$$

$$\delta\underline{z}(t) = H(t)\delta\underline{x}(t) + M(t)\delta\underline{u}(t) + N(t)\delta\underline{v}(t) \qquad \text{Linearized Observer}$$

The relevant Jacobians are

$$F(t) = \left.\frac{\partial}{\partial \underline{x}}\underline{f}\right|_{\underline{x}} \qquad G(t) = L(t) = \left.\frac{\partial}{\partial \underline{u}}\underline{f}\right|_{\underline{x}}$$

$$H(t) = \left.\frac{\partial}{\partial \underline{x}}\underline{h}\right|_{\underline{x}} \qquad M(t) = N(t) = \left.\frac{\partial}{\partial \underline{w}}\underline{h}\right|_{\underline{x}}$$

The process Jacobian $F(t)$ is used to derive the transition matrix for the system. Then, using this and the Jacobians, the solution equations are the vector and matrix convolution integrals:

$$\underline{x}(t) = \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)\underline{u}(\tau)d\tau$$

$$P(t) = \Phi(t, t_0)P(t_0)\Phi^T(t, t_0) + \int_{t_0}^{t} \Phi(t, \tau)L(\tau)Q(\tau)L^T(\tau)\Phi^T(t, \tau)d\tau$$

In all cases considered, the echelon form of the equations means that they can be solved in closed form by back substitution. However, the process is tedious and not very illuminating and it does not eliminate all self-references as does a transition matrix solution, so we will pursue a transition matrix solution.

5.1.1 Direct Heading Odometry

We will consider first the direct heading case of equation (14).:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} V(t)\,cos\,\theta(t) \\ V(t)\,sin\,\theta(t) \end{bmatrix}$$

The Jacobians are[1]:

$$F(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad G(t) = L(t) = \begin{bmatrix} c\theta(t) & -V(t)s\theta(t) \\ s\theta(t) & V(t)s\theta(t) \end{bmatrix} \qquad \textbf{(86)}$$

The transition matrix is trivial for this memoryless case because the system Jacobian $F(\zeta)$ vanishes:

$$\Phi(t, \tau) = exp\left(\int_{\tau}^{t} F(\zeta)d\zeta\right) = exp[0] = I$$

1. Throughout the rest of the document we will at times use the notation $c\theta = cos\theta \quad s\theta = sin\theta$ and explicit expression of dependence on time will at times be suppressed to reduce clutter.

So the matrix form of the general solution is:

$$\delta \underline{x}(t) = \delta \underline{x}(0) + \int_0^t \begin{bmatrix} c\theta & -Vs\theta \\ s\theta & Vc\theta \end{bmatrix} \begin{bmatrix} \delta V(t) \\ \delta\theta(t) \end{bmatrix} d\tau$$

$$P(t) = P(0) + \int_{t_0}^t \begin{bmatrix} c\theta & -Vs\theta \\ s\theta & Vc\theta \end{bmatrix} \begin{bmatrix} \sigma_{vv} & \sigma_{v\theta} \\ \sigma_{v\theta} & \sigma_{\theta\theta} \end{bmatrix} \begin{bmatrix} c\theta & -Vs\theta \\ s\theta & Vc\theta \end{bmatrix}^T d\tau \tag{87}$$

Here, because the system has a vanishing system Jacobian, we could have written this solution by inspection. Separating contributions from various error sources[1] leads to:

$$\delta \underline{x}(t) = \delta \underline{x}(0) + \int_0^t \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} \delta V d\tau + \int_0^t V \begin{bmatrix} -s\theta \\ c\theta \end{bmatrix} \delta\theta d\tau$$

$$P(t) = P(0) + \int_0^t \begin{bmatrix} c^2\theta & c\theta s\theta \\ c\theta s\theta & s^2\theta \end{bmatrix} \sigma_{vv} d\tau + \int_0^t \begin{bmatrix} -2c\theta s\theta & 1 \\ 1 & 2c\theta s\theta \end{bmatrix} V\sigma_{v\theta} d\tau$$

$$+ \int_0^t \begin{bmatrix} s^2\theta & -c\theta s\theta \\ -c\theta s\theta & c^2\theta \end{bmatrix} V^2 \sigma_{\theta\theta} d\tau \tag{88}$$

Substituting the original differential equations into this allows us to make the substitutions:

$$dx = V(\tau) cos\theta(\tau) d\tau$$
$$dy = V(\tau) sin\theta(\tau) d\tau$$

which convert the second integral in the first equation to a line integral. Both forms are the general (linearized) solution for the propagation of systematic and random error in 2D direct heading odometry for any trajectory and any error model.

1. A straightforward mechanism exists to do this separation based on the columns of the "input transition matrix". See the journal paper [7] for details.

5.1.2 Integrated Heading Odometry

We will consider next the integrated heading case of equation (15):

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\mathit{cos}\theta(t) \\ V(t)\mathit{sin}\theta(t) \\ \omega(t) \end{bmatrix}$$

The Jacobians are:

$$F(t) = \begin{bmatrix} 0 & 0 & -V(t)s\theta(t) \\ 0 & 0 & V(t)c\theta(t) \\ 0 & 0 & 0 \end{bmatrix} \qquad G(t) = L(t) = \begin{bmatrix} c\theta(t) & 0 \\ s\theta(t) & 0 \\ 0 & 1 \end{bmatrix} \tag{89}$$

This F matrix is of the form of equation (56) and it therefore has two extremely important properties. First, its second power vanishes. Second, it satisfies the commutative dynamics condition.

The dynamics integral matrix is:

$$R(t,\tau) = \int_{\tau}^{t} F(\zeta)d\zeta = \begin{bmatrix} 0 & 0 & -\Delta y(t,\tau) \\ 0 & 0 & \Delta x(t,\tau) \\ 0 & 0 & 0 \end{bmatrix}$$

Next, the transition matrix is:

$$\Phi(t,\tau) = \mathit{exp}[R(t,\tau)] = I + R(t,\tau) = \begin{bmatrix} 1 & 0 & -\Delta y(t,\tau) \\ 0 & 1 & \Delta x(t,\tau) \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix $\Phi(t,\tau)$ is just the Jacobian of the transformation relating changes in the pose $[x(t), y(t), \theta(t)]$ to errors $[\delta x(\tau), \delta y(\tau), \delta\theta(\tau)]$ occurring at the pose $[x(\tau), y(\tau), \theta(\tau)]$.

Therefore, we have the product:

$$\Phi(t,\tau)G(t) = \begin{bmatrix} 1 & 0 & -\Delta y(t,\tau) \\ 0 & 1 & \Delta x(t,\tau) \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c\theta(t) & 0 \\ s\theta(t) & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix}$$

Henceforth we will use the following notation to save space:

$$\underline{IC}_d = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix} \delta \underline{x}(0) \qquad IC_s = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx}(0) & \sigma_{xy}(0) & \sigma_{x\theta}(0) \\ \sigma_{xy}(0) & \sigma_{yy}(0) & \sigma_{y\theta}(0) \\ \sigma_{x\theta}(0) & \sigma_{y\theta}(0) & \sigma_{\theta\theta}(0) \end{bmatrix} \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$\underline{IC}_s = \begin{bmatrix} 1 & 0 & 0 & -2y(t) & 0 & y(t)^2 \\ 0 & 1 & 0 & 0 & 2x(t) & x(t)^2 \\ 0 & 0 & 1 & x(t) & -y(t) & -x(t)y(t) \\ 0 & 0 & 0 & 1 & 0 & -y(t) \\ 0 & 0 & 0 & 0 & 1 & x(t) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx}(0) \\ \sigma_{yy}(0) \\ \sigma_{xy}(0) \\ \sigma_{x\theta}(0) \\ \sigma_{y\theta}(0) \\ \sigma_{\theta\theta}(0) \end{bmatrix}$$

The matrix form of the general solution is:

$$\delta \underline{x}(t) = \underline{IC}_d + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta V(\tau) \\ \delta \omega(\tau) \end{bmatrix} d\tau$$

$$P(t) = IC_s + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix} \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix}^T d\tau \qquad \textbf{(90)}$$

Separating contributions from various error sources leads to:

$$\delta \underline{x}(t) = \underline{IC}_d + \int_0^t \begin{bmatrix} c\theta \\ s\theta \\ 0 \end{bmatrix} \delta V d\tau + \int_0^t \begin{bmatrix} -\Delta y(t,\tau) \\ \Delta x(t,\tau) \\ 1 \end{bmatrix} \delta \omega d\tau$$

$$P(t) = IC_s + \int_0^t \begin{bmatrix} -2c\theta\Delta y & c\theta\Delta x - s\theta\Delta y & c\theta \\ c\theta\Delta x - s\theta\Delta y & 2s\theta\Delta x & s\theta \\ c\theta & s\theta & 0 \end{bmatrix} \sigma_{v\omega} d\tau \qquad \textbf{(91)}$$

$$+ \int_0^t \begin{bmatrix} c^2\theta & c\theta s\theta & 0 \\ c\theta s\theta & s^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \sigma_{vv} d\tau + \int_0^t \begin{bmatrix} \Delta y^2 & -\Delta x\Delta y & -\Delta y \\ -\Delta x\Delta y & \Delta x^2 & \Delta x \\ -\Delta y & \Delta x & 1 \end{bmatrix} \sigma_{\omega\omega} d\tau$$

All three forms are the general (linearized) solution for the propagation of systematic and random error in 2D integrated heading odometry for any trajectory and any error model.

5.1.3 Differential Heading Odometry

We will consider next the integrated heading case. There are several potential routes to a solution. The most tedious is to define a new input vector and associated new F and G matrices, and recompute the result from scratch. Next, we could use the states of the integrated heading case and solve for an equivalent set of integrated inputs. This last case is an ad hoc approach to using an observer, so instead this case will be used to illustrate the use of the observer formulation.

The system dynamics are therefore the same as the integrated heading case given in equation (15):

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\mathit{cos}\theta(t) \\ V(t)\mathit{sin}\theta(t) \\ \omega(t) \end{bmatrix}$$

The observer is of the form of equation (16):

$$\begin{bmatrix} V_r(t) \\ V_l(t) \end{bmatrix} = \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} \tag{92}$$

The relevant Jacobians are:

$$F(t) = \begin{bmatrix} 0 & 0 & -V(t)s\theta(t) \\ 0 & 0 & V(t)c\theta(t) \\ 0 & 0 & 0 \end{bmatrix} \qquad G(t) = L(t) = \begin{bmatrix} c\theta(t) & 0 \\ s\theta(t) & 0 \\ 0 & 1 \end{bmatrix}$$

$$H(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M(t) = N(t) = \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \tag{93}$$

The left pseudoinverse (LI) reduces to the inverse in this case of a square M matrix:

$$M(t)^{LI} = M(t)^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/W & -1/W \end{bmatrix}$$

Referring to equation (65), the F matrix is unchanged whereas the new G matrix is:

$$\tilde{G}(t) = G(t)M^{LI}$$

Using this allows us to treat the measurement vector $\delta\underline{z}(t)$ as the input.

Similarly, following equation (81), we can use the measurement covariance R instead of the process noise covariance Q if we substitute the following for L:

$$\tilde{L}(t) = LN^{-1} = \tilde{G}(t)$$

These new matrices are:

$$\tilde{G}(t) = \tilde{L}(t) = \begin{bmatrix} c\theta & 0 \\ s\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/W & -1/W \end{bmatrix} = \begin{bmatrix} c\theta/2 & c\theta/2 \\ s\theta/2 & s\theta/2 \\ 1/W & -1/W \end{bmatrix}$$

Therefore we have the product:

$$\Phi(t, \tau)\tilde{G}(t) = \begin{bmatrix} 1 & 0 & -\Delta y(t, \tau) \\ 0 & 1 & \Delta x(t, \tau) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 \\ s\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/W & -1/W \end{bmatrix} = \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix}$$

The matrix form of the general solution is:

$$\delta\underline{x}(t) = \underline{IC}_d + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \delta V_r \\ \delta V_l \end{bmatrix} d\tau \quad \textbf{(94)}$$

$$P(t) = IC_s + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \sigma_{rr} & \sigma_{rl} \\ \sigma_{rl} & \sigma_{ll} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix}^T \begin{bmatrix} c\theta & -\Delta y(t, \tau) \\ s\theta & \Delta x(t, \tau) \\ 0 & 1 \end{bmatrix}^T d\tau$$

Comparing this with equation (90), we clearly have the equivalent integrated heading errors:

$$\begin{bmatrix} \delta V \\ \delta\omega \end{bmatrix}\Bigg|_{equiv} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \delta V_r \\ \delta V_l \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix}\Bigg|_{equiv} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \sigma_{rr} & \sigma_{rl} \\ \sigma_{rl} & \sigma_{ll} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix}^T \quad \textbf{(95)}$$

This leads to:

$$\delta\underline{x}(t) = \underline{IC}_d + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta V \\ \delta\omega \end{bmatrix}\Bigg|_{equiv} d\tau \tag{96}$$

$$P(t) = IC_s + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix}\Bigg|_{equiv} \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix}^T d\tau$$

It was shown earlier in Equations (42) and (45) that these equivalent errors can, in turn, be defined in terms of equivalent error parameters.

For systematic errors:

$$\delta V(t) = \delta V_v V(t) \qquad \delta\omega(t) = \delta\omega_v V(t)$$

$$\delta V_v\big|_{equiv} = \left(\frac{\delta r_r + \delta l_l}{2}\right) + \frac{(\delta r_r - \delta l_l)}{4R(t)} W$$

$$\delta\omega\big|_{equiv} = \delta\omega_v V(t) = \left\{\frac{(\delta r_r - \delta l_l)}{W} + \left(\frac{\delta r_r + \delta l_l}{2R(t)}\right)\right\} V(t) \tag{97}$$

For stochastic errors:

$$\begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix}\Bigg|_{DH} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} |r(t)| \\ |l(t)| \end{bmatrix} = \begin{bmatrix} \frac{|r(t)| + |l(t)|}{2} \\ \frac{|r(t)| - |l(t)|}{W} \end{bmatrix}$$

$$\sigma_{vv}\big|_{equiv} = \sigma_{vv}^{(v)} V(t)\big|_{DH} = \left\{\frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{4} \frac{W}{2R(t)\big|_{DH}}\right\} V(t)\big|_{DH} \tag{98}$$

$$\sigma_{v\omega}\big|_{equiv} = \sigma_{v\omega}^{(v)} V(t)\big|_{DH} = \left\{\frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{2W} + \frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{4R(t)\big|_{DH}}\right\} V(t)\big|_{DH}$$

$$\sigma_{\omega\omega}\big|_{equiv} = \sigma_{\omega\omega}^{(v)} V(t)\big|_{DH} = \left\{\frac{(\sigma_{rr}^{(r)} + \sigma_{ll}^{(l)})}{W^2} + \frac{(\sigma_{rr}^{(r)} - \sigma_{ll}^{(l)})}{W^2} \frac{W}{2R(t)\big|_{DH}}\right\} V(t)\big|_{DH}$$

Separating contributions from various error sources leads to:

$$\delta \underline{x}(s) = \underline{IC}_d + \int_0^s \begin{bmatrix} c\theta \\ s\theta \\ 0 \end{bmatrix} \delta V_v(s) ds + \int_0^s \begin{bmatrix} -\Delta y \\ \Delta x \\ 1 \end{bmatrix} \delta \omega_v(s) ds$$

$$P(s) = IC_s + \int_0^s \begin{bmatrix} -2c\theta\Delta y & c\theta\Delta x - s\theta\Delta y & c\theta \\ c\theta\Delta x - s\theta\Delta y & 2s\theta\Delta x & s\theta \\ c\theta & s\theta & 0 \end{bmatrix} \sigma_{v\omega}^{(v)}(s) ds \qquad \textbf{(99)}$$

$$+ \int_0^s \begin{bmatrix} c^2\theta & c\theta s\theta & 0 \\ c\theta s\theta & s^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \sigma_{vv}^{(v)}(s) ds + \int_0^s \begin{bmatrix} \Delta y^2 & -\Delta x\Delta y & -\Delta y \\ -\Delta x\Delta y & \Delta x^2 & \Delta x \\ -\Delta y & \Delta x & 1 \end{bmatrix} \sigma_{\omega\omega}^{(v)}(s) ds$$

All forms above are the general (linearized) solution for the propagation of systematic and random error in 2D differential heading odometry for any trajectory and any error model. All terms in this solution are motion dependent as expected. The covariance integrals must be interpreted carefully to ensure that the error accumulation is consistent with unsigned wheel velocities.

5.2 Discussion

The integrated heading case will be used to illustrate some general conclusions. The solutions are repeated here for reference:

$$\delta \underline{x}(t) = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix} \delta \underline{x}(0) + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta V \\ \delta \omega \end{bmatrix} d\tau$$

$$P(t) = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx}(0) & \sigma_{xy}(0) & \sigma_{x\theta}(0) \\ \sigma_{xy}(0) & \sigma_{yy}(0) & \sigma_{y\theta}(0) \\ \sigma_{x\theta}(0) & \sigma_{y\theta}(0) & \sigma_{\theta\theta}(0) \end{bmatrix} \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix}^T + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix} \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix}^T d\tau$$

5.2.1 General Interpretation

Each solution consists of a state (initial conditions) response as well as an input response. The solutions are vectors and matrices consisting of scalars which are line integrals evaluated on the reference trajectory. These line integrals are also functionals which may or may not be integrable in closed form. In the event they are, path independent errors result. In the event they are not, path dependent errors result.

5.2.2 Path Independence

Recall that path independent terms must vanish upon return to the start point. The terms involving the initial conditions in all solutions are path independent. In the case of a large initial heading error, for example, all traces of arbitrarily large errors during excursions which are linear in excursion will be gone when returning to the start.

5.2.3 Path Dependence

For path dependent errors, their influence can be likened to moments of error evaluated over the trajectory. Integrals like:

$$\int_0^t [x(t) - x(\tau)]\delta\omega(\tau)d\tau$$

are equivalent to the first moment of rotation error evaluated on the path. Integrals like:

$$\int_0^t \delta V(\tau) cos\theta(\tau)d\tau$$

are equivalent to the first coefficient of the Fourier series of linear velocity error. Such moments (at least the first order ones) can potentially vanish on symmetric trajectories. The integrals in stochastic results are similar but of second order.

When the indicated errors are constant, the functionals become properties of the state trajectory itself. Such functionals can, of course be treated with the calculus of variations to compute trajectories for which error is maximized and minimized.

5.3 Derivation by Inspection

Now that the solution is written out, it is clear that it could have been written by inspection. The initial conditions affect the endpoint error in a predictable manner and the remaining terms amount to an addition of the effects felt at the endpoint at time t of the linear and angular errors occurring at each time τ between the start and end as illustrated below:

$$\delta\vec{r}_V = [cos\theta(\tau)\hat{i} + sin\theta(\tau)\hat{j}]\delta V(\tau)d\tau$$

$$\delta\vec{r}_\omega = [-\Delta y(t, \tau)\hat{i} + \Delta x(t, \tau)\hat{j}]\delta\omega(\tau)d\tau$$

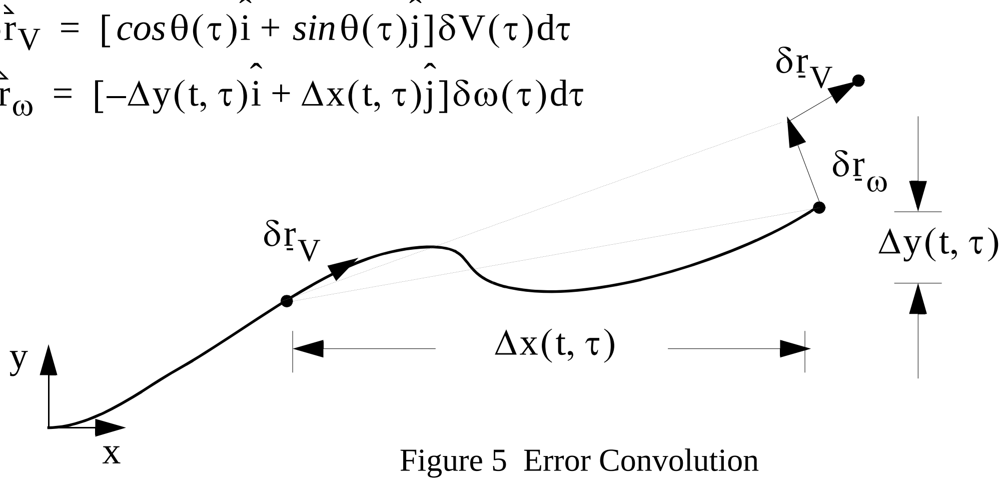


Figure 5 Error Convolution

Linear velocity errors are projected onto the x and y axes and shifted to the endpoint. Angular velocity errors are projected to the endpoint by multiplying by the component of radius perpendicular to the associated axis. The matrix relating input errors occurring at time τ to their later effect at time t is:

$$\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta\theta(t) \end{bmatrix} = \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta V(\tau) \\ \delta\omega(\tau) \end{bmatrix} d\tau$$

And this is exactly what equation (90) is integrating. The extension to covariance is also clear. Using this geometric formulation of error propagation, it is clear how it might be extended to 3D and higher order approximations beyond the linear.

6. Moments of Error

The main results of the last section include integrals which cannot be solved in closed form because there is no such thing as the general solution of an integral for an arbitrary integrand. These integrals are functionals evaluated on the reference trajectory and they are responsible for the path dependent components of error in odometry. They will be called **moments of error** because they are analogous to the moments of mechanics (moment of inertia), statistics (variance) and calculus (Fourier coefficients).

Just as they are used in these other fields, moments are convenient concepts that permit the suppression of explicit integrals in our results while notationally encapsulating integral properties of functions, curves, areas and volumes into an equivalent quantity. Just as the transition matrix was tantamount to a general solution to odometry error propagation, knowledge of these moments is tantamount to a specific solution for a given error model and a given trajectory.

6.1 General Moment of Error

The main moments are easy to isolate in equations (88), (91), and (99). All moments take the form:

$$U(t) = \int_0^t \Delta x^a(t,\tau)\Delta y^b(t,\tau)\,cos\theta(\tau)^c\,sin\theta(\tau)^d\,\delta u(\tau)d\tau \qquad \textbf{(100)}$$

where a, b, c, and d are integers and the error model $\delta u(\tau)$ may be deterministic or statistics of a probability distribution. As in earlier sections, the following notation is used:

$$\begin{aligned}\Delta x(t,\tau) &= [x(t) - x(\tau)]\\ \Delta y(t,\tau) &= [y(t) - y(\tau)]\end{aligned} \qquad \textbf{(101)}$$

The moments of mechanics are volume or area integrals of differential mass multiplied by distance coordinates. Error moments are path integrals of error multiplied by distance coordinates or trigonometric functions of the path.

They arise as elemental scalar equations of the vector and matrix convolution integrals which form the total solutions to deterministic and stochastic error propagation respectively. Each moment computes, for a differential error $\delta u(\tau)$ suffered along the trajectory, the contribution of that input error to the total state error $\delta x(t)$ resulting at the endpoint.

In the case of errors in heading, the expressions $\Delta x(t,\tau)$ and $\Delta y(t,\tau)$ compute the projections of the radius vector to the endpoint onto the x and y axes. In the case of errors in distance, the expressions $cos\theta(\tau)$ and $sin\theta(\tau)$ compute the projections of the differential translation error vector onto the x and y axes.

6.1.1 Order of a Moment

The order of the moment is defined as:

$$O[U(t)] = a + b + c + d$$

We will only need moments of the first and second order. First order moments apply to deterministic errors and second order moments apply to stochastic errors.

6.1.2 Moments of Duration, Excursion, and Rotation

When the error model $\delta u(\tau)$ takes simple forms, error moments become intrinsic properties of reference trajectories. In this case, they are also analogous to the integral transforms of calculus like the Laplace and Fourier transforms - and they can be tabulated for any trajectory.

When the error model is proportional to time, we have:

$$\delta u(\tau) d\tau = k \times d\tau$$

Ignoring the constant factor, the associated moment will be called a **duration moment** and takes the form:

$$T(t) = \int_0^t \Delta x^a(t, \tau) \Delta y^b(t, \tau) \cos\theta(\tau)^c \sin\theta(\tau)^d d\tau \quad \textbf{(102)}$$

When the error model is proportional to linear velocity $d\xi/d\tau$, we have:

$$\delta u(\tau) d\tau = k \times (d\xi/d\tau) \times d\tau = k \times d\xi$$

Ignoring the constant factor, the associated moment will be called an **excursion moment** and takes the form:

$$S(s) = \int_0^s \Delta x^a(s, \xi) \Delta y^b(s, \xi) \cos\theta(\xi)^c \sin\theta(\xi)^d d\xi \quad \textbf{(103)}$$

When the error model is proportional to angular velocity $d\zeta/d\tau$, we have:

$$\delta u(\tau) d\tau = k \times (d\zeta/d\tau) \times d\tau = k \times d\zeta$$

Ignoring the constant factor, the associated moment will be called a **rotation moment** and takes the form:

$$\Theta(\theta) = \int_0^\theta \Delta x^a(\theta, \zeta) \Delta y^b(\theta, \zeta) \cos\zeta^c \sin\zeta^d d\zeta \quad \textbf{(104)}$$

6.1.3 Moment Notation

Following standard notation for the moments of mechanics and statistics, we will write strings of potentially repeated subscripts to indicate the exponents a, b, c and d. For example, a first order excursion moment is:

$$S_x = \int_0^s [x(s) - x(\xi)]d\xi$$

Whereas two second order excursion moments are:

$$S_{yy} = \int_0^s [y(s) - y(\xi)]^2 d\xi \qquad S_{xc} = \int_0^s [x(s) - x(\xi)]\cos\theta(\xi)d\xi$$

The moment at the left is a principal second order moment because its subscripts are equal. Such moments have special properties. The one on the right is a cross moment.

6.1.4 Classes of Moments

We will differentiate three different classes of moment:

- **spatial moments**: depend only on the spatial coordinates $\Delta x(t, \tau)$ and $\Delta y(t, \tau)$.
- **Fourier moments**: depend only on the rotation coordinate through $\cos\theta(\tau)$ and $\sin\theta(\tau)$.
- **hybrid moments**: depend on both spatial and rotation coordinates.

Spatial moments are related to power series coefficients of the path whereas Fourier moments are actually equal to the coefficients of the discrete Fourier series of the error model.

6.1.5 Treatment of Dummy Variables

The dummy variables of integration τ, ξ and ζ in the spatial moments must be treated correctly. They are used to differentiate the variable of integration from the variable appearing in the limits of integration.

For example, the spatial moments incorporate the projections of the radius vector **from the endpoint** rather than from the initial position, and this means a second dummy variable is needed to distinguish the endpoint from the integration variable. The first spatial moment is evaluated as shown in the following example:

$$T_x(t) = \int_0^t \Delta x(t, \tau)d\tau = \int_0^t [x(t) - x(\tau)]d\tau = tx(t) - \int_0^t x(\tau)d\tau \tag{105}$$

Such moments can, as shown above, be converted to moments from the initial position by subtracting the total moment of the endpoint ($tx(t)$ in the example) from the initial point moment.

These relationships are also useful for implementing an efficient recursive numerical computation based on the analog of the statistical accumulators used in calculators. The relevent formulae up to second order are:

$$T_x(t) = \int_0^t \Delta x(t,\tau)d\tau = \int_0^t [x(t) - x(\tau)]d\tau = tx(t) - \int_0^t x(\tau)d\tau$$

$$T_y(t) = \int_0^t \Delta y(t,\tau)d\tau = \int_0^t [y(t) - y(\tau)]d\tau = ty(t) - \int_0^t y(\tau)d\tau$$

$$\Gamma_{xx}(t) = \int_0^t \Delta x^2(t,\tau)d\tau = \int_0^t [x^2(t) - 2x(t)x(\tau) + x^2(\tau)]d\tau = tx^2(t) - 2x(t)\int_0^t x(\tau)d\tau + \int_0^t x^2(\tau)d\tau$$

$$\Gamma_{yy}(t) = \int_0^t \Delta y^2(t,\tau)d\tau = \int_0^t [y^2(t) - 2y(t)y(\tau) + y^2(\tau)]d\tau = ty^2(t) - 2y(t)\int_0^t y(\tau)d\tau + \int_0^t y^2(\tau)d\tau$$

$$T_{xy}(t) = \int_0^t \Delta x(t,\tau)\Delta y(t,\tau)d\tau = \int_0^t [x(t)y(t) - x(t)y(\tau) - y(t)x(\tau) + x(\tau)y(\tau)]d\tau$$

$$T_{xy}(t) = tx(t)y(t) - x(t)\int_0^t y(\tau)d\tau - y(t)\int_0^t x(\tau)d\tau + \int_0^t x(\tau)y(\tau)d\tau$$

6.2 Moment Definitions

This section provides explicit definitions for all first and second order moments that we will require.

6.2.1 Spatial Moments of Trajectories

The spatial excursion, rotation, and duration moments are:

The second order rotation moments are plotted below for a constant curvature arc trajectory in

$$
\begin{aligned}
S_x &= \int_0^s \Delta x(s, \xi)d\xi & \Theta_x &= \int_0^\theta \Delta x(\theta, \zeta)d\zeta & T_x &= \int_0^t \Delta x(t, \tau)d\tau \\
S_y &= \int_0^s \Delta y(s, \xi)d\xi & \Theta_y &= \int_0^\theta \Delta y(\theta, \zeta)d\zeta & T_y &= \int_0^t \Delta y(t, \tau)d\tau \\
S_{xx} &= \int_0^s \Delta x(s, \xi)^2 d\xi & \Theta_{xx} &= \int_0^\theta \Delta x(\theta, \zeta)^2 d\zeta & T_{xx} &= \int_0^t \Delta x(t, \tau)^2 d\tau \\
S_{yy} &= \int_0^s \Delta y(s, \xi)^2 d\xi & \Theta_{yy} &= \int_0^\theta \Delta y(\theta, \zeta)^2 d\zeta & T_{yy} &= \int_0^t \Delta y(t, \tau)^2 d\tau \\
S_{xy} &= \int_0^s \Delta x(s, \xi)\Delta y(s, \xi)d\xi & \Theta_{xy} &= \int_0^\theta \Delta x(\theta, \zeta)\Delta y(\theta, \zeta)d\zeta & T_{xy} &= \int_0^t \Delta x(t, \tau)\Delta y(t, \tau)d\tau
\end{aligned}
\quad \textbf{(106)}
$$

order to illustrate the behavior of the spatial moments.

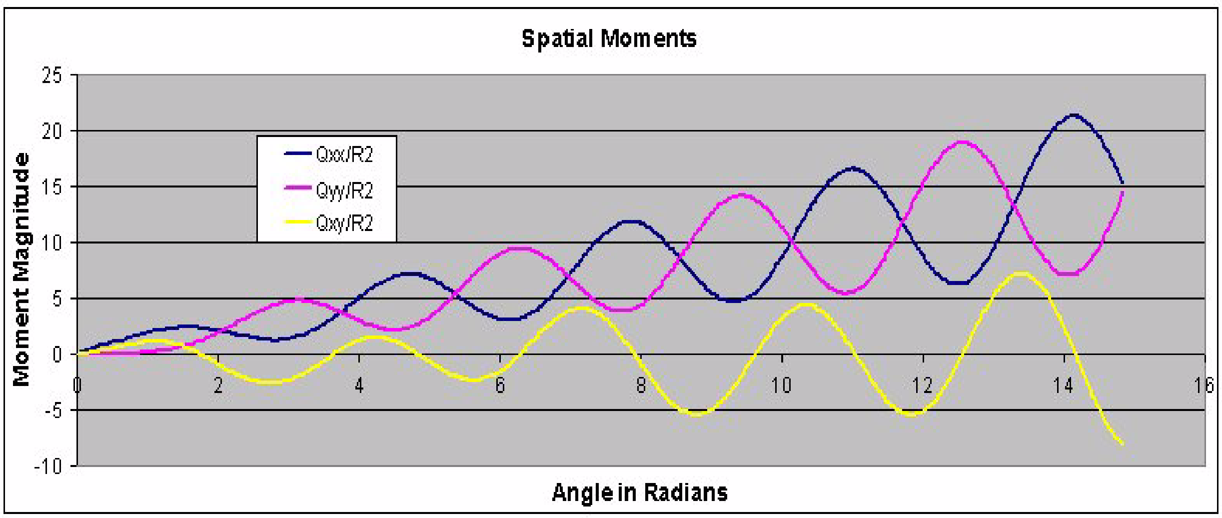


Figure 6 Spatial Moments

In the figure the letter Q is the plotter's rendition of the symbol Θ. Clearly, **these moments are not monotone** over a trajectory, so they can exhibit local extrema. The two principal moments seem to trade places in achieving local maxima and minima.

6.2.2 Fourier Moments of Trajectories

The Fourier excursion, rotation, and duration moments are:

$$S_c = \int_0^s cos\theta ds = x(s) \qquad \Theta_c = \int_0^\theta cos\theta d\theta \qquad T_c = \int_0^t cos\theta dt$$

$$S_s = \int_0^s sin\theta ds = y(s) \qquad \Theta_s = \int_0^\theta sin\theta d\theta \qquad T_s = \int_0^t sin\theta dt$$

$$S_{cc} = \int_0^s c^2\theta ds \qquad \Theta_{cc} = \int_0^\theta c^2\theta d\theta \qquad T_{cc} = \int_0^t c^2\theta dt \qquad \textbf{(107)}$$

$$S_{ss} = \int_0^s s^2\theta ds \qquad \Theta_{ss} = \int_0^\theta s^2\theta d\theta \qquad T_{ss} = \int_0^t s^2\theta dt$$

$$S_{sc} = \int_0^s s\theta c\theta ds \qquad \Theta_{sc} = \int_0^\theta s\theta c\theta d\theta \qquad T_{sc} = \int_0^t s\theta c\theta dt$$

The second order rotation moments are plotted below for a constant curvature arc trajectory in order to illustrate the behavior of the Fourier moments.

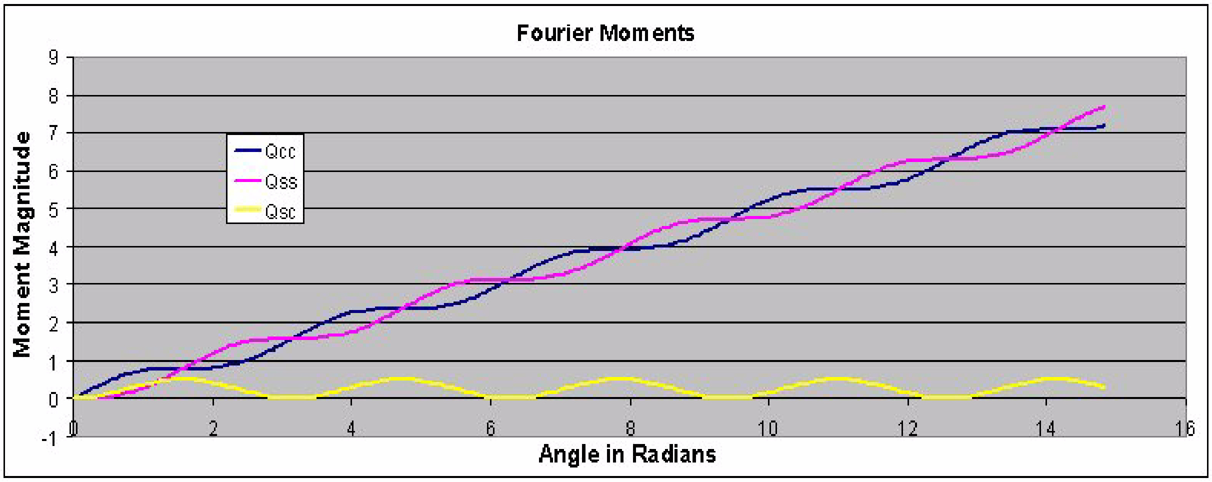


Figure 7 Fourier Moments

The principal moments here are monotone as would be expected because their integrands are always positive.

Certain double angle moments can also be defined for convenience, but they are not independent of the above moments:

$$S_{s2} = \int_0^s sin(2\theta) ds = 2\int_0^s s\theta c\theta ds = 2S_{cs}$$

$$\Theta_{s2} = \int_0^\theta sin(2\theta) d\theta = 2\int_0^\theta s\theta c\theta d\theta = 2\Theta_{cs}$$

$$T_{s2} = \int_0^t sin(2\theta) dt = 2\int_0^t s\theta c\theta dt = 2T_{cs}$$

$$S_{c2} = \int_0^s cos(2\theta) ds = \int_0^s [c^2\theta - s^2\theta] ds = S_{cc} - S_{ss}$$

$$\Theta_{c2} = \int_0^\theta cos(2\theta) d\theta = \int_0^\theta [c^2\theta - s^2\theta] d\theta = \Theta_{cc} - \Theta_{ss}$$

$$T_{c2} = \int_0^t cos(2\theta) dt = \int_0^t [c^2\theta - s^2\theta] dt = T_{cc} - T_{ss}$$

6.2.3 Hybrid Moments

The Hybrid excursion, rotation, and duration moments are:

$$S_{xc} = \int_0^s \Delta x(s,\xi) c\theta d\xi \qquad \Theta_{xc} = \int_0^\theta \Delta x(\theta,\zeta) c\theta d\theta \qquad T_{xc} = \int_0^t \Delta x(t,\tau) c\theta d\tau$$

$$S_{xs} = \int_0^s \Delta x(s,\xi) s\theta d\xi \qquad \Theta_{xs} = \int_0^\theta \Delta x(\theta,\zeta) s\theta d\theta \qquad T_{xs} = \int_0^t \Delta x(t,\tau) s\theta d\tau$$

$$S_{yc} = \int_0^s \Delta y(s,\xi) c\theta d\xi \qquad \Theta_{yc} = \int_0^\theta \Delta y(\theta,\zeta) c\theta d\theta \qquad T_{yc} = \int_0^t \Delta y(t,\tau) c\theta d\tau \qquad \textbf{(108)}$$

$$S_{ys} = \int_0^s \Delta y(s,\xi) s\theta d\xi \qquad \Theta_{ys} = \int_0^\theta \Delta y(\theta,\zeta) s\theta d\theta \qquad T_{ys} = \int_0^t \Delta y(t,\tau) s\theta d\tau$$

The second order rotation moments are plotted below for a constant curvature arc trajectory in order to illustrate the behavior of the Hybrid moments.

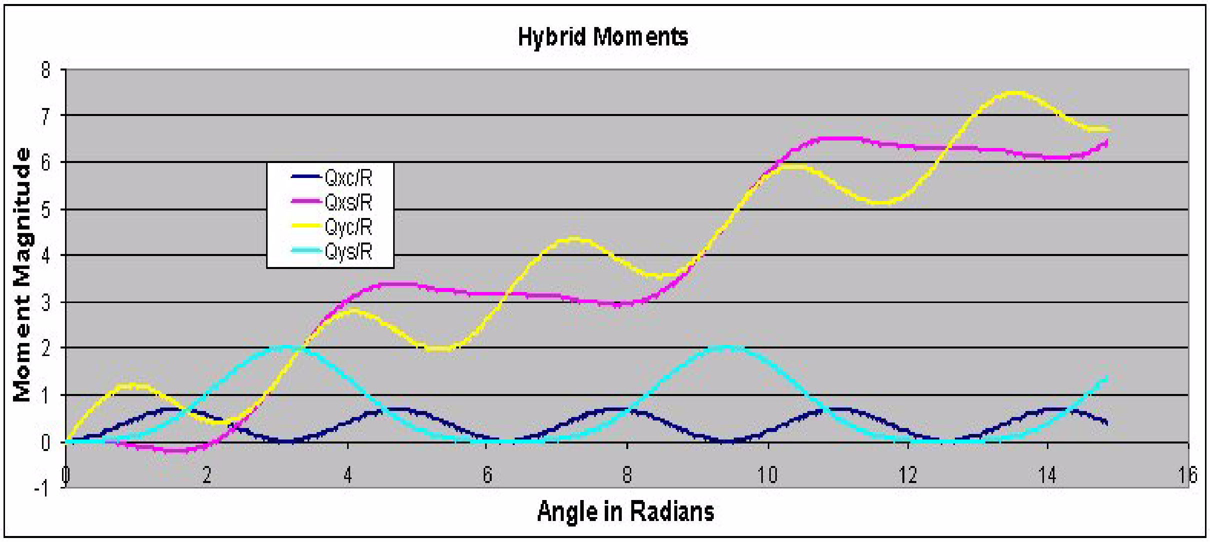


Figure 8 Hybrid Moments

None of these moments are monotone. They exhibit harmonic behavior resulting in distortions of the pure input frequency.

6.3 Interrelationships

We can expect error moments to be related to each other in all of the diverse ways that Fourier coefficients are related to each other. Some important relationships are outlined below. In order to save space, only the excursion moments will be covered but analogous relationships exist for the duration and rotation moments.

6.3.1 Time-Space-Angle Relationships

When linear or angular velocity is constant with respect to the integration variable, many obvious relationships exist between these moments. When curvature is constant, the following relationship holds for any rotation and excursion moment with equal subscripts:

$$\Theta_M = \kappa S_M \tag{109}$$

Similarly, when linear velocity is constant, the following relationship holds for any duration and excursion moment with equal subscripts:

$$T_M = \frac{1}{V} S_M \tag{110}$$

6.3.2 Rotation Moments

The rotation moments require no assumptions to integrate their trigonometric integrands so they are interrelated in diverse ways. For example:

$$\Theta_{cc} = \int_0^\theta c^2\theta d\theta = \int_0^\theta c\theta c\theta d\theta = c\theta s\theta + \int_0^\theta s\theta s\theta d\theta = c\theta s\theta + \Theta_{ss}$$

6.3.3 Special Trajectories

Also, when linear or angular velocity or curvature are constant with respect to the integration variable, many moments yield to integration by parts in a manner that is intrinsic to the rotation moments. For example, when curvature is constant

$$S_{cc} = \int_0^s c\theta c\theta d\xi = Rc\theta s\theta - \int_0^s c\theta c\theta d\xi = Rc\theta s\theta + S_{ss}$$

6.3.4 Double Angle Related to Fourier

A relationship between double angle Fourier moments and second order moments was mentioned earlier. It is repeated here:

$$S_{s2} = \int_0^s sin(2\theta)ds = 2\int_0^s s\theta c\theta ds = 2S_{cs}$$

$$S_{c2} = \int_0^s cos(2\theta)ds = \int_0^s [c^2\theta - s^2\theta]ds = S_{cc} - S_{ss}$$

6.3.5 Derivatives of First Order Spatial Moments

Consider the first order spatial excursion moments. These can be written:

$$S_x = \int_0^s \Delta x(s, \xi)d\xi = \int_0^s [x(s) - x(\xi)]d\xi = sx(s) - \int_0^s x(\xi)d\xi$$

$$S_y = \int_0^s \Delta y(s, \xi)d\xi = \int_0^s [y(s) - y(\xi)]d\xi = sy(s) - \int_0^s y(\xi)d\xi$$

Using Leibnitz rule[1] on the second terms, the first derivatives are:

$$\frac{dS_x}{ds} = x(s) + sx'(s) - x(s) = sx'(s) = s\,\mathit{cos}\theta$$
$$\frac{dS_y}{ds} = y(s) + sy'(s) - y(s) = sy'(s) = s\,\mathit{sin}\theta \tag{111}$$

In general then, the first moment gradient is the projection of distance travelled onto the axis associated with the moment. Increase occurs fastest when travelling parallel to that axis and critical points occur when travelling normal to the axis associated with the moment.

The second derivatives evaluated at critical points are:

$$\left.\frac{d^2 S_x}{ds^2}\right|_{\mathit{cos}\theta = 0} = \{\mathit{cos}\theta - \kappa s\,\mathit{sin}\theta\}\big|_{\mathit{cos}\theta = 0} = \pm\kappa s$$
$$\left.\frac{d^2 S_y}{ds^2}\right|_{\mathit{sin}\theta = 0} = \{\mathit{sin}\theta + \kappa s\,\mathit{cos}\theta\}\big|_{\mathit{sin}\theta = 0} = \pm\kappa s \tag{112}$$

Hence, the sign of the second derivative at critical points depends on the sign of the curvature.

6.3.6 Derivatives of First Order Spatial Fourier Moments

Consider now the first order spatial Fourier moments. The first derivatives are:

$$\frac{dS_c}{ds} = \frac{d}{ds}\int_0^s \mathit{cos}\theta ds = \mathit{cos}\theta$$
$$\frac{dS_s}{ds} = \frac{d}{ds}\int_0^s \mathit{sin}\theta ds = \mathit{sin}\theta \tag{113}$$

These behave identically to the spatial excursion moments in terms of directions where the first

1. Leibnitz rule gives how to differentiate an integral. A simple form needed here is:

$$\frac{d}{dx}\int_0^x f(u)du = f(x)$$

derivative vanishes. The second derivatives evaluated at critical points are:

$$\left.\frac{d^2 S_c}{ds^2}\right|_{cos\theta = 0} = -\kappa s\, sin\theta|_{cos\theta = 0} = \pm\kappa s$$
$$\left.\frac{d^2 S_s}{ds^2}\right|_{sin\theta = 0} = \kappa s\, cos\theta|_{sin\theta = 0} = \pm\kappa s \quad \textbf{(114)}$$

Again, the sign of the second derivative at critical points depends on the sign of the curvature.

6.3.7 Derivatives of Second Order Spatial Excursion Moments

Consider now the second order spatial excursion moments. These can be written:

$$S_{xx} = \int_0^s \Delta x(s,\xi)^2 d\xi = \int_0^s [x(s) - x(\xi)]^2 d\xi$$
$$S_{yy} = \int_0^s \Delta y(s,\xi)^2 d\xi = \int_0^s [y(s) - y(\xi)]^2 d\xi$$
$$S_{xy} = \int_0^s \Delta x(s,\xi)\Delta y(s,\xi) d\xi = \int_0^s [x(s) - x(\xi)][y(s) - y(\xi)] d\xi$$

Leibnitz rule[1] applied here gives the first derivatives as:

$$\frac{dS_{xx}}{ds} = \int_0^s 2[x(s) - x(\xi)]x'(s) d\xi = 2cos\theta \int_0^s [x(s) - x(\xi)] d\xi = 2cos\theta S_x(s)$$
$$\frac{dS_{yy}}{ds} = \int_0^s 2[y(s) - y(\xi)]y'(s) d\xi = 2sin\theta \int_0^s [y(s) - y(\xi)] d\xi = 2sin\theta S_y(s) \quad \textbf{(115)}$$
$$\frac{dS_{xy}}{ds} = \int_0^s [x(s) - x(\xi)]y'(s) + [y(s) - y(\xi)]x'(s) d\xi = sin\theta S_x(s) + cos\theta S_y(s)$$

So the first derivatives of second order spatial excursion moments are related to the first order spatial excursion moments. Critical points occur when travelling normal to the associated axes or when the associated first moment vanishes.

1. A more complicated version of the rule, applicable here, is:

$$\frac{d}{ds}\int_0^s f(s,\xi) d\xi = \int_0^s \frac{\partial}{\partial s} f(s,\xi) d\xi + f(s,s)$$

Lets consider just the principal moments S_{xx} and S_{yy}. Consider the case where the first moment vanishes. The second derivatives of the principal moments evaluated at these critical points are:

$$\left.\frac{d^2S_{xx}}{ds^2}\right|_{S_x = 0} = \left.2\frac{d}{ds}cos\theta S_x(s)\right|_{S_x = 0} = \left.2[-\kappa sin\theta S_x(s) + s cos\theta^2]\right|_{S_x = 0} = s cos\theta^2$$

$$\left.\frac{d^2S_{yy}}{ds^2}\right|_{S_y = 0} = \left.2\frac{d}{ds}sin\theta S_y(s)\right|_{S_y = 0} = \left.2[\kappa cos\theta S_y(s) + s sin\theta^2]\right|_{S_y = 0} = s sin\theta^2 \qquad \textbf{(116)}$$

Both of these are nonnegative. When one is increasing most rapidly, the other is zero. This result says that the moments achieve maxima at points where the first moment vanishes.

Now lets consider just the case when travelling normal to the associated axis. The second derivatives of the principal moments evaluated at these critical points are:

$$\left.\frac{d^2S_{xx}}{ds^2}\right|_{cos\theta = 0} = \left.2\frac{d}{ds}cos\theta S_x(s)\right|_{cos\theta = 0} = \left.2[-\kappa sin\theta S_x(s) + s cos\theta^2]\right|_{cos\theta = 0} = \pm 2\kappa S_x(s)$$

$$\left.\frac{d^2S_{yy}}{ds^2}\right|_{sin\theta = 0} = \left.2\frac{d}{ds}sin\theta S_y(s)\right|_{sin\theta = 0} = \left.2[\kappa cos\theta S_y(s) + s sin\theta^2]\right|_{sin\theta = 0} = \pm 2\kappa S_y(s) \qquad \textbf{(117)}$$

This result says that either maxima or minima may be achieved when travelling normal to the associated axis, at points where the first moments do not vanish, provided curvature is not zero. In other words, the second moments are always critical at a local extremity of the associated axis.

6.3.8 Derivatives of Second Order Spatial Fourier Moments

The second order spatial Fourier moments again are:

$$S_{cc} = \int_0^s c^2\theta ds \qquad S_{ss} = \int_0^s s^2\theta ds \qquad S_{sc} = \int_0^s s\theta c\theta ds \qquad \textbf{(118)}$$

The first derivatives are:

$$\frac{dS_{cc}}{ds} = \frac{d}{ds}\int_0^s c^2\theta ds = c^2\theta$$
$$\frac{dS_{ss}}{ds} = \frac{d}{ds}\int_0^s s^2\theta ds = s^2\theta \quad \textbf{(119)}$$
$$\frac{dS_{sc}}{ds} = \frac{d}{ds}\int_0^s c\theta s\theta ds = c\theta s\theta$$

Critical points occur when travelling normal to the associated axis.

The principal second derivatives evaluated at the critical points are:

$$\left.\frac{d^2S_{cc}}{ds^2}\right|_{cos\theta = 0} = \left.\frac{d}{ds}c^2\theta\right|_{cos\theta = 0} = \left.-2\kappa c\theta s\theta\right|_{cos\theta = 0} = 0$$
$$\left.\frac{d^2S_{ss}}{ds^2}\right|_{sin\theta = 0} = \left.\frac{d}{ds}s^2\theta\right|_{sin\theta = 0} = \left.2\kappa s\theta c\theta\right|_{sin\theta = 0} = 0 \quad \textbf{(120)}$$

so the critical points may be saddle points as earlier graphs have suggested. The third derivatives evaluated at the critical points are:

$$\left.\frac{d^3S_{cc}}{ds^3}\right|_{cos\theta = 0} = \left.-\frac{d}{ds}2\kappa c\theta s\theta\right|_{cos\theta = 0} = \left.-2[\kappa'(s)c\theta s\theta + \kappa(c^2\theta - s^2\theta)]\right|_{cos\theta = 0} = 2\kappa s^2\theta$$
$$\textbf{(121)}$$
$$\left.\frac{d^3S_{ss}}{ds^3}\right|_{sin\theta = 0} = \left.\frac{d}{ds}2\kappa c\theta s\theta\right|_{sin\theta = 0} = \left.2[\kappa'(s)c\theta s\theta + \kappa(c^2\theta - s^2\theta)]\right|_{sin\theta = 0} = 2\kappa c^2\theta$$

These are always positive on non-straight trajectories, so they are indeed saddle points.

6.4 Properties of Trajectory Moments

The behavior of error moments determines the behavior of errors associated with them. This section will present some general conclusions about the behavior of errors on arbitrary, reversed, closed and symmetric trajectories.

6.4.1 Reversibility and Irreversibility

While time advances monotonically, the differentials ds and $d\theta$ get their signs from the signs of the associated linear and angular velocities because:

$$ds = Vdt \qquad\qquad d\theta = \omega dt$$

For first order moments, in situations where the moment integrand is an even function of distance or angle and the variable of integration reverses at the midpoint, the associated moment will vanish upon return to the start point.

For principal second order moments, the differentials are usually considered positive in both directions of motion in reflection of the fact that the associated sensor variances must remain positive. Hence, reversal for such moments is impossible.

6.4.2 Monotonicity

The principal second order Fourier moments have integrands that are always positive, so these moments are monotonic over intervals where the variable of integration does not change sign. As a result, any variances that are wholly dependent upon them never decrease. Covariances tend to depend on cross moments and these may vary in sign.

6.4.3 Zeros and Centroids

Moments will be said to have **zeros** at places where they have a value of zero. For the first spatial moments, define the spatial trajectory **centroids** as follows:

$$\bar{x}(s) = \int_0^s x(\xi)d\xi \Big/ \int_0^s d\xi = S_x / s \qquad\qquad \bar{y}(s) = \int_0^s y(\xi)d\xi \Big/ \int_0^s d\xi = S_y / s$$

Then, the first spatial moments can be written purely in terms of the path length and distance from the centroid:

$$S_x = \int_0^s \Delta x(s, \xi)d\xi = \int_0^s [x(s) - x(\xi)]d\xi = sx(s) - \int_0^s x(\xi)d\xi = s[x(s) - \bar{x}(s)]$$

$$S_y = \int_0^s \Delta y(s, \xi)d\xi = \int_0^s [y(s) - y(\xi)]d\xi = sy(s) - \int_0^s y(\xi)d\xi = s[y(s) - \bar{y}(s)]$$

We can conclude that all first spatial excursion moments have a zero point at the centroid of the coordinate signal when expressed as a function of distance. Clearly, centroids for the first rotation $\bar{x}(\theta)$ and duration $\bar{x}(t)$ spatial moments can also be defined. Some examples for a zero of the x coordinate moment occurring at the endpoint are shown below:

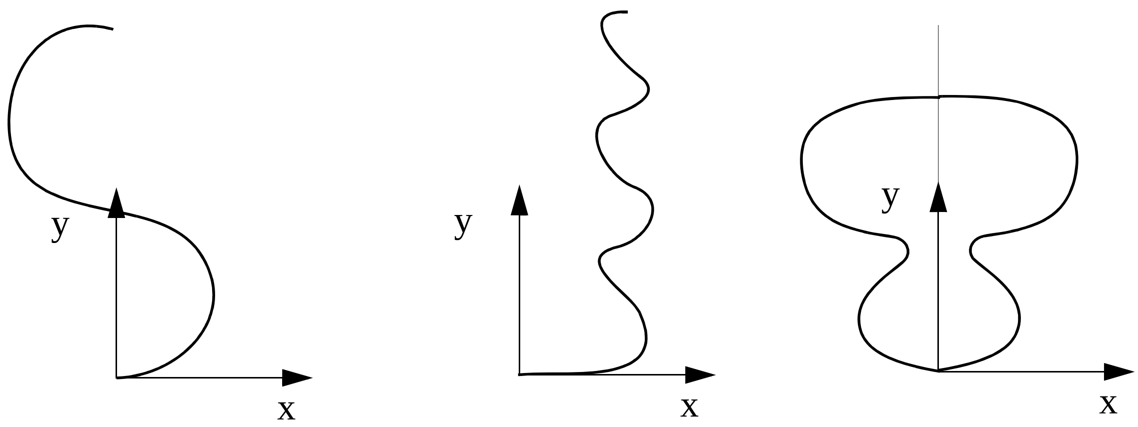


Figure 9 Some Cases for zeros of S_x

The first Fourier moment zeros are immediate because they are just the trajectory spatial coordinates themselves:

$$S_c = \int_0^s cos\theta ds = x(s) \qquad S_s = \int_0^s sin\theta ds = y(s)$$

These moments clearly have zeros on the coordinate axes and both vanish at the origin.

6.4.4 Behavior on Symmetric Trajectories

Appropriate symmetry is a sufficient (not a necessary) condition for the origin to be a zero of the first spatial moments. As a result, systematic error components which can be reduced to such moments will vanish on any path which is symmetric about the origin. The second order cross moments of all classes will also vanish on closed symmetric paths.

6.4.5 Local Extrema

Critical points of the principal second spatial moments S_{xx} and S_{yy}, occur at local extremes of the trajectory in the direction of the associated coordinate axes and at zeros of the associated first moments. As a result, random error (in one direction) that is wholly dependent on such moments can be expected to exhibit such local extrema.

These conditions are satisfied in particular at the point of closure of any path which is tangent to an axis and at the point of closure of a symmetric path regardless of axis tangency.

The principal second Fourier moments S_{cc} and S_{ss} do not exhibit extrema but the hybrid moments S_{xc}, S_{yc}, S_{xs} and S_{ys} and the cross moments S_{xy} and S_{sc} do.

6.4.6 Conservation

In the next section, we will see that many corresponding pairs of moments have similar terms of opposite signs - at least on constant curvature trajectories. This is a hint that combinations have simple forms. For example:

$$\Theta_{cc} + \Theta_{ss} = \int_0^{\theta} [cos\theta^2 + sin\theta^2] d\theta = \theta$$

So, these moments tend to trade value back and forth while conserving their sum so that it grows precisely linearly for any trajectory.

A more complicated example:

$$S_{xx} + S_{yy} = \int_0^{s} [\Delta x^2 + \Delta y^2] ds = \int_0^{s} [\Delta x^2 + \Delta y^2] ds$$

Another example:

$$\Theta_c^2 + \Theta_s^2 = \left(\int_0^{\theta} cos\theta d\theta \right)^2 + \left(\int_0^{\theta} sin\theta d\theta \right)^2 = 2(1 - cos\theta)$$

6.5 Moments of Straight Trajectories

This section tabulates the moments for the straight line trajectory developed in equation (18). The detailed derivations are provided in the Appendices. The trajectory is given by:

$$x(t) = s(t) \qquad y(t) = 0 \qquad \theta(t) = 0$$

$$x(s) = s \qquad y(s) = 0 \qquad \theta(s) = 0$$

$$x(\theta) = s(\theta) \qquad y(\theta) = 0 \qquad \theta(\theta) = 0$$

The following table gives expressions for the spatial moments

Table 4: Spatial Moments of Straight Trajectory

Moment	Excursion Moment S		Duration Moment T		Rotation Moment Θ	
M_x	$\frac{s^2}{2}$	$\frac{x^2}{2}$	$\frac{Vt^2}{2}$ (*)	$\mathbf{\frac{xt}{2}}$ (*)	0	0
M_y	0	0	0	0	0	0
M_{xx}	$\frac{s^3}{3}$	$\frac{x^3}{3}$	$\frac{V^2t^3}{6}$ (*)	$\frac{x^2t}{6}$ (*)	0	0
M_{yy}	0	0	0	0	0	0
M_{xy}	0	0	0	0	0	0
(*) means constant linear velocity assumed						

The following table gives expressions for the Fourier moments

Table 5: Fourier Moments of Straight Trajectory

Moment	Excursion Moment S		Duration Moment T		Rotation Moment Θ	
M_c	s	x	t (*)	$\frac{x}{V}$ (*)	0	0
M_s	0	0	0	0	0	0
M_{cc}	s	x	t (*)	$\frac{x}{V}$ (*)	0	0
M_{ss}	0	0	0	0	0	0
M_{sc}	0	0	0	0	0	0
(*) means constant linear velocity assumed						

The following table gives expressions for the Hybrid moments

Table 6: Hybrid Moments of Straight Trajectory

Moment	Excursion Moment S		Duration Moment T		Rotation Moment Θ	
M_{xc}	$\frac{s^2}{2}$	$\frac{x^2}{2}$	$\frac{st}{2}$ (*)	$\frac{x^2}{2V}$ (*)	0	0
M_{yc}	0	0	0	0	0	0
M_{xs}	0	0	0	0	0	0
M_{ys}	0	0	0	0	0	0
(*) means constant linear velocity assumed						

6.6 Moments of Arc Trajectories

This section tabulates the moments for the arc line trajectory developed in equation (19). The detailed derivations as well as a convenient table of trigonometric integrals is provided in the Appendices. The trajectory is given by:

$$\begin{aligned} \theta(s) &= \kappa s \\ x(s) &= R\,sin(\kappa s) \\ y(s) &= R[1 - cos(\kappa s)] \end{aligned} \qquad \kappa \cong \frac{1}{R}$$

The following table gives expressions for the spatial excursion moments

Table 7: Spatial Excursion Moments of Arc Trajectory

Moment	Excursion Moment S	
S_x	$R^2[\kappa s\, sin(\kappa s) + (cos(\kappa s) - 1)]$	$sx(s) - Ry(s)$
S_y	$R^2[sin(\kappa s) - \kappa s\, cos(\kappa s)]$	$Rx(s) + s[y(s) - R]$
S_{xx}	$\frac{R^2}{\kappa}\left[\kappa s\left(1 - \frac{cos2\kappa s}{2}\right) + \frac{3}{4}sin2\kappa s - 2sin(\kappa s)\right]$	$s\left(\frac{R^2}{2} + x^2\right) - Rx\left(\frac{R}{2} - \frac{3y}{2}\right)$
S_{yy}	$\frac{R^2}{\kappa}\left[\kappa s\left(1 + \frac{cos2\kappa s}{2}\right) - \frac{3}{4}sin2\kappa s\right]$	$s\left(\frac{R^2}{2} + (R-y)^2\right) - 3xR\left(\frac{R}{2} - \frac{y}{2}\right)$
S_{xy}	$\frac{R^2}{\kappa}\left[-\frac{\kappa s}{2}sin(2\kappa s) - \frac{3}{4}cos(2\kappa s) + cos(\kappa s) - \frac{1}{4}\right]$	$xs(y-R) + R\left(\frac{3}{2}x^2 - Ry\right)$
(*) means constant linear and angular velocity assumed		

The following table gives expressions for the spatial rotation moments

Table 8: Spatial Rotation Moments of Arc Trajectory

Moment	Rotation Moment Θ	
Θ_x	$R[\theta sin(\theta) + (cos(\theta) - 1)]$	$\theta x(\theta) - y(\theta)$
Θ_y	$R[sin(\theta) - \theta cos(\theta)]$	$x(\theta) + \theta[y(\theta) - R]$
Θ_{xx}	$R^2\left[\theta\left(1 - \frac{cos2\theta}{2}\right) + \frac{3}{4}sin2\theta - 2sin\theta\right]$	$\theta\left(\frac{R^2}{2} + x^2\right) - x\left(\frac{R}{2} - \frac{3y}{2}\right)$
Θ_{yy}	$R^2\left[\theta\left(1 + \frac{cos2\theta}{2}\right) - \frac{3}{4}sin2\theta\right]$	$\left[\theta\left(\frac{R^2}{2} + (R-y)^2\right) - 3x\left(\frac{R}{2} - \frac{y}{2}\right)\right]$

Table 8: Spatial Rotation Moments of Arc Trajectory

Moment	Rotation Moment Θ	
Θ_{xy}	$R^2\left[-\frac{\theta}{2}sin(2\theta) - \frac{3}{4}cos(2\theta) + cos(\theta) - \frac{1}{4}\right]$	$x\theta(y - R) + \left(\frac{3}{2}x^2 - Ry\right)$
(*) means constant linear and angular velocity assumed		

When linear and angular velocity are constant, the trajectory is:

$$\theta(t) = \omega t \qquad \theta(\theta) = \theta$$

$$x(t) = R\,sin(\omega t) \qquad \omega = \frac{V}{R} = \frac{1}{T} \qquad x(\theta) = R\,sin(\theta)$$

$$y(t) = R[1 - cos(\omega t)] \qquad y(\theta) = R[1 - cos(\theta)]$$

Under this assumption[1], the following table gives expressions for the spatial duration moments.

Table 9: Spatial Duration Moments of Arc Trajectory

Moment	Duration Moment T	
T_x	$\frac{R}{\omega}[\omega t\,sin(\omega t) + (cos(\omega t) - 1)]$ (*)	$tx(t) - Ty(t)$ (*)
T_y	$\frac{R}{\omega}[sin(\omega t) - \omega t\,cos(\omega t)]$ (*)	$Tx(t) + t[y(t) - R]$ (*)
T_{xx}	$\frac{R^2}{\omega}\left[\omega t\left(1 - \frac{cos 2\omega t}{2}\right) + \frac{3}{4}sin 2\omega t - 2 sin \omega t\right]$ (*)	$t\left(\frac{R^2}{2} + x^2\right) - Tx\left(\frac{R}{2} - \frac{3y}{2}\right)$ (*)
T_{yy}	$\frac{R^2}{\omega}\left[\omega t\left(1 + \frac{cos 2\omega t}{2}\right) - \frac{3}{4}sin 2\omega t\right]$ (*)	$t\left(\frac{R^2}{2} + (R - y)^2\right) - 3Tx\left(\frac{R}{2} - \frac{y}{2}\right)$ (*)
T_{xy}	$\frac{R^2}{\omega}\left[-\frac{\omega t}{2}sin(2\omega t) - \frac{3}{4}cos(2\omega t) + cos(\omega t) - \frac{1}{4}\right]$ (*)	$xt(y - R) + T\left(\frac{3}{2}x^2 - Ry\right)$ (*)
(*) means constant linear and angular velocity assumed		

The following table gives expressions for the Fourier moments

Table 10: Fourier Moments of Arc Trajectory

Moment	Excursion Moment S		Duration Moment T		Rotation Moment Θ	
M_c	$Rs\theta$	x	$Ts\theta$ (*)	$\frac{Tx}{R}$ (*)	$s\theta$	$\frac{x}{R}$
M_s	$R(1 - c\theta)$	y	$T(1 - c\theta)$ (*)	$\frac{Ty}{R}$ (*)	$(1 - c\theta)$	$\frac{y}{R}$
M_{cc}	$R\left[\frac{\theta}{2} + \frac{s2\theta}{4}\right]$	$\frac{s}{2} + \frac{x}{2}\left(1 - \frac{y}{R}\right)$	$T\left[\frac{\theta}{2} + \frac{s2\theta}{4}\right]$ (*)	$\frac{T}{R}\left[\frac{s}{2} + \frac{x}{2}\left(1 - \frac{y}{R}\right)\right]$ (*)	$\frac{\theta}{2} + \frac{s2\theta}{4}$	$\frac{\theta}{2} + \frac{x}{2R}\left(1 - \frac{y}{R}\right)$
M_{ss}	$R\left[\frac{\theta}{2} - \frac{s2\theta}{4}\right]$	$\frac{s}{2} - \frac{x}{2}\left(1 - \frac{y}{R}\right)$	$T\left[\frac{\theta}{2} - \frac{s2\theta}{4}\right]$ (*)	$\frac{T}{R}\left[\frac{s}{2} - \frac{x}{2}\left(1 - \frac{y}{R}\right)\right]$ (*)	$\frac{\theta}{2} - \frac{s2\theta}{4}$	$\frac{\theta}{2} - \frac{x}{2R}\left(1 - \frac{y}{R}\right)$
M_{sc}	$\frac{Rs^2\theta}{2}$	$\frac{x^2}{2R}$	$T\left(\frac{s^2\theta}{2}\right)$ (*)	$\frac{T}{R}\left(\frac{x^2}{2R}\right)$ (*)	$\frac{s^2\theta}{2}$	$\frac{x^2}{2R^2}$
(*) means constant linear and angular velocity assumed						

The following table gives expressions for the Hybrid Excursion moments

Table 11: Hybrid Excursion Moments of Arc Trajectory

Moment	Excursion Moment S	
M_{xc}	$\frac{1}{2}R^2 \mathit{sin}(\kappa s)^2$	$\frac{x^2}{2}$
M_{yc}	$R^2\left\{\frac{\kappa s}{2} - \frac{\mathit{sin}(2\kappa s)}{4}\right\}$	$\frac{x}{2}(R - y) + R\left(\frac{s}{2}\right)$

1. The situation with regard to the constant assumptions required to integrate sinusoids like $\mathit{cos}(\kappa s)$ and $\mathit{cos}(\omega t)$ is asymmetric. The assumption $\omega = \mathrm{const}$ (and hence $V = \mathrm{const}$) must be explicitly mentioned here while the assumption $\kappa = \mathrm{const}$ is built into the constant curvature trajectory so it does not require mention.

Table 11: Hybrid Excursion Moments of Arc Trajectory

Moment	Excursion Moment S	
M_{xs}	$R^2\left\{-\frac{\kappa s}{2} + sin(\kappa s) - \frac{sin(2\kappa s)}{4}\right\}$	$R\left(-\frac{s}{2} + x\right) - \frac{x}{2}(R - y)$
M_{ys}	$\frac{R^2}{2}\{cos(\kappa s) - 1\}^2$	$\frac{y^2}{2}$
(*) means constant linear and angular velocity assumed		

The following table gives expressions for the hybrid rotation moments

Table 12: Hybrid Rotation Moments of Arc Trajectory

Moment	Rotation Moment Θ	
M_{xc}	$\frac{1}{2}R^2 sin(\theta)^2$	$\frac{1}{2}\frac{x^2}{R}$
M_{yc}	$R^2\left\{\frac{\theta}{2} - \frac{sin(2\theta)}{4}\right\}$	$\frac{x}{2}(R - y) + R\left(\frac{s}{2}\right)$
M_{xs}	$R^2\left\{-\frac{\theta}{2} + sin(\theta) - \frac{sin(2\theta)}{4}\right\}$	$R\left(-\frac{s}{2} + x\right) - \frac{x}{2}(R - y)$
M_{ys}	$\frac{R}{2}\{cos(\kappa s) - 1\}^2$	$\frac{y^2}{2R}$
(*) means constant linear velocity assumed		

The following table gives expressions for the Hybrid Duration moments

Table 13: Hybrid Duration Moments of Arc Trajectory

Moment	Duration Moment T	
M_{xc}	$\frac{1}{2}TR\,sin(\omega t)^2\,(*)$	$\frac{x^2}{2V}\,(*)$
M_{yc}	$TR\left\{\frac{\omega t}{2} - \frac{sin(2\omega t)}{4}\right\}(*)$	$\frac{x}{2V}(R - y) + T\left(\frac{s}{2}\right)\,(*)$
M_{xs}	$TR\left\{-\frac{\omega t}{2} + sin(\omega t) - \frac{sin 2\omega t}{4}\right\}(*)$	$T\left(-\frac{s}{2} + x\right) - \frac{x}{2V}(R - y)\,(*)$
M_{ys}	$\frac{TR}{2}\{cos(\omega t) - 1\}^2\,(*)$	$\frac{y^2}{2V}\,(*)$
(*) means constant linear and angular velocity assumed		

7. Error Propagation For Specific Error Models

Once specific error models are chosen, error propagation in our earlier general solutions becomes dependent only on the reference trajectory. When the error models are constant or motion dependent, errors become expressible in terms of the trajectory moments of the last section. This section will re-express earlier results in terms of trajectory moments.

7.1 Direct Heading Odometry

Substituting the error models of equations (26) and (27) into equation (88) leads to:

$$\delta \underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \end{bmatrix} + \delta V_v \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + \begin{bmatrix} -\delta\theta_c S_{sc}(s) - \delta\theta_s S_{ss}(s) \\ \delta\theta_s S_{sc}(s) + \delta\theta_c S_{cc}(s) \end{bmatrix}$$

$$P(s) = P(0) + \sigma_{vv}^{(v)} \begin{bmatrix} S_{cc}(s) & S_{sc}(s) \\ S_{sc}(s) & S_{ss}(s) \end{bmatrix} + |V|\sigma_{\theta\theta} \begin{bmatrix} S_{ss}(s) & -S_{sc}(s) \\ -S_{sc}(s) & S_{cc}(s) \end{bmatrix} \quad \textbf{(122)}$$

Odometry error for direct heading case for an arbitrary trajectory with scale factor translation errors and sinusoidal compass error.

Clearly, since the first order moments are the endpoint coordinates, the second order Fourier moments tell us everything there is to know about error propagation in this case. Systematic error is reversible. The arc length s is signed in the systematic case and unsigned in the stochastic case.

7.2 Integrated Heading Odometry

Substituting the error models of equations (28) and (29) into equation (91) gives:

$$\delta \underline{x}(t) = \underline{IC}_d + \delta V_v \begin{bmatrix} S_c(s) \\ S_s(s) \\ 0 \end{bmatrix} + \delta\omega \begin{bmatrix} -T_y(t) \\ T_x(t) \\ t \end{bmatrix}$$

$$P(t) = IC_s + \sigma_{vv}^{(v)} \begin{bmatrix} S_{cc}(s) & S_{sc}(s) & 0 \\ S_{sc}(s) & S_{ss}(s) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_{\omega\omega} \begin{bmatrix} T_{yy}(t) & -T_{xy}(t) & -T_y(t) \\ -T_{xy}(t) & T_{xx}(t) & T_x(t) \\ -T_y(t) & T_x(t) & t \end{bmatrix} \quad \textbf{(123)}$$

Odometry error for integrated heading case for an arbitrary trajectory with scale factor translation errors and gyro bias.

In this case, first order moments determine systematic error whereas second order moments determine random error. Careful scrutiny of equation (91) indicates that all 3 second order Fourier moments $S_{cc}(t)$, $S_{ss}(t)$, and $S_{cs}(t)$ must be evaluated for unsigned distance.

7.3 Differential Heading Odometry

When the trajectory is of constant curvature, the trajectory moment matrices can be isolated and expressed in terms of the scalar moment notation. Substituting the error models of equations (42) and (45) into equation (99) gives:

$$\delta\underline{x}(s) = \underline{IC}_d + \delta V_v \begin{bmatrix} S_c(s) \\ S_s(s) \\ 0 \end{bmatrix} + \delta\omega_v \begin{bmatrix} -S_y(s) \\ S_x(s) \\ s \end{bmatrix}$$

$$P(t) = IC_s + \sigma_{v\omega}^{(v)} \begin{bmatrix} -2S_{yc}(s) & S_{xc}(s) - S_{ys}(s) & S_c(s) \\ S_{xc}(s) - S_{ys}(s) & 2S_{xs}(s) & S_s(s) \\ S_c(s) & S_s(s) & 0 \end{bmatrix} \tag{124}$$

$$+ \sigma_{vv}^{(v)} \begin{bmatrix} S_{cc}(s) & S_{sc}(s) & 0 \\ S_{sc}(s) & S_{ss}(s) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_{\omega\omega}^{(v)} \begin{bmatrix} S_{yy}(s) & -S_{xy}(s) & -S_y(s) \\ -S_{xy}(s) & S_{xx}(s) & S_x(s) \\ -S_y(s) & S_x(s) & s \end{bmatrix}$$

The second integral is subject to the velocity definitions of equation (98). The extra matrix associated with $\sigma_{v\omega}^{(v)}$ arises from the fact that the errors in linear and angular velocities are correlated due to their common basis on two underlying measurements of wheel velocities.

The alternate form is also available:

$$\delta\underline{x}(\theta) = \underline{IC}_d + \int_0^\theta \delta V_\omega(\theta) \begin{bmatrix} c\theta \\ s\theta \\ 0 \end{bmatrix} d\theta + \int_0^\theta \delta\omega_\omega(\theta) \begin{bmatrix} -\Delta y \\ \Delta x \\ 1 \end{bmatrix} d\theta$$

$$P(\theta) = IC_s + \int_0^\theta \begin{bmatrix} -2c\theta\Delta y & c\theta\Delta x - s\theta\Delta y & c\theta \\ c\theta\Delta x - s\theta\Delta y & 2s\theta\Delta x & s\theta \\ c\theta & s\theta & 0 \end{bmatrix} \sigma_{v\omega}^{(\omega)}(\theta) d\theta \tag{125}$$

$$+ \int_0^\theta \begin{bmatrix} c^2\theta & c\theta s\theta & 0 \\ c\theta s\theta & s^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \sigma_{vv}^{(\omega)}(\theta) d\theta + \int_0^\theta \begin{bmatrix} \Delta y^2 & -\Delta x\Delta y & -\Delta y \\ -\Delta x\Delta y & \Delta x^2 & \Delta x \\ -\Delta y & \Delta x & 1 \end{bmatrix} \sigma_{\omega\omega}^{(\omega)}(\theta) d\theta$$

When the trajectory is of constant curvature, the trajectory moment matrices can be isolated and expressed in terms of the scalar moment notation:

$$\delta\underline{x}(\theta) = \underline{IC}_d + \delta V_\omega \begin{bmatrix} \Theta_c(\theta) \\ \Theta_s(\theta) \\ 0 \end{bmatrix} + \delta\omega_\omega \begin{bmatrix} -\Theta_y(\theta) \\ \Theta_x(\theta) \\ \theta \end{bmatrix}$$

$$P(\theta) = IC_s + \sigma_{v\omega}^{(\omega)} \begin{bmatrix} -2\Theta_{yc}(\theta) & \Theta_{xc}(\theta) - \Theta_{ys}(\theta) & \Theta_c(\theta) \\ \Theta_{xc}(\theta) - \Theta_{ys}(\theta) & 2\Theta_{xs}(\theta) & \Theta_s(\theta) \\ \Theta_c(\theta) & \Theta_s(\theta) & 0 \end{bmatrix} \tag{126}$$

$$+ \sigma_{vv}^{(\omega)} \begin{bmatrix} \Theta_{cc}(\theta) & \Theta_{sc}(\theta) & 0 \\ \Theta_{sc}(\theta) & \Theta_{ss}(\theta) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_{\omega\omega}^{(\omega)} \begin{bmatrix} \Theta_{yy}(\theta) & -\Theta_{xy}(\theta) & -\Theta_y(\theta) \\ -\Theta_{xy}(\theta) & \Theta_{xx}(\theta) & \Theta_x(\theta) \\ -\Theta_y(\theta) & \Theta_x(\theta) & \theta \end{bmatrix}$$

8. Error Propagation On Specific Trajectories

We have seen that once the error model is specified, error propagation depends only on the initial conditions and the moments of error evaluated on the trajectory. For the error models we have chosen in our three examples, the error moments reduce to trajectory moments - intrinsic properties of the trajectory itself. In such cases, fixing the trajectory followed completely determines the error experienced along the trajectory. This section will derive the propagation of error on straight and constant curvature trajectories.

8.1 Straight Trajectory

Expressions for a straight line trajectory were given in equation (18):

$$x(t) = s(t) \qquad y(t) = 0 \qquad \theta(t) = 0$$

$$x(s) = s \qquad y(s) = 0 \qquad \theta(s) = 0$$

$$x(\theta) = s(\theta) \qquad y(\theta) = 0 \qquad \theta(\theta) = 0$$

8.1.1 Direct Heading Odometry

Substituting the trajectory moments for this trajectory into equation (122) gives the result:

$$\delta\underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \end{bmatrix} + \delta V_v \begin{bmatrix} s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \delta\theta_c s \end{bmatrix}$$

$$P(t) = P(0) + \sigma_{vv}^{(v)} \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} + |V|\sigma_{\theta\theta} \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix}$$

Odometry error for direct heading case for a straight trajectory with scale factor translation errors and sinusoidal compass error. **(127)**

For systematic error, both translational errors are linear in distance - but for different reasons. The x error is due to the translational scale error while the y error is due to the constant heading error that exists for a straight trajectory.

For random error, the covariance matrix remains diagonal. Alongtrack variance increases linearly with distance. Crosstrack variance also increases linearly under the assumption that velocity is constant.

8.1.2 Integrated Heading Odometry

Substituting the trajectory moments for this trajectory into equation (123) gives the result:

$$\delta \underline{x}(t) = \underline{IC}_d + \delta V_v \begin{bmatrix} s(t) \\ 0 \\ 0 \end{bmatrix} + \delta\omega \begin{bmatrix} 0 \\ Vt^2/2 \\ t \end{bmatrix}$$

$$P(t) = IC_s + \sigma_{vv}^{(v)} \begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_{\omega\omega} \begin{bmatrix} 0 & 0 & 0 \\ 0 & (s^2 t)/3 & (st)/2 \\ 0 & (st)/2 & t \end{bmatrix} \quad \textbf{(128)}$$

Odometry error for integrated heading case for a straight trajectory with scale factor translation errors and gyro bias.

For systematic error, constant velocity was assumed for the T_x and T_{xx} duration moments. Alongtrack error is linear in distance while heading error is linear in time. Crosstrack error includes a term linear in distance and another term proportional to the duration moment which is quadratic in time or distance, or proportional to the product of time and distance, for constant velocity.

For stochastic error, heading variance is linear in time as was intended. Constant velocity was assumed for the T_x and T_{xx} duration moments. Heading covariance with crosstrack is linear in distance and time (or quadratic in either for constant velocity) in a manner identical to crosstrack in the deterministic case. Notice that the alongtrack variance is (to first order) linear in distance rather than time whereas crosstrack variance is cubic in time (or distance for constant velocity). This result echoes are earlier results for systematic error in that crosstrack error grows with a higher order polynomial than alongtrack.

8.1.3 Differential Heading Odometry

Substituting the trajectory moments for this trajectory into equation (124) gives the result:

$$\delta \underline{x}(s) = \underline{IC}_d + \delta V_v(s) \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} + \delta\omega_v(s) \begin{bmatrix} 0 \\ \frac{s^2}{2} \\ s \end{bmatrix}$$

$$P(s) = IC_s + \sigma_{v\omega}^{(v)} \begin{bmatrix} 0 & s^2/2 & s \\ s^2/2 & 0 & 0 \\ s & 0 & 0 \end{bmatrix} + \sigma_{vv}^{(v)} \begin{bmatrix} s & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \sigma_{\omega\omega}^{(v)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & s^3/3 & s^2/2 \\ 0 & s^2/2 & s \end{bmatrix} \quad \textbf{(129)}$$

Odometry error for differential heading case for arbitrary trajectory with scale factor errors on both encoders.

For systematic error, all terms are motion dependent. As in the integrated heading case, alongtrack error is linear in distance. However, in this case the heading error is also linear in distance (rather than time). The crosstrack error involves an excursion moment term so it is quadratic in distance.

Recall that σ_{gg}' vanishes when both encoders have identical error statistics so some of these terms are due only to a differential in the characteristics of the encoder noises. When the encoders have identical noises, this solution is analogous (after replacing time with distance) to the integrated heading case. Heading variance and along track variance are linear in distance whereas crosstrack variance is cubic.

8.2 Constant Curvature Trajectory

Expressions for a constant curvature trajectory were given in equation (19):

$$\begin{aligned} \theta(s) &= \kappa s \\ x(s) &= R\,sin(\kappa s) \\ y(s) &= R[1 - cos(\kappa s)] \end{aligned} \qquad \kappa \cong \frac{1}{R}$$

8.2.1 Direct Heading Odometry

Substituting the trajectory moments for this trajectory into equation (122) gives the result:

$$\delta \underline{x}(s) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \end{bmatrix} + \delta V_v \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} + R \begin{bmatrix} -\delta\theta_c/4 \\ \delta\theta_s/4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\delta\theta_s s \\ \delta\theta_c s \end{bmatrix} + \frac{R}{4}\begin{bmatrix} \delta\theta_c c2\theta + \delta\theta_s s2\theta \\ -\delta\theta_s c2\theta + \delta\theta_c s2\theta \end{bmatrix}$$

$$P(t) = P(0) + \frac{\sigma_{vv}^{(v)} R}{2}\begin{bmatrix} \theta + \frac{s2\theta}{2} & s^2\theta \\ s^2\theta & \theta - \frac{s2\theta}{2} \end{bmatrix} + |V|\sigma_{\theta\theta}\begin{bmatrix} \theta - \frac{s2\theta}{2} & -s^2\theta \\ -s^2\theta & \theta + \frac{s2\theta}{2} \end{bmatrix} \qquad \textbf{(130)}$$

For systematic error, in addition to linear terms relating to position coordinates and distance travelled, there are pure oscillation terms that depend only on the distance travelled along the arc but which cycle twice per orbit.

For stochastic error, the x-y covariance cycles at twice the frequency of the original trajectory but does vanish 4 times per orbit (at the top, bottom, left and right “edges” of the circular trajectory). The variances are composed of linearly increasing terms with additional oscillatory second harmonic components impressed on the steady growth. Overall, the character of the result is that a constant probability ellipse will steadily increase in size while rotating twice per orbit of the original trajectory. The overall influence of each source of error is clear from the constants in each term. There is a particular radius when

$$\frac{\sigma_{vv}^{(v)} R}{2} = |V|\sigma_{\theta\theta}$$

where the principal variances become purely linear because the oscillations cancel.

8.2.2 Integrated Heading Odometry

For integrated heading, let us define the following moment matrices for arc trajectories:

$$\underline{f}_v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \qquad \underline{f}_\omega = \begin{bmatrix} -\frac{R}{\omega}[s\theta - \theta c\theta] \\ \frac{R}{\omega}[\theta s\theta + c\theta - 1] \\ t \end{bmatrix} = \begin{bmatrix} -(Tx(t) + t[y(t) - R]) \\ tx(t) - Ty(t) \\ t \end{bmatrix}$$

$$F_{vv} = \begin{bmatrix} R\left[\frac{\theta}{2} + \frac{s2\theta}{4}\right] & \frac{Rs^2\theta}{2} & 0 \\ \frac{Rs^2\theta}{2} & R\left[\frac{\theta}{2} - \frac{s2\theta}{4}\right] & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{s}{2} + \frac{x}{2}\left(1 - \frac{y}{R}\right) & \frac{x^2}{2R} & 0 \\ \frac{x^2}{2R} & \frac{s}{2} - \frac{x}{2}\left(1 - \frac{y}{R}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_{\omega\omega} = \begin{bmatrix} \frac{R^2}{\omega}\left[\theta\left(1 + \frac{c2\theta}{2}\right) - \frac{3}{2}\left(\frac{s2\theta}{2}\right)\right] & -\frac{R^2}{\omega}\left[\theta\left(\frac{s2\theta}{2}\right) + \frac{3}{2}\left(\frac{c2\theta}{2}\right) - c\theta + \frac{1}{4}\right] & -\frac{R}{\omega}[s\theta - \theta c\theta] \\ \frac{R^2}{\omega}\left[\theta\left(\frac{s2\theta}{2}\right) + \frac{3}{2}\left(\frac{c2\theta}{2}\right) - c\theta + \frac{1}{4}\right] & \frac{R^2}{\omega}\left[\theta\left(1 - \frac{c2\theta}{2}\right) + \frac{3}{2}\left(\frac{s2\theta}{2}\right) - 2s\theta\right] & \frac{R}{\omega}[\theta s\theta + c\theta - 1] \\ -\frac{R}{\omega}[s\theta - \theta c\theta] & \frac{R}{\omega}[\theta s\theta + c\theta - 1] & t \end{bmatrix}$$

$$F_{\omega\omega} = \begin{bmatrix} t\left(\frac{R^2}{2} + (R - y)^2\right) - 3Tx\left(\frac{R}{2} - \frac{y}{2}\right) & -\left(xt(y - R) + T\left(\frac{3}{2}x^2 - Ry\right)\right) & -(Tx(t) + t[y(t) - R]) \\ -\left(xt(y - R) + T\left(\frac{3}{2}x^2 - Ry\right)\right) & T_{xx}(t) & T_x(t) \\ -(Tx(t) + t[y(t) - R]) & T_x(t) & t \end{bmatrix}$$

Many of the component expressions can be interpreted as either time or distance dependent because, of course

$$\theta = \omega t = \kappa s$$

Under these definitions, the solution is:

Substituting the trajectory moments for this trajectory into equation (123) gives the result:

$$\delta\underline{x}(t) = \underline{IC}_d + \delta V_v \underline{f}_v + \delta\omega \underline{f}_\omega$$
$$P(t) = IC_s + \sigma_{vv}^{(v)} F_{vv} + \sigma_{\omega\omega} F_{\omega\omega} \qquad \textbf{(131)}$$

Odometry error for integrated heading case for a constant curvature trajectory with scale factor translation errors and gyro bias.

Constant velocity was assumed for all duration moments. For deterministic error, the heading error is linear in time whereas the position errors are oscillations of increasing amplitude as time increases. The linear increase in amplitude is a first order approximation to the true behavior of a beat frequency derived in equation (37). Eventually, the amplitude decreases again for the exact solution. The y error also includes a linear growth with time.

For stochastic error, heading variance $\sigma_{\theta\theta}$ increases linearly with time as was intended. The covariances of translation with heading $\sigma_{y\theta}$ and $\sigma_{x\theta}$ include a pure oscillation plus another oscillation at the fundamental frequency whose amplitude increases linearly with heading, distance, or time. Translational covariance σ_{xy} includes pure oscillations at the fundamental and second harmonic frequencies. One term is a second harmonic oscillation whose amplitude grows linearly with heading, distance, or time. The translational variances σ_{yy} and σ_{xx} include terms of similar character to σ_{xy} (but there is no fundamental term) but they also include a pure linear term in distance, heading, or time which does not oscillate. Both the gyro and the encoder variances cause these linear terms in the translational variances.

8.2.3 Differential Heading Odometry

For integrated heading, let us define the following moment matrices for arc trajectories:

$$\underline{f}_v = \begin{bmatrix} x(s) \\ y(s) \\ 0 \end{bmatrix} \qquad \underline{f}_\omega = \begin{bmatrix} -R^2[s\theta - \theta c\theta] \\ R^2[\theta s\theta + c\theta - 1] \\ s \end{bmatrix} = \begin{bmatrix} -(Rx(s) + s[y(s) - R]) \\ sx(s) - Ry(s) \\ s \end{bmatrix}$$

$$F_{vv} = \begin{bmatrix} R\left[\frac{\theta}{2} + \frac{s2\theta}{4}\right] & \frac{Rs^2\theta}{2} & 0 \\ \frac{Rs^2\theta}{2} & R\left[\frac{\theta}{2} - \frac{s2\theta}{4}\right] & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{s}{2} + \frac{x}{2}\left(1 - \frac{y}{R}\right) & \frac{x^2}{2R} & 0 \\ \frac{x^2}{2R} & \frac{s}{2} - \frac{x}{2}\left(1 - \frac{y}{R}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F_{v\omega} = \begin{bmatrix} -2R^2\left[\frac{\theta}{2} - \frac{s2\theta}{4}\right] & -R^2\left(\frac{1}{2} - c\theta + \frac{c2\theta}{2}\right) & Rs\theta \\ -R^2\left(\frac{1}{2} - c\theta + \frac{c2\theta}{2}\right) & 2R^2\left[-\frac{\theta}{2} + s\theta - \frac{s2\theta}{4}\right] & R(1 - c\theta) \\ Rs\theta & R(1 - c\theta) & 0 \end{bmatrix}$$

$$F_{v\omega} = \begin{bmatrix} -2\left(\frac{x}{2}(R - y) + R\left(\frac{s}{2}\right)\right) & \frac{1}{2}(x^2 - y^2) & x \\ \frac{1}{2}(x^2 - y^2) & R\left(-\frac{s}{2} + x\right) - \frac{x}{2}(R - y) & y \\ x & y & 0 \end{bmatrix}$$

$$F_{\omega\omega} = \begin{bmatrix} \frac{R^2}{\kappa}\left[\theta\left(1 + \frac{c2\theta}{2}\right) - \frac{3}{2}\left(\frac{s2\theta}{2}\right)\right] & \frac{R^2}{\kappa}\left[\theta\left(\frac{s2\theta}{2}\right) + \frac{3}{2}\left(\frac{c2\theta}{2}\right) - c\theta + \frac{1}{4}\right] & -R^2[s\theta - \theta c\theta] \\ \frac{R^2}{\kappa}\left[\theta\left(\frac{s2\theta}{2}\right) + \frac{3}{2}\left(\frac{c2\theta}{2}\right) - c\theta + \frac{1}{4}\right] & \frac{R^2}{\kappa}\left[\theta\left(1 - \frac{c2\theta}{2}\right) + \frac{3}{2}\left(\frac{s2\theta}{2}\right) - 2s\theta\right] & R^2[\theta s\theta + c\theta - 1] \\ -R^2[s\theta - \theta c\theta] & R^2[\theta s\theta + c\theta - 1] & R\theta \end{bmatrix}$$

$$F_{\omega\omega} = \begin{bmatrix} s\left(\frac{R^2}{2} + (R - y)^2\right) - 3xR\left(\frac{R}{2} - \frac{y}{2}\right) & -\left(xs(y - R) + R\left(\frac{3}{2}x^2 - Ry\right)\right) & -(Rx(s) + s[y(s) - R]) \\ -\left(xs(y - R) + R\left(\frac{3}{2}x^2 - Ry\right)\right) & s\left(\frac{R^2}{2} + x^2\right) - Rx\left(\frac{R}{2} - \frac{3y}{2}\right) & sx(s) - Ry(s) \\ -(Rx(s) + s[y(s) - R]) & sx(s) - Ry(s) & s \end{bmatrix}$$

For the stochastic integrals, the differential ds is based on the equivalent velocity:

$$V\big|_{DH} = (|V_r| + |V_l|)/2$$

Many of the component expressions can be interpreted as either time or distance dependent because, of course

$$\theta = \omega t = \kappa s$$

Substituting the trajectory moments for this trajectory into equation (124) gives the result:

$$\delta\underline{x}(t) = \underline{IC}_d + \delta V_v \underline{f}_v + \delta\omega \underline{f}_\omega$$

$$P(t) = IC_s + \sigma_{vv}^{(v)} F_{vv} + \sigma_{v\omega}^{(v)} F_{v\omega} + \sigma_{\omega\omega}^{(v)} F_{\omega\omega} \quad \textbf{(132)}$$

Odometry error for differential heading case for arbitrary trajectory with scale factor errors on both encoders.

All terms are motion dependent. The structure is analogous to the integrated heading case. For deterministic error, the heading error is linear in distance whereas the position errors are entirely oscillatory but of increasing amplitude as distance increases. Again, some of these terms are due only to a differential in the characteristics of the encoder noises.

For stochastic error, heading variance $\sigma_{\theta\theta}$ increases linearly with distance as was intended. The covariances of translation with heading $\sigma_{y\theta}$ and $\sigma_{x\theta}$ include a pure oscillation as well as another oscillation at the fundamental frequency whose amplitude increases linearly with heading, distance, or time. Translational covariance σ_{xy} includes pure oscillations at the fundamental and second harmonic frequencies. One term is a second harmonic oscillation whose amplitude grows linearly with heading, distance, or time. The translational variances σ_{yy} and σ_{xx} include terms of similar character to σ_{xy} (but there is no fundamental term) but they also include a pure linear term in distance, heading, or time which does not oscillate. These linear terms arise from both the mean and differential encoder variances.

9. Selected Analyses

This section provides a few short analyses which demonstrate the practical utility of our results.

9.1 Local Behavior on Curved Trajectory

Refer to the deterministic error result for integrated heading on a circular trajectory given as equation (131):

$$\delta\underline{x}(t) = \underline{IC}_d + \delta V_v \underline{f}_v + \delta\omega \underline{f}_\omega$$

$$\delta\underline{x}(t) = \underline{IC}_d + \delta V_v \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix} + \delta\omega \begin{bmatrix} -(Tx(t) + t[y(t) - R]) \\ tx(t) - Ty(t) \\ t \end{bmatrix} \qquad \textbf{(133)}$$

$$\delta\underline{x}(t) = \underline{IC}_d + \delta V_v \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix} + \delta\omega \begin{bmatrix} -T[\sin(\omega t) - \omega t\cos(\omega t)] \\ T[\omega t\sin(\omega t) + \cos(\omega t) - 1] \\ t \end{bmatrix}$$

The reference trajectory is a circle of radius R centered at the point $(0, R)$. This trajectory has a vanishing x duration moment at the origin but the y duration moment is finite at the origin. The solution reflects this asymmetry. To first order, the y error vanishes upon return to the origin whereas the x coordinate of the perturbed trajectory lags or leads the unperturbed trajectory by an additional distance of $R\delta\omega t = 2\pi R(\delta\omega / \omega)$ for each orbit of the unperturbed trajectory. This lag or lead is equal to the gyro bias times the value of the y duration moment for a single orbit.

For small excursions (θ small), and no initial errors, the translational errors reduce to:

$$\delta x(t) = \left[\delta V_v - \frac{\delta\omega}{\omega}\right]R\sin\theta + [\delta\omega t]R\cos\theta = \left[\delta V_v - \frac{\delta\omega}{\omega}\right]R\theta + \left[\delta\omega\frac{\theta}{\omega}\right]R = \delta V_v R\theta = \delta V_v s$$

$$\delta y(t) = \left[\delta V_v - \frac{\delta\omega}{\omega}\right]R(1 - \cos\theta) + [\delta\omega t]R\sin\theta = \left[\delta V_v - \frac{\delta\omega}{\omega}\right]R\theta^2 + \left[\delta\omega\frac{\theta}{\omega}\right]R\theta = \frac{\delta V_v}{R}s^2$$

which echoes the linear alongtrack and quadratic crosstrack errors of the straight trajectory case.

9.2 Calibrating Integrated Heading Odometry

The two systematic results for integrated heading on known trajectories are:

$$\delta\underline{x}(t) = \underline{IC}_d + \delta V_v \begin{bmatrix} s(t) \\ 0 \\ 0 \end{bmatrix} + \delta\omega \begin{bmatrix} 0 \\ Vt^2/2 \\ t \end{bmatrix} \qquad \text{Straight Trajectory}$$

$$\delta\underline{x}(t) = \underline{IC}_d + \delta V_v \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix} + \delta\omega \begin{bmatrix} -T[\sin(\omega t) - \omega t\cos(\omega t)] \\ T[\omega t\sin(\omega t) + \cos(\omega t) - 1] \\ t \end{bmatrix} \qquad \text{Arc Trajectory}$$

Dependence on initial conditions can be eliminated by defining the start point as the origin.

Calibrating gyro bias can be easily accomplished by integrating the heading output for null input (stationary vehicle) over a long time period and dividing by the time period. The simplest way to calibrate the scale factor seems to be the x equation from the straight line trajectory case.

9.3 Calibrating Differential Heading Odometry

The two systematic results for differential heading on known trajectories are:

$$\delta \underline{x}(s) = \underline{IC}_d + \delta V_v(s)\begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} + \delta\omega_v(s)\begin{bmatrix} 0 \\ \frac{s^2}{2} \\ s \end{bmatrix} \qquad \text{Straight Trajectory}$$

$$\delta \underline{x}(s) = \underline{IC}_d + \delta V_v \begin{bmatrix} x(s) \\ y(s) \\ 0 \end{bmatrix} + \delta\omega \begin{bmatrix} -R^2[s\theta - \theta c\theta] \\ R^2[\theta s\theta + c\theta - 1] \\ s \end{bmatrix} \qquad \text{Arc Trajectory}$$

For the linear case, the endpoint errors are indicated in the figure below:

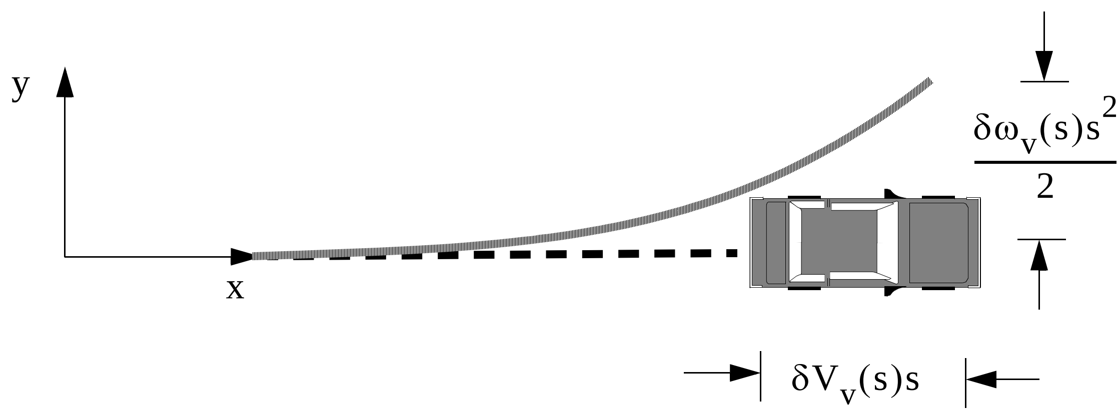


Figure 10 Differential Heading Odometry

A vehicle can be driven on a straight trajectory and actual crosstrack δy and alongtrack δx errors can be measured. When these values are available, we can solve:

$$\delta x = \delta V_v(s)s = \left(\frac{\delta r_r + \delta l_l}{2}\right)s + \frac{(\delta r_r - \delta l_l)}{4R(t)}Ws = (\delta r_r + \delta l_l)s/2$$

$$\delta y = \frac{\delta\omega_v(s)s^2}{2} = \left\{\frac{(\delta r_r - \delta l_l)}{W} + \left(\frac{\delta r_r + \delta l_l}{2R(t)}\right)\right\}s^2/2 = (\delta r_r - \delta l_l)s^2/(2W)$$

$$\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \frac{s}{2}\begin{bmatrix}1 & 1\\ \frac{s}{W} & -\frac{s}{W}\end{bmatrix}\begin{bmatrix}\delta r_r\\ \delta l_l\end{bmatrix}$$

We have used the fact that $R(t)$ is infinite on a straight trajectory. Inverting the matrix is easy:

$$\begin{bmatrix}\delta r_r\\ \delta l_l\end{bmatrix} = \frac{1}{s}\begin{bmatrix}1 & \frac{W}{s}\\ 1 & -\frac{W}{s}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix}$$

9.4 Emergence of Quadratic Behavior

When do the quadratic crosstrack systematic error terms become important? For integrated heading, if we equate the alongtrack and crosstrack terms we have:

$$\delta V_v s = \delta\omega st/2$$

Solving for time:

$$t = 2\delta V_v/\delta\omega$$

If $\delta V_v = 0.01$ and $\delta\omega$ is 3 degrees per second[1] (typical of MEMS gyros today), it takes 0.38 seconds for the crosstrack term to exceed the alongtrack term in integrated heading odometry.

For differential heading, if we equate the alongtrack and crosstrack terms we have:

$$\delta V_v(s)s = \frac{\delta\omega_v(s)s^2}{2}$$

Solving for distance:

$$s = 2\delta V_v(s)/\delta\omega_v(s) = (\delta r_r + \delta l_l)/\left[\frac{(\delta r_r - \delta l_l)}{W}\right]$$

1. Earlier versions of the report erroneously used 3 degrees per hour. While FOG gyros achieve this performance level, it is about 3600 times too small for a MEMS gyro.

If $\delta r_r = 0.01$ and $\delta l_l = 0.005$, and the wheel tread is 1 m, it takes just 3 meters of motion for the crosstrack term to dominate.

9.5 Instantaneous Gyro Equivalent

Given a particular gyro bias and transmission encoder scale error and a particular set of differential encoder scale errors, which is better? Consider systematic error only. The answer is contained in equation (42). The instantaneous "integrated heading equivalent" of a particular set of differential encoder scale errors is:

$$\delta V_v = \left(\frac{\delta r_r + \delta l_l}{2}\right) + \frac{(\delta r_r - \delta l_l)}{4R(t)} W$$
$$\delta \omega_v = \left\{\frac{(\delta r_r - \delta l_l)}{W} + \left(\frac{\delta r_r + \delta l_l}{2R(t)}\right)\right\} \quad \textbf{(134)}$$

Hence, there is no definitive answer because the result depends on curvature and velocity. Differential heading is most competitive on a straight trajectory because error grows at the slowest rate. For this trajectory, we get:

$$\delta V_v = \frac{(\delta r_r + \delta l_l)}{2}$$
$$\delta \omega_v = \frac{(\delta r_r - \delta l_l)}{W}$$

The gyro bias which is equivalent to a given differential odometry system on straight trajectories is:

$$\delta \omega(t) = \delta \omega_v(t) V(t) = \frac{(\delta r_r - \delta l_l)}{W} V(t)$$

The motion dependent nature of differential heading implies that if time is not important, the velocity can always be reduced enough to beat the performance of any gyro. However, in a realistic context, the velocity needs to be on the order of 1 m/s and the wheelbase is also on the order of 1 m so both W and V disappear. Equating to a gyro bias gives:

$$(\delta r_r - \delta l_l) = \delta \omega(t) \frac{W}{V} = \delta \omega(t)$$

Hence the difference in encoder scale errors (a nondimensional number) corresponds to an equivalent gyro bias (in radians per second). On the presumption that the encoder scale errors can be calibrated to a residual 0.5%, the difference between the residual errors cannot exceed 1%. Solving for the equivalent residual gyro bias leads to:

$$\delta \omega(t) = (\delta r_r - \delta l_l) = 1\% = 0.01 \frac{\text{rads}}{\text{sec}} = 2063 \frac{\text{deg}}{\text{hr}}$$

While this level of performance (0.6 deg/sec) is comparable to the best contemporary MEMS gyros, contemporary fiber optic gyros exceed it by several orders of magnitude. On this basis alone, such gyros can be expected to significantly outperform differential heading in practice even before accounting for their relative immunity to floor irregularities.

The figure below illustrates a simulated runoff between the two options on a straight trajectory at a speed of 0.25 meters/sec. The 3 dege/sec gyro is the clear loser. A 3 deg / hour fiber optic gyro would have been a clear winner.

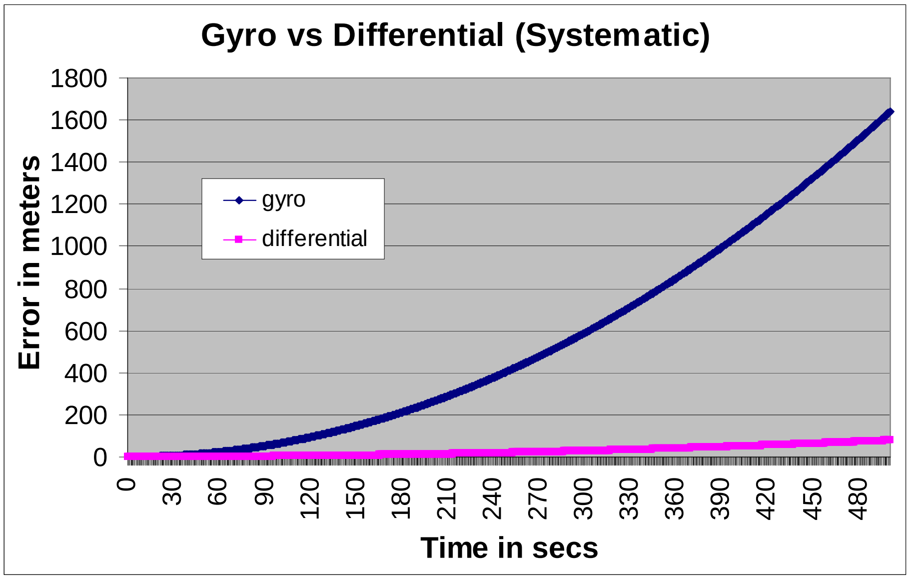


Figure 11 Integrated - Differential Runoff

9.6 Circular Test Trajectory

The closed-form results contained in equation (132) can be written out in detail as follows:

$$
\begin{aligned}
\sigma_{xx} &= \sigma_{vv}^{(v)} R\left[\frac{\theta}{2}+\frac{s2\theta}{4}\right] - \sigma_{v\omega}^{(v)} 2R^2\left[\frac{\theta}{2}-\frac{s2\theta}{4}\right] + \sigma_{\omega\omega}^{(v)} R\frac{R^2}{\kappa}\left[\theta\left(1+\frac{c2\theta}{2}\right)-\frac{3}{2}\left(\frac{s2\theta}{2}\right)\right] \\
\sigma_{yy} &= \sigma_{vv}^{(v)} R\left[\frac{\theta}{2}-\frac{s2\theta}{4}\right] + \sigma_{v\omega}^{(v)} 2R^2\left[-\frac{\theta}{2}+s\theta-\frac{s2\theta}{4}\right] + \sigma_{\omega\omega}^{(v)} \frac{R^2}{\kappa}\left[\theta\left(1-\frac{c2\theta}{2}\right)+\frac{3}{2}\left(\frac{s2\theta}{2}\right)-2s\theta\right] \\
\sigma_{xy} &= \sigma_{vv}^{(v)} R\left[\frac{s^2\theta}{2}\right] - \sigma_{v\omega}^{(v)} R^2\left(\frac{1}{2}-c\theta+\frac{c2\theta}{2}\right) + \sigma_{\omega\omega}^{(v)} \frac{R^2}{\kappa}\left[\theta\left(\frac{s2\theta}{2}\right)+\frac{3}{2}\left(\frac{c2\theta}{2}\right)-c\theta+\frac{1}{4}\right] \\
\sigma_{x\theta} &= \sigma_{vv}^{(v)}[0] + \sigma_{v\omega}^{(v)} R[s\theta] - \sigma_{\omega\omega}^{(v)} R^2[s\theta-\theta c\theta] \\
\sigma_{y\theta} &= \sigma_{vv}^{(v)}[0] + \sigma_{v\omega}^{(v)} R(1-c\theta) + \sigma_{\omega\omega}^{(v)} R^2[\theta s\theta+c\theta-1] \\
\sigma_{\theta\theta} &= \sigma_{vv}^{(v)}[0] + \sigma_{v\omega}^{(v)}[0] + \sigma_{\omega\omega}^{(v)} R[\theta]
\end{aligned}
\qquad \textbf{(135)}
$$

In order to verify the conclusions of the report, these equations were compared against a direct numerical integration of the linear variance equation for this case. The resulting trajectories agreed to within expected numerical integration error. The above equations are the exact solution. these are plotted below:

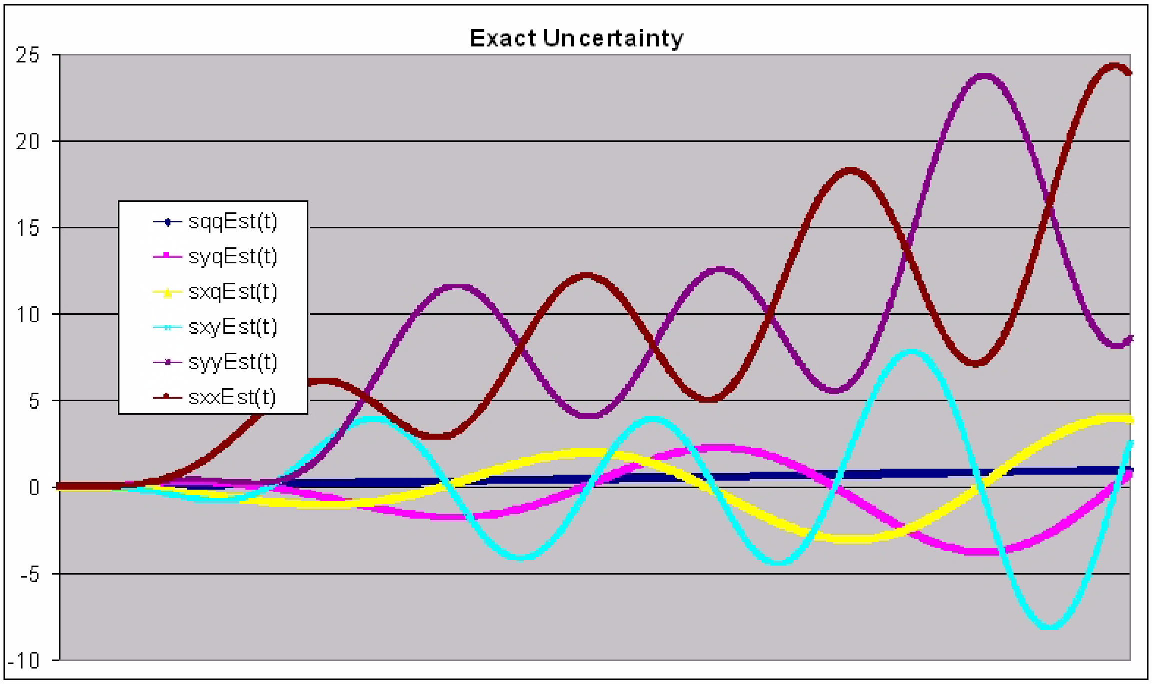


Figure 12 Simulated Differential Heading Test Run

In the figure Sqq is the plotter's rendition of the symbol $\sigma_{\theta\theta}$. In order from lowest equation to highest, the behaviors are linear for $\sigma_{\theta\theta}$, first harmonic with linearly increasing amplitude for $\sigma_{y\theta}$ and $\sigma_{x\theta}$, second harmonic with linearly increasing amplitude for σ_{xy} and second harmonic upon linearly increasing amplitude for σ_{yy} and σ_{xx}.

9.7 Validation on an Asymmetric Test Trajectory

Because so many sources of error cancel on closed and symmetric trajectories, the following arbitrary trajectory was chosen to assess the linearization error on deterministic errors. This case was done for integrated heading odometry at a speed of 0.25 m/sec with error characteristics:

$$\delta V = 0.05V \qquad \delta\omega = 30\deg/hr$$

Such error magnitudes are considerably larger than might be expected in a practical situation. The intention here is both to stress the linearity assumption and to provide a common error magnitude for both systematic and random sources which is large enough to be noticeable in the following figures. In the figure below, the nominal trajectory is black and the perturbed is white.

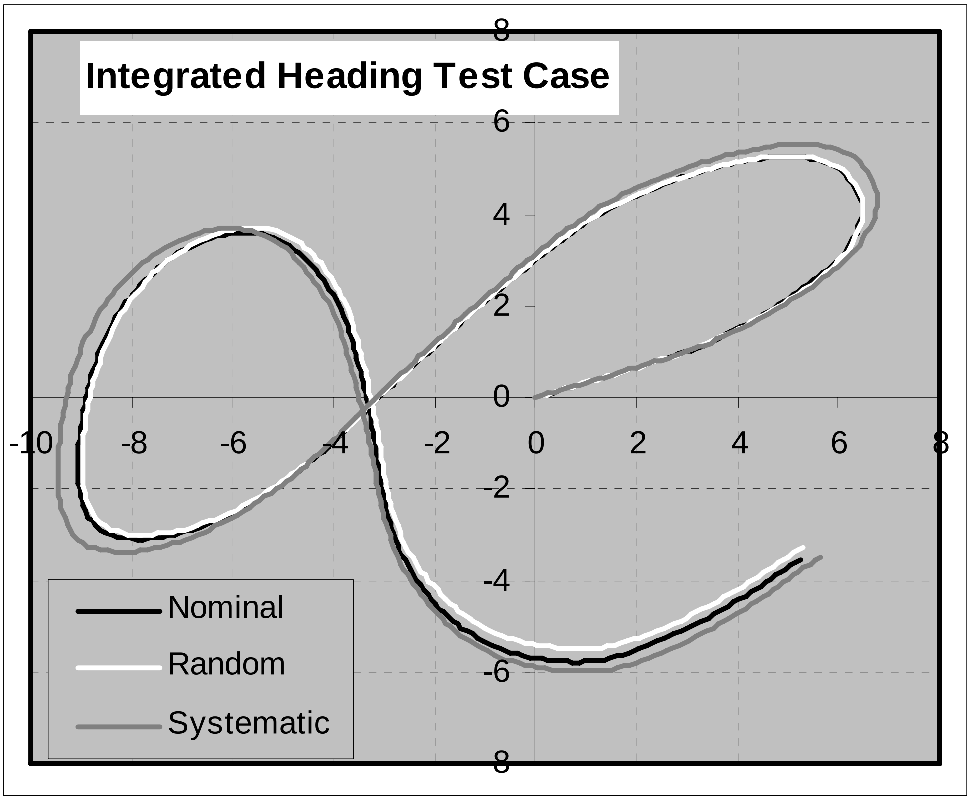


Figure 13 Simulated Integrated Heading Test Run

The exact nonlinear error was computed by corrupting the inputs, solving the nonlinear system numerically, and subtracting the reference trajectory. Estimated error was computed using the vector convolution integral. According to it, error propagates according to equation (123).

$$\delta\underline{x}(t) = \underline{IC}_d + \alpha \begin{bmatrix} S_c(t) \\ S_s(t) \\ 0 \end{bmatrix} + b \begin{bmatrix} -T_y(t) \\ T_x(t) \\ t \end{bmatrix}$$

This can be written for null initial conditions as:

$$\delta\underline{x}(t) = \alpha \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix} + bt \begin{bmatrix} -y(t) \\ x(t) \\ 0 \end{bmatrix} + b\int_0^t \begin{bmatrix} y(\tau) \\ -x(\tau) \\ t \end{bmatrix} d\tau$$

This equation illustrates how to use the general results for a trajectory which is not analytically known. We simply compute the moments which appear (in this case, the last term) numerically.

Figure 14 shows the true and estimated error curves. On this scale, the difference between linearized and nonlinear error propagation in not detectable. Angular error is omitted because it is linear by construction. The point of loop closure is the last place where both translational error curves cross.

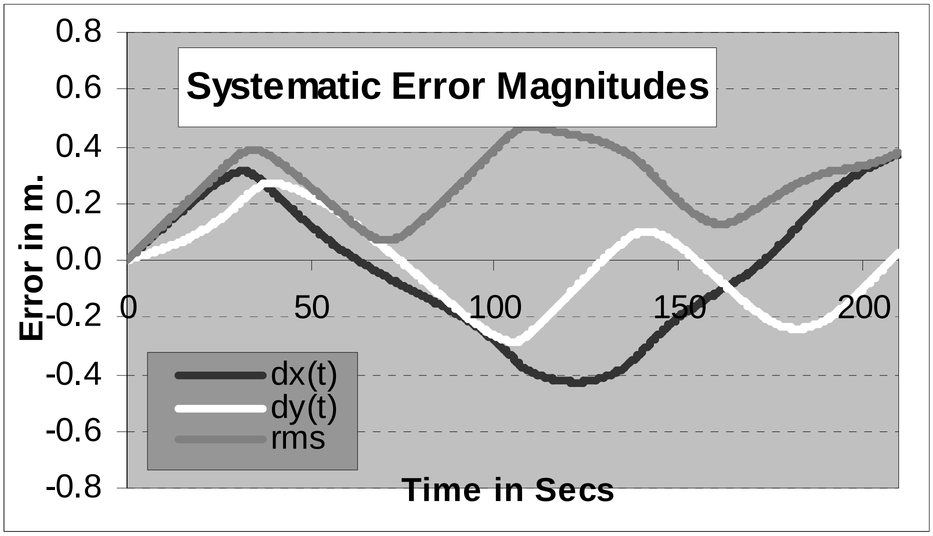


Figure 14 Simulated Integrated Heading Test Run State Errors

The only way to see the difference between the true and estimated error signals (that is, the linearization error) is to display their difference directly. As shown in Figure 15, the linearization error is less than 1 cm throughout the test run.

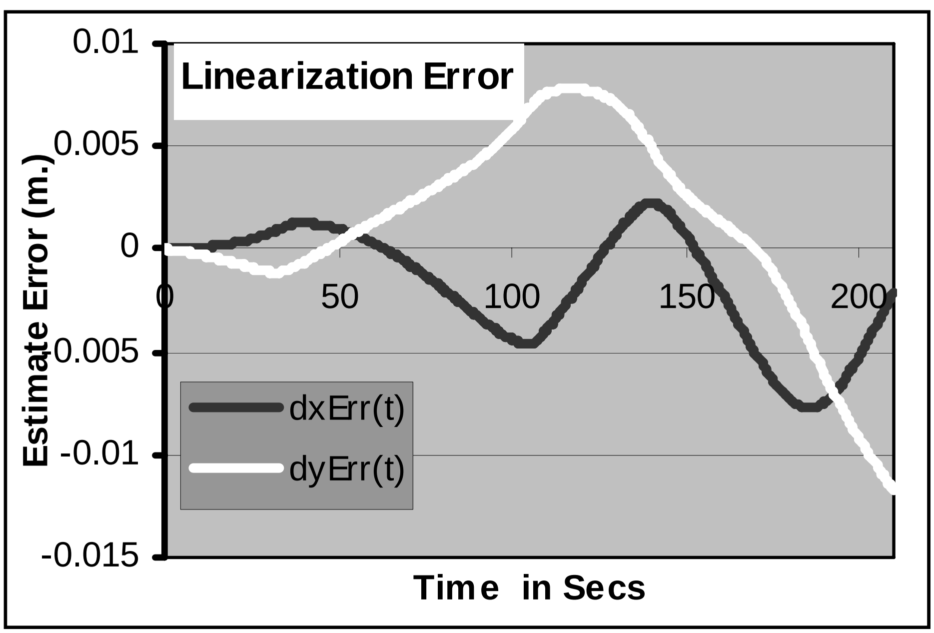


Figure 15 Linearization Error

The fact that the error models used are representative of real odometry systems indicates that linear error models are so good that nonlinear techniques are hardly worth the effort in most applications.

Monte Carlo simulation was used to generate a large number of separately and randomly perturbed solutions to the system dynamics whose statistics could then be computed and compared with the linearized solution of the article.

Roughly 500,000 independent, unbiased, unit variance Gaussian random variables were first generated. Half were designated for linear velocity error $\delta V(t)$ and the other half for angular velocity error $\delta\omega(t)$. Each set was divided into 250 discrete time random signals which were then scaled appropriately and presented to the nonlinear system as random corruptions to the nominal inputs. The results of the Monte Carlo simulation are provided in figure Figure 16: for the translational variances and co-variances. Rotational variance is linear by construction and translational-rotational covariances agree similarly with theory.

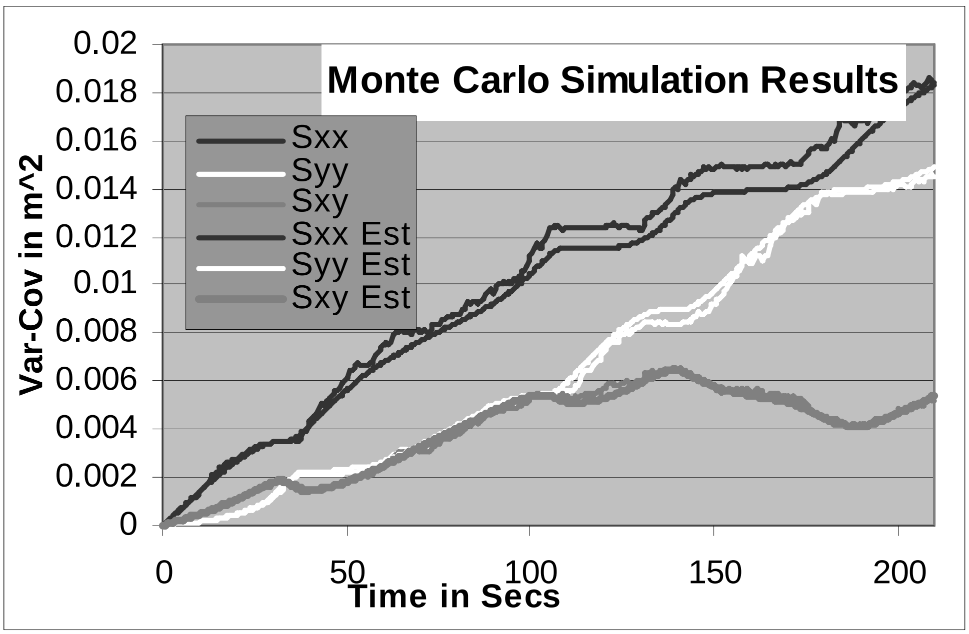


Figure 16: Monte Carlo Simulation Compared to Theory

The agreement between theory and simulation is excellent. Note how the two translational variances exhibit conservation behavior by varying symmetrically about an average steady growth curve.

Overall, three classes of error can be expected in the stochastic case: linearization, discretization, and sampling error.

10. Summary and Conclusions

One of the most important distinctions in position estimation is the distinction between triangulation and dead reckoning. These names are no longer as descriptive as they were when they were coined centuries ago before mathematics developed past geometry. The essential difference from a mathematics perspective is whether the available observations project onto the states of interest, or onto their derivatives.

Odometry is a form of dead reckoning and it can be described by a nonlinear system of differential equations. A related distinction between dead reckoning and triangulation is that errors in triangulation are felt when they occur whereas errors in dead reckoning are felt forever thereafter. This is equivalent to saying that errors in odometry have dynamics - that error propagation in dynamical systems is also described by a differential equation.

The precise description of this error propagation is another nonlinear differential equation, and this is bad news because very few such equations are solvable in general. The goal of this report is to understand the behavior of odometry error for any error model while moving over any trajectory. This goal is incompatible with the mathematical reality that solutions to the equations formally do not exist.

When faced with such issues, a practical alternative is to linearize. A new system of equations is generated which is linear and which approximates the behavior of the original system. Using this tool, we have found that a surprising amount can be said about the general case, as outlined below.

Linear systematic error propagates according to the linear perturbation equation and random error propagates according to the linear variance equation. Even in the case of time-varying coefficients, the solutions to these equations are known to exist and to be given by the vector and matrix convolution integrals respectively. Elaboration of both of these solution integrals rests on determining a special matrix called the transition matrix.

Fortunately, the equations of odometry as formulated here have two key properties that allow us to find approximate solutions through the process of linearization. First, the linearized equations have commutable dynamics. This means that the transition matrix is given by a particular infinite matrix series, called the matrix exponential, evaluated on an argument given by a particular definite integral of the system Jacobian. This is the key to the results of this report.

The resulting approximate linear solutions satify the property of superposition. This permits us to consider the errors casued by different sources in an additive, independent manner.

From a convenience perspective, it is also important that the odometry equations are in echelon form because this property implies that the infinite matrix series cannot exceed three additive terms in length. Both properties taken together mean that a solution to our linearized problem not only exists, but is easy to find.

Certain other properties of systems are important. Motion dependence means that errors evolve as distance increases instead of as time increases. In odometry, this means that errors stop accumulating when motion stops. Reversibility means that errors can be erased by driving back over the path that caused the error. Some systematic errors are reversible whereas the principal variances tend to be monotone.

Given that a general linearized solution exists for any set of sensors, any trajectory, and any error model, we have elaborated the solution for three cases of odometry called direct, integrated, and differential heading. The general solutions for each of these cases are given respectively in equations (87), (90), and (94).

In elaborating the solution for the general case, other properties appear that are important. Response to initial conditions (initial error) is always path independent in both the deterministic and the random case, but the response does depend on the endpoint of the trajectory. Response to one particular type of input - translational scale errors - is path independent and there are many other path independent integrals. The accumulated effect of path independent errors vanishes on any closed trajectory.

Most other errors have a path dependent influence meaning they are not integrable in closed form. When these errors have simple forms, their influence can be reduced to constant multiples of path functionals evaluated on the reference trajectory. These functionals are called moments by analogy to the moments of mechanics and statistics. They can be tabulated for commonly useful trajectories once and for all so that the difficulty of integration is removed from the problem. First order moments determine the path dependent propagation of systematic error whereas second order moments determine the propagation of random error.

Moments have properties of interest too. Most important is the fact that many first order moments vanish at the centroid of the trajectory they are evaluated on. This means that sometimes the only remaining nonvanishing error component of systematic error will vanish on closed symmetric trajectories. The Fourier second order principal moments are monotone while the spatial and hybrid moments and all cross moments can exhibit local maxima and minima.

Second order principal moments often have critical points at the zeros of the first moments and at local coordinate extremities of the axis associated with the moment. Certain corresponding pairs of moments exchange magnitude while their combination grows in a simple way. Now that error propagation has been reduced to path functionals, the calculus of variations can, of course, now be used to design trajectories which are

All of these results have multiple practical uses. The goal has been to provide the tools necessary to answer many important questions on paper without resorting to numerical simulation. Using the results of this report, answers can be obtained to questions like the following:

- which of two sets of sensors will provide better position estimation accuracy
- what error can be expected on a given trajectory
- how to specify sensor performance in order to meet a given localization specification
- how can systematic error source X be calibrated
- what trajectory maximizes or minimizes exposure to error source X

11. Appendix A - Drawing Covariance

A multivariate normal distribution for an unbiased random variable $\underline{u}$ of dimension n and covariance C is given by[1]:

$$P(\underline{u}) = \frac{1}{\sqrt{(2\pi)^n |C|}} exp\left(-\frac{1}{2}\underline{u}^T C^{-1} \underline{u}\right) \tag{136}$$

where $|C|$ is the determinant of the covariance matrix.

When n exceeds 1, a contour of constant probability is a n-1 degree of freedom surface embedded in an n dimensional space. When $n = 2$, the contour is an ellipse. An algorithm for drawing such ellipses has been presented in [1], but it has been difficult to debug, so a hopefully simpler algorithm has been developed and presented here which is based on diagonalization of the covariance matrix.

11.1 Rotating Covariance

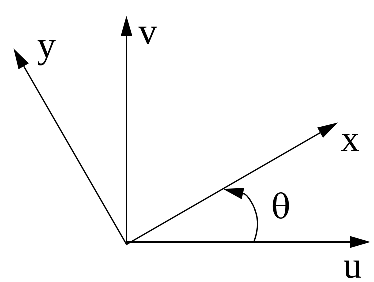
Suppose frame X is a counterclockwise rotation of frame U. The the matrix:

$$R \cong R_x^u$$

convert the coordinates of points from their expression in frame X to their expression in frame U:

$${}^u\underline{r} = R\,{}^x\underline{r}$$

$$R = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \tag{137}$$



Let $\underline{x} = \underline{r}^x$ be an unbiased random vector of covariance C_x which is expressed in frame X. The covariance C_u of the same vector $\underline{u} = \underline{r}^u$ expressed in frame U is:

$$C_u = Exp(\underline{u}\underline{u}^T) = Exp(R\underline{x}\underline{x}^T R^T) = RC_x R^T \tag{138}$$

We can interpret the R matrix either as an operator on $\underline{x}$ which produces $\underline{u}$ or as a conversion of coordinates. The result means that covariance is dependent on the choice of coordinate system. In the latter (coordinate conversion) case, note that both C_u and C_x designate the same uncertainty region.

1. Be sure to distinguish exp() - the exponential function, from Exp() - the expectation operator. The near collision of notation, both of which occur on this page, is regrettably standard in probability theory.

11.2 Diagonalizing Covariance

Due to their definition, all covariance matrices are symmetric. All symmetric matrices can be diagonalized with a matrix similarity transform based on an orthonormal (rotation) matrix R. This means that there is always a rotation of coordinates which renders a covariance matrix diagonal. Using the above result for rotating covariance, lets require that covariance be diagonal in the U frame and search for the necessary rotation matrix $R = R_x^u$:

$$C_u = \begin{bmatrix} \sigma_{uu} & 0 \\ 0 & \sigma_{vv} \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix} = RC_xR^T \quad \textbf{(139)}$$

Multiplying out the matrices leads to:

$$\begin{bmatrix} \sigma_{uu} & 0 \\ 0 & \sigma_{vv} \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx}c\theta - \sigma_{xy}s\theta & \sigma_{xx}s\theta + \sigma_{xy}c\theta \\ \sigma_{xy}c\theta - \sigma_{yy}s\theta & \sigma_{xy}s\theta + \sigma_{yy}c\theta \end{bmatrix} \quad \textbf{(140)}$$

$$\begin{bmatrix} \sigma_{uu} & 0 \\ 0 & \sigma_{vv} \end{bmatrix} = \begin{bmatrix} \sigma_{xx}c^2\theta - 2\sigma_{xy}c\theta s\theta + \sigma_{yy}s^2\theta & \sigma_{xx}c\theta s\theta + \sigma_{xy}c^2\theta - \sigma_{xy}s^2\theta - \sigma_{yy}c\theta s\theta \\ \sigma_{xx}c\theta s\theta + \sigma_{xy}c^2\theta - \sigma_{xy}s^2\theta - \sigma_{yy}c\theta s\theta & \sigma_{xx}c^2\theta + 2\sigma_{xy}c\theta s\theta + \sigma_{yy}s^2\theta \end{bmatrix}$$

From the off-diagonal equation at location (0,1) we have:

$$\sigma_{xx}c\theta s\theta + \sigma_{xy}c^2\theta - \sigma_{xy}s^2\theta - \sigma_{yy}c\theta s\theta = 0$$

Which reduces using the double angle trig identities to:

$$\sigma_{xy}c2\theta + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})s2\theta = 0$$

Giving the required rotation angle as:

$$\theta = \frac{1}{2}atan2(2\sigma_{xy}, \sigma_{yy} - \sigma_{xx}) \quad \textbf{(141)}$$

Note that:

- if $\sigma_{xy} = 0$, the matrix is already zero and $\theta = 0$ is correctly computed.
- if $\sigma_{yy} = \sigma_{xx}$, then $\theta = \pi/4$ regardless of the value of σ_{xy}.
- if both arguments are zero, θ is arbitrary, so detect this case and set θ to zero.
- if $\sigma_{yy} = \sigma_{xx} = \sigma_{xy} = 0$, this perfect certainty case must be detected and θ set to zero.

The diagonal covariances are related to the major and minor axes of constant uncertainty ellipses:

$$\sigma_{uu} = \sigma_{xx}c^2\theta - 2\sigma_{xy}c\theta s\theta + \sigma_{yy}s^2\theta$$
$$\sigma_{vv} = \sigma_{xx}c^2\theta + 2\sigma_{xy}c\theta s\theta + \sigma_{yy}s^2\theta \quad \textbf{(142)}$$

11.3 Drawing Equiprobability Ellipses

Let us concentrate now on a diagonal covariance matrix $C_u = \mathrm{diag}(\sigma_{uu}, \sigma_{vv})$. Equiprobability contours of a multivariate normal distribution, such as equation (136) are also contours of the exponent:

$$\underline{u}^T C^{-1} \underline{u} = k^2(p)$$

For $n = 2$ the probability of a sample falling within the contour is:

$$p = 1 - e^{-k^2/2} \quad \textbf{(143)}$$

Hence, the constant value of the exponent is:

$$k^2(p) = -2\,ln(1-p)$$

where p is the area under the probability density (corresponding to all points inside the contour).

Since the covariance is diagonal by assumption, we can write:

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix} = k^2$$

$$\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2} = k^2$$

Dividing by k^2 gives an unrotated ellipse in standard form:

$$\frac{u^2}{k^2\sigma_u^2} + \frac{v^2}{k^2\sigma_v^2} = 1 \quad \textbf{(144)}$$

Consider the U frame to be the model frame of the ellipse. In this frame, the parametric equations of the ellipse are:

$$u = a\cos\phi \qquad v = a\sin\phi$$

Points (u, v) can be converted to world coordinates by using the rotation matrix determined from the negative of the angle given by equation (141) when applied to the original (nondiagonal) covariance matrix.

11.4 Function diag_covariance()

Following is a C code fragment for the diagonalization function:

```
int diag_covariance(dblmatrix cov, double *theta, double *Suu, double *Svv)
{
  double Sxx,Syy,Sxy;
  double Ct,Ctt,St,Stt;
  double Yarg,Xarg;
  double epsilon = 1.0e-27;

  Sxx = m_el(cov,0,0);
  Sxy = m_el(cov,0,1);
  Syy = m_el(cov,1,1);

  Yarg = 2.0*Sxy;
  Xarg = Syy - Sxx;
/*
** Detect if all are zero
*/
  if(Sxx < epsilon && Sxy < epsilon && Syy < epsilon)
    {*theta = 0.0; *Suu = Sxx; *Svv = Syy; return(0);}
/*
** Detect if circle
*/
  if( fabs(Sxx-Syy) < epsilon )
    {*theta = 3.14159/2.0; *Suu = Sxx; *Svv = Syy; return(1);}
/*
** General case
*/
  *theta = 0.5 * atan2(Yarg,Xarg);
  Ct = cos(*theta); Ctt = Ct*Ct;
  St = sin(*theta); Stt = St*St;
  *Suu = Sxx*Ctt-2.0*Sxy*Ct*St+Syy*Stt;
  *Svv = Sxx*Stt+2.0*Sxy*Ct*St+Syy*Ctt;
  return(1);
}
```

11.5 Function diag_covariance()

Following is a C code fragment for the ellipse drawing function:

```
void renderEllipse(double x, double y, dblmatrix cov, double pr)
{
double a,b,kk;
```

```
double th,Suu,Svv;
double xx[40],yy[40];
int    ff[40];
int    i;
int debug = 0;

diag_covariance(cov,&th,&Suu,&Svv);

kk = -2.0*log(1.0-pr);
a = sqrt(kk*Suu);
b = sqrt(kk*Svv);
// Thin ellipses do not draw,
// so make sure they are large and wide enough to see.
if( a < 0.05 ) a = 0.05;
if( b < 0.05 ) b = 0.05;
for(i=0 ; i<40 ; i++)
  {
  xx[i] = a * cos(2*A_PI*i/39.0);
  yy[i] = b * sin(2*A_PI*i/39.0);
  }
render_polyline(x,y,-th,xx,yy,40);
}
```

12. Appendix B - Trig Integrals

The following double angle trig identity is useful:

$$c2\theta = c^2\theta - s^2\theta = 1 - 2s^2\theta = 2c^2\theta - 1$$

The following trigonometric integrals are provided for reference

$$
\begin{aligned}
&\int_0^x \cos ax\,dx = \left[\frac{1}{a}\sin ax\right]\Bigg|_0^x = \frac{1}{a}\sin ax \\
&\int_0^x \sin ax\,dx = \left[-\frac{1}{a}\cos ax\right]\Bigg|_0^x = \frac{1}{a}[1 - \cos ax] \\
&\int_0^x x\cos ax\,ds = \left[\frac{1}{a}\left(\frac{\cos ax}{a} + x\sin ax\right)\right]\Bigg|_0^x = \left[\frac{1}{a}\left(\frac{(\cos ax - 1)}{a} + x\sin ax\right)\right] \\
&\int_0^x x\sin ax\,ds = \left[\frac{1}{a}\left(\frac{\sin ax}{a} - x\cos ax\right)\right]\Bigg|_0^x = \left[\frac{1}{a}\left(\frac{\sin ax}{a} - x\cos ax\right)\right] \\
&\int_0^x \cos ax^2\,ds = \left[\frac{x}{2} + \frac{\sin 2ax}{4a}\right]\Bigg|_0^x = \left[\frac{x}{2} + \frac{\sin 2ax}{4a}\right] \\
&\int_0^x \sin ax^2\,ds = \left[\frac{x}{2} - \frac{\sin 2ax}{4a}\right]\Bigg|_0^x = \left[\frac{x}{2} - \frac{\sin 2ax}{4a}\right] \\
&\int_0^x \cos ax \sin ax\,ds = \left[\frac{\sin ax^2}{2a}\right]\Bigg|_0^x = \frac{\sin ax^2}{2a} \\
&\int_0^x x\cos ax^2\,ds = \left[\frac{x^2}{4} + x\frac{\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}\right]\Bigg|_0^x = \frac{x^2}{4} + x\frac{\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2} - \frac{1}{8a^2} \\
&\int_0^x x\sin ax^2\,ds = \left[\frac{x^2}{4} - x\frac{\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}\right]\Bigg|_0^x = \frac{x^2}{4} - x\frac{\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2} + \frac{1}{8a^2} \\
&\int_0^x x\cos ax \sin ax\,ds = \left[\frac{1}{4a}\left(\frac{\sin 2ax}{2a} - x\cos 2ax\right)\right]\Bigg|_0^x = \frac{1}{4a}\left(\frac{\sin 2ax}{2a} - x\cos 2ax\right)
\end{aligned}
\qquad \textbf{(145)}
$$

13. Appendix C - Trajectory Moments

13.1 Straight Trajectory

This section derives the moments for the straight line trajectory developed in equation (18):

$$x(t) = s(t) \qquad y(t) = 0 \qquad \theta(t) = 0$$
$$x(s) = s \qquad y(s) = 0 \qquad \theta(s) = 0 \tag{146}$$
$$x(\theta) = s(\theta) \qquad y(\theta) = 0 \qquad \theta(\theta) = 0$$

13.1.1 Spatial Moments

The spatial excursion moments are:

$$S_x = \int_0^s [s-\xi]d\xi = \frac{s^2}{2} = \frac{x^2}{2} \qquad S_y = 0$$
$$S_{xx} = \int_0^s [s-\xi]^2 d\xi = \frac{s^3}{3} = \frac{x^3}{3} \qquad S_{yy} = 0 \tag{147}$$
$$S_{xy} = 0$$

Since ω is zero, there is no rotation and the spatial rotation moments are:

$$\Theta_x = \Theta_y = \Theta_{xx} = \Theta_{yy} = \Theta_{xy} = 0 \tag{148}$$

When velocity is constant, the spatial duration moments are:

$$T_x = \int_0^t [s(t)-s(\tau)]d\tau = \frac{Vt^2}{2} = \frac{st}{2} = \frac{xt}{2} \qquad T_y = 0$$
$$T_{xx} = \int_0^t [s(t)-s(\tau)]^2 d\tau = \frac{V^2t^3}{3} = \frac{s^2t}{3} = \frac{x^2t}{3} \qquad T_{yy} = 0 \tag{149}$$
$$T_{xy} = 0$$

13.1.2 Fourier Moments

The Fourier excursion moments are:

$$S_c = \int_0^s cos\theta ds = s = x \qquad S_s = \int_0^s sin\theta ds = 0$$

$$S_{cc} = \int_0^s c^2\theta ds = s = x \qquad S_{sc} = \int_0^s s\theta c\theta ds = 0 \qquad S_{ss} = \int_0^s s^2\theta ds = 0 \tag{150}$$

$$S_{s2} = 2S_{cs} = 0 \qquad S_{c2} = S_{cc} - S_{ss} = s = x$$

Since ω is zero, there is no rotation and the Fourier rotation moments are:

$$\Theta_c = \Theta_s = \Theta_{cc} = \Theta_{ss} = \Theta_{sc} = \Theta_{s2} = \Theta_{c2} = 0 \tag{151}$$

When velocity is constant, the Fourier duration moments are:

$$T_c = \frac{S_c}{V} = t = \frac{x}{V} \qquad T_s = \frac{S_s}{V} = 0$$

$$T_{cc} = \frac{S_{cc}}{V} = t = \frac{x}{V} \qquad T_{xy} = 0 \qquad T_{ss} = \frac{S_{ss}}{V} = 0 \tag{152}$$

$$T_{s2} = \frac{S_{s2}}{V} = 0 \qquad T_{c2} = \frac{S_{c2}}{V} = t = \frac{x}{V}$$

13.1.3 Hybrid Moments

The hybrid excursion moments are:

$$S_{xc} = \int_0^s \Delta x(s, \xi) c\theta ds = \int_0^s [s-\xi] c\theta d\xi = \frac{s^2}{2} = \frac{x^2}{2}$$

$$S_{xs} = \int_0^s \Delta x(s, \xi) s\theta d\xi = 0$$

$$S_{ys} = \int_0^s \Delta y(s, \xi) s\theta d\xi = 0 \qquad \textbf{(153)}$$

$$S_{yc} = \int_0^s \Delta y(s, \xi) c\theta d\xi = 0$$

Since ω is zero, there is no rotation and the Fourier rotation moments are:

$$\Theta_{xc} = \Theta_{xs} = \Theta_{ys} = \Theta_{yc} = 0 \qquad \textbf{(154)}$$

When velocity is constant, the hybrid duration moments are:

$$T_{xc} = \frac{S_{xc}}{V} = \frac{st}{2} = \frac{x^2}{2V} \qquad T_{xs} = \frac{S_{xs}}{V} = 0$$

$$T_{yc} = \frac{S_{yc}}{V} = 0 \qquad T_{ys} = \frac{S_{ys}}{V} = 0 \qquad \textbf{(155)}$$

13.2 Arc Trajectories

13.2.1 Spatial Moments

Consider the constant curvature arc trajectory developed in equation (19):

$$\theta(s) = \kappa s \qquad \kappa \cong \frac{1}{R} \qquad (156)$$
$$x(s) = R\,sin(\kappa s)$$
$$y(s) = R[1 - cos(\kappa s)]$$

When linear and angular velocity are constant, the trajectory is:

$$\theta(t) = \omega t \qquad \omega = \frac{V}{R} = \frac{1}{T} \qquad \theta(\theta) = \theta \qquad (157)$$
$$x(t) = R\,sin(\omega t) \qquad x(\theta) = R\,sin(\theta)$$
$$y(t) = R[1 - cos(\omega t)] \qquad y(\theta) = R[1 - cos(\theta)]$$

Some useful expressions, valid on this trajectory are:

$$x/R = s\theta \qquad x^2/R^2 = s^2\theta = 1 - c^2\theta = (1 - c2\theta)/2 \qquad (158)$$
$$y/R = 1 - c\theta \qquad y^2/R^2 = (1 - c\theta)^2 = 1 - 2c\theta + c^2\theta$$
$$(xy)/R^2 = s\theta(1 - c\theta) = s\theta - s\theta c\theta = -s\theta - s2\theta/2$$

Also, some reverse sense expressions are:

$$c\theta = 1 - y/R \qquad c^2\theta = 1 - s^2\theta = 1 - (x/R)^2$$
$$s\theta = x/R \qquad c^2\theta = (y/R)^2 + 2c\theta - 1 = \text{more} \qquad (159)$$
$$c2\theta = 1 - 2x^2 \qquad s^2\theta = (x/R)^2$$
$$s2\theta = 2x(1 - y)$$

The S_x moment is:

$$S_x = R\int_0^s [sin(\kappa s) - sin(\kappa\xi)]d\xi = R\left[s\,sin(\kappa s) + \frac{cos(\kappa\xi)}{\kappa}\right]_0^s \qquad (160)$$
$$S_x = R^2[\kappa s\,sin(\kappa s) + (cos(\kappa s) - 1)] = sx(s) - Ry(s)$$

The S_y moment is:

$$S_y = R\int_0^s [cos(\kappa\xi) - cos(\kappa s)]d\xi = R\left[\frac{sin(\kappa\xi)}{\kappa} - s\,cos(\kappa s)\right]_0^s$$

$$S_y = R^2[sin(\kappa s) - \kappa s\,cos(\kappa s)] = Rx(s) + s[y(s) - R] \tag{161}$$

The S_{xx} moment is:

$$S_{xx} = R^2\int_0^s [sin(\kappa s) - sin(\kappa\xi)]^2 d\xi$$

$$S_{xx} = R^2\left\{\int_0^s [sin(\kappa s)^2 - 2sin(\kappa s)sin(\kappa\xi) + sin(\kappa\xi)^2]d\xi\right\}$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\kappa s\,sin(\kappa s)^2 + 2sin(\kappa s)cos(\kappa\xi) + \frac{\kappa\xi}{2} - \frac{sin2\kappa\xi}{4}\right]_0^s \tag{162}$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\kappa s\,sin(\kappa s)^2 + 2sin(\kappa s)cos(\kappa s) - 2sin(\kappa s) + \frac{\kappa s}{2} - \frac{sin2\kappa s}{4}\right]$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\kappa s\frac{(1 + 2sin(\kappa s)^2)}{2} + \frac{3}{4}sin2\kappa s - 2sin(\kappa s)\right]$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\kappa s\left(1 - \frac{cos2\kappa s}{2}\right) + \frac{3}{4}sin2\kappa s - 2sin(\kappa s)\right]$$

Using earlier rewrites in terms of the endpoint coordinates, we can write:

$$S_{xx} = \frac{R^2}{\kappa}\left[\kappa s\,sin(\kappa s)^2 + 2sin(\kappa s)cos(\kappa s) - 2sin(\kappa s) + \frac{\kappa s}{2} - \frac{sin2\kappa s}{4}\right]$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\frac{\kappa s}{2}(1 + 2sin(\kappa s)^2) + \frac{3}{2}sin\kappa s\,cos(\kappa s) - 2sin(\kappa s)\right] \tag{163}$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\frac{\theta}{2}(1 + 2sin\theta^2) + \frac{3}{2}sin\theta\,cos\theta - 2sin\theta\right]$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\frac{\theta}{2}\left(1 + 2\frac{x^2}{R^2}\right) + \frac{3}{2}\frac{x}{R}\left(1 - \frac{y}{R}\right) - 2\frac{x}{R}\right]$$

$$S_{xx} = \frac{R^2}{\kappa}\left[\theta\left(\frac{1}{2} + \frac{x^2}{R^2}\right) - \frac{x}{R}\left(\frac{1}{2} + \frac{3}{2}\frac{y}{R}\right)\right]$$

$$S_{xx} = \frac{1}{\kappa}\left[\theta\left(\frac{R^2}{2} + x^2\right) - x\left(\frac{R}{2} - \frac{3y}{2}\right)\right]$$

$$S_{xx} = \left[s\left(\frac{R^2}{2} + x^2\right) - Rx\left(\frac{R}{2} - \frac{3y}{2}\right)\right]$$

The S_{yy} moment is:

$$S_{yy} = R^2 \int_0^s [cos(\kappa\xi) - cos(\kappa s)]^2 d\xi$$

$$S_{yy} = R^2 \left\{ \int_0^s [cos(\kappa\xi)^2 - 2cos(\kappa\xi)cos(\kappa s) + cos(\kappa s)^2] d\xi \right\}$$

$$S_{yy} = \frac{R^2}{\kappa}\left[\frac{\kappa s}{2} + \frac{sin 2\kappa s}{4} - 2sin(\kappa\xi)cos(\kappa s) + \kappa s\, cos(\kappa s)^2\right]_0^s \qquad \textbf{(164)}$$

$$S_{yy} = \frac{R^2}{\kappa}\left[\kappa s\left(\frac{1 + 2cos(\kappa s)^2}{2}\right) - \frac{3}{4}sin 2\kappa s\right]$$

$$S_{yy} = \frac{R^2}{\kappa}\left[\kappa s\left(1 + \frac{cos 2\kappa s}{2}\right) - \frac{3}{4}sin 2\kappa s\right]$$

$$S_{yy} = \frac{R^2}{\kappa}\left[\kappa s\left(1 + \frac{cos 2\kappa s}{2}\right) - \frac{3}{4}sin 2\kappa s\right]$$

Using earlier rewrites in terms of the endpoint coordinates, we can write:

$$S_{yy} = \frac{R^2}{\kappa}\left[\kappa s\left(\frac{1 + 2cos(\kappa s)^2}{2}\right) - \frac{3}{4}sin 2\kappa s\right]$$

$$S_{yy} = \frac{R^2}{\kappa}\left[\frac{\kappa s}{2}\left(1 + 2\left(1 - \frac{y}{R}\right)^2\right) - \frac{3}{2}\frac{x}{R}\left(1 - \frac{y}{R}\right)\right] \qquad \textbf{(165)}$$

$$S_{yy} = \frac{1}{\kappa}\left[\frac{\kappa s}{2}(R^2 + 2(R - y)^2) - \frac{3}{2}x(R - y)\right]$$

$$S_{yy} = \left[s\left(\frac{R^2}{2} + (R - y)^2\right) - 3xR\left(\frac{R}{2} - \frac{y}{2}\right)\right]$$

The S_{xy} moment is:

$$S_{xy} = R^2\int_0^s [sin(\kappa s) - sin(\kappa\xi)][cos(\kappa\xi) - cos(\kappa s)]d\xi$$

$$S_{xy} = R^2\left\{\int_0^s [sin(\kappa s)cos(\kappa\xi) - sin(\kappa s)cos(\kappa s) - sin(\kappa\xi)cos(\kappa\xi) + sin(\kappa\xi)cos(\kappa s)]d\xi\right\}$$

$$S_{xy} = \frac{R^2}{\kappa}\left[sin(\kappa s)sin(\kappa\xi) - \kappa s\, sin(\kappa s)cos(\kappa s) - \frac{sin\kappa\xi^2}{2} - cos(\kappa\xi)cos(\kappa s)\right]_0^s \qquad \textbf{(166)}$$

$$S_{xy} = \frac{R^2}{\kappa}\left[sin(\kappa s)sin(\kappa s) - \kappa s\, sin(\kappa s)cos(\kappa s) - \frac{sin\kappa s^2}{2} - cos(\kappa s)cos(\kappa s) + cos(\kappa s)\right]$$

$$S_{xy} = \frac{R^2}{\kappa}\left[-\frac{\kappa s}{2}sin(2\kappa s) - \frac{sin\kappa s^2}{2} - cos(2\kappa s) + cos(\kappa s)\right]$$

$$S_{xy} = \frac{R^2}{\kappa}\left[-\frac{\kappa s}{2}sin(2\kappa s) - \frac{1 - cos(2\kappa s)}{4} - cos(2\kappa s) + cos(\kappa s)\right]$$

$$S_{xy} = \frac{R^2}{\kappa}\left[-\frac{\kappa s}{2}sin(2\kappa s) - \frac{3}{4}cos(2\kappa s) + cos(\kappa s) - \frac{1}{4}\right]$$

Using earlier rewrites in terms of the endpoint coordinates, we can write:

$$S_{xy} = \frac{R^2}{\kappa}\left[-\frac{\kappa s}{2}sin(2\kappa s) - \frac{sin\kappa s^2}{2} - cos(2\kappa s) + cos(\kappa s)\right]$$

$$S_{xy} = \frac{R^2}{\kappa}\left[-\Theta\frac{x}{R}\left(1 - \frac{y}{R}\right) - \frac{x^2}{2R^2} - \left(1 - 2\frac{x^2}{R^2}\right) + \left(1 - \frac{y}{R}\right)\right]$$

$$S_{xy} = \frac{R^2}{\kappa}\left[-\Theta\frac{x}{R} + \Theta\frac{xy}{R^2} + \frac{3x^2}{2R^2} - 1 + 1 - \frac{y}{R}\right] \qquad \textbf{(167)}$$

$$S_{xy} = \frac{R^2}{\kappa}\left[\frac{y}{R}\left(\frac{x}{R}\Theta - 1\right) + \frac{x}{R}\left(\frac{3}{2}\frac{x}{R} - \theta\right)\right]$$

$$S_{xy} = \left[y(xs - R^2) + xR\left(\frac{3}{2}x - s\right)\right]$$

$$S_{xy} = xs(y - R) + R\left(\frac{3}{2}x^2 - Ry\right)$$

The spatial rotation moments are:

$$\Theta_x = \kappa S_x = R[\theta sin(\theta) + (cos(\theta) - 1)] = \theta x(\theta) - y(\theta)$$

$$\Theta_y = \kappa S_y = R[sin(\theta) - \theta cos(\theta)] = x(\theta) + \theta[y(\theta) - R]$$

$$\Theta_{xx} = \kappa S_{xx} = R^2\left[\theta\left(1 - \frac{cos2\theta}{2}\right) + \frac{3}{4}sin2\theta - 2sin\theta\right] = \theta\left(\frac{R^2}{2} + x^2\right) - x\left(\frac{R}{2} - \frac{3y}{2}\right) \quad \textbf{(168)}$$

$$\Theta_{yy} = \kappa S_{yy} = R^2\left[\theta\left(1 + \frac{cos2\theta}{2}\right) - \frac{3}{4}sin2\theta\right] = \left[\theta\left(\frac{R^2}{2} + (R-y)^2\right) - 3x\left(\frac{R}{2} - \frac{y}{2}\right)\right]$$

$$\Theta_{xy} = \kappa S_{xy} = R^2\left[-\frac{\theta}{2}sin(2\theta) - \frac{3}{4}cos(2\theta) + cos(\theta) - \frac{1}{4}\right] = x\theta(y-R) + \left(\frac{3}{2}x^2 - Ry\right)$$

When linear and angular velocity are constant, the spatial duration moments are:

$$T_x = \frac{S_x}{V} = \frac{R}{\omega}[\omega t sin(\omega t) + (cos(\omega t) - 1)] = tx(t) - Ty(t)$$

$$T_y = \frac{S_y}{V} = \frac{R}{\omega}[sin(\omega t) - \omega t cos(\omega t)] = Tx(t) + t[y(t) - R]$$

$$T_{xx} = \frac{S_{xx}}{V} = \frac{R^2}{\omega}\left[\omega t\left(1 - \frac{cos2\omega t}{2}\right) + \frac{3}{4}sin2\omega t - 2sin\omega t\right] = t\left(\frac{R^2}{2} + x^2\right) - Tx\left(\frac{R}{2} - \frac{3y}{2}\right) \quad \textbf{(169)}$$

$$T_{yy} = \frac{S_{yy}}{V} = \frac{R^2}{\omega}\left[\omega t\left(1 + \frac{cos2\omega t}{2}\right) - \frac{3}{4}sin2\omega t\right] = \left[t\left(\frac{R^2}{2} + (R-y)^2\right) - 3Tx\left(\frac{R}{2} - \frac{y}{2}\right)\right]$$

$$T_{xy} = \frac{S_{xy}}{V} = \frac{R^2}{\omega}\left[-\frac{\omega t}{2}sin(2\omega t) - \frac{3}{4}cos(2\omega t) + cos(\omega t) - \frac{1}{4}\right] = xt(y-R) + T\left(\frac{3}{2}x^2 - Ry\right)$$

13.2.2 Fourier Moments

The Fourier excursion moments are:

$$
\begin{aligned}
S_c &= \int_0^s c\theta ds = R\int_0^\theta c\theta d\theta = Rs\theta = x \\
S_s &= \int_0^s s\theta ds = R\int_0^\theta c\theta d\theta = R(1-c\theta) = y \\
S_{cc} &= \int_0^s c^2\theta ds = R\int_0^\theta c^2\theta d\theta = R\left(\frac{\theta}{2}+\frac{s\theta c\theta}{2}\right) = R\left[\frac{\theta}{2}+\frac{s2\theta}{4}\right] = \frac{s}{2}+\frac{x}{2R}(R-y) \\
S_{ss} &= \int_0^s s^2\theta ds = R\int_0^\theta s^2\theta d\theta = R\left(\frac{\theta}{2}-\frac{s\theta c\theta}{2}\right) = R\left[\frac{\theta}{2}-\frac{s2\theta}{4}\right] = \frac{s}{2}-\frac{x}{2R}(R-y) \\
S_{sc} &= \int_0^s s\theta c\theta ds = R\int_0^\theta s\theta c\theta ds = \frac{Rs^2\theta}{2} = R\left(\frac{1}{4}-\frac{c2\theta}{4}\right) = \frac{x^2}{(2R)} \\
S_{s2} &= 2S_{sc} = s^2\theta = R\left(\frac{1}{2}-\frac{c2\theta}{2}\right) = \frac{x^2}{R} \\
S_{c2} &= S_{cc}-S_{ss} = Rs\theta c\theta = \frac{Rs2\theta}{2} = \frac{x}{R}(R-y)
\end{aligned}
\tag{170}
$$

The Fourier rotation moments are:

$$
\begin{aligned}
\Theta_c &= \kappa S_c = s\theta = \frac{x}{R} \\
\Theta_s &= \kappa S_s = (1-c\theta) = \frac{y}{R} \\
\Theta_{cc} &= \kappa S_{cc} = \left[\frac{\theta}{2}+\frac{s2\theta}{4}\right] = \frac{\theta}{2}+\frac{x}{2R}\left(1-\frac{y}{R}\right) \\
\Theta_{ss} &= \kappa S_{ss} = \left[\frac{\theta}{2}-\frac{s2\theta}{4}\right] = \frac{\theta}{2}-\frac{x}{2R}\left(1-\frac{y}{R}\right) \\
\Theta_{sc} &= \kappa S_{sc} = \frac{s^2\theta}{2} = \frac{x^2}{(2R^2)} \\
\Theta_{s2} &= \kappa S_{s2} = s^2\theta = \frac{x^2}{R^2} \\
\Theta_{c2} &= \kappa S_{c2} = \frac{s2\theta}{2} = \left(\frac{x}{R}\right)\left(1-\frac{y}{R}\right)
\end{aligned}
\tag{171}
$$

When linear and angular velocity are constant, the Fourier duration moments are:

$$
\begin{aligned}
T_c &= S_c/V = Ts\theta = \frac{Tx}{R} \\
T_s &= S_s/V = T(1 - c\theta) = \frac{Ty}{R} \\
T_{cc} &= S_{cc}/V = T\left[\frac{\theta}{2} + \frac{s2\theta}{4}\right] = \frac{T}{2}\left[\frac{s}{R} + \frac{x}{R}\left(1 - \frac{y}{R}\right)\right] \\
T_{ss} &= S_{ss}/V = T\left[\frac{\theta}{2} - \frac{s2\theta}{2}\right] = \frac{T}{2}\left[\frac{s}{R} - \frac{x}{R}\left(1 - \frac{y}{R}\right)\right] \\
T_{sc} &= S_{sc}/V = T\left(\frac{s^2\theta}{2}\right) = \frac{T}{2}\left(\frac{x^2}{R^2}\right) \\
T_{s2} &= S_{s2}/V = T(s^2\theta) = (T)\left(\frac{x^2}{R^2}\right) \\
T_{c2} &= S_{c2}/V = T\frac{s2\theta}{2} = \left(\frac{T}{2}\right)\left(\frac{x}{R}\right)\left(1 - \frac{y}{R}\right)
\end{aligned}
\tag{172}
$$

13.2.3 Hybrid Moments

The S_{xc} moment is:

$$
\begin{aligned}
S_{xc} &= \int_0^s \Delta x(s,\xi) c\theta d\xi = R\int_0^s [\mathit{sin}(\kappa s) - \mathit{sin}(\kappa\xi)]\mathit{cos}(\kappa\xi) d\xi \\
S_{xc} &= R\,\mathit{sin}(\kappa s)\int_0^s \mathit{cos}(\kappa\xi) d\xi - R\int_0^s \mathit{sin}(\kappa\xi)\mathit{cos}(\kappa\xi) d\xi \\
S_{xc} &= R\,\mathit{sin}(\kappa s) S_c(s) - RS_{sc}(s) = R[\mathit{sin}(\kappa s) R\,\mathit{sin}(\kappa s) - R\,\mathit{sin}(\kappa s)^2/2] \\
S_{xc} &= \frac{1}{2}R^2 \mathit{sin}(\kappa s)^2 = \frac{x^2}{2}
\end{aligned}
\tag{173}
$$

The S_{xs} moment is:

$$S_{xs} = \int_0^s \Delta x(s,\xi) s\theta d\xi = R\int_0^s [sin(\kappa s) - sin(\kappa\xi)] sin(\kappa\xi) d\xi$$

$$S_{xs} = R sin(\kappa s)\int_0^s sin(\kappa\xi) d\xi - R\int_0^s sin(\kappa\xi) sin(\kappa\xi) d\xi = R sin(\kappa s) S_s(s) - R S_{ss}(s)$$

$$S_{xs} = R sin(\kappa s) R(1 - cos(\kappa s)) - R^2(\kappa s - sin(\kappa s) cos(\kappa s))/2 \tag{174}$$

$$S_{xs} = R^2\left\{ sin(\kappa s) - sin(\kappa s) cos(\kappa s) - \frac{\kappa s}{2} + \frac{sin(\kappa s) cos(\kappa s)}{2} \right\}$$

$$S_{xs} = R^2\left\{ -\frac{\kappa s}{2} + sin(\kappa s) - \frac{1}{2} sin(\kappa s) cos(\kappa s) \right\} = R^2\left\{ -\frac{\kappa s}{2} + sin(\kappa s) - \frac{sin(2\kappa s)}{4} \right\}$$

$$S_{xs} = R\left(-\frac{s}{2} + x\right) - \frac{x}{2}(R - y)$$

The S_{yc} moment is:

$$S_{yc} = \int_0^s \Delta y(s,\xi) c\theta d\xi = R\int_0^s [cos(\kappa\xi) - cos(\kappa s)] cos(\kappa\xi) d\xi$$

$$S_{yc} = -R cos(\kappa s)\int_0^s cos(\kappa\xi) d\xi + R\int_0^s cos(\kappa\xi) cos(\kappa\xi) d\xi$$

$$S_{yc} = -R cos(\kappa s) S_c(s) + R S_{cc}(s)$$

$$S_{yc} = R[-cos(\kappa s) R sin(\kappa s) + R(\kappa s + sin(\kappa s) cos(\kappa s))/2]$$

$$S_{yc} = R^2\left\{ \frac{\kappa s}{2} - \frac{1}{2} sin(\kappa s) cos(\kappa s) \right\} = R^2\left\{ \frac{\kappa s}{2} - \frac{sin(2\kappa s)}{4} \right\} \tag{175}$$

$$S_{yc} = \frac{x}{2}(R - y) + R\left(\frac{s}{2}\right)$$

The S_{ys} moment is:

$$S_{ys} = \int_0^s \Delta y(s,\xi)s\theta d\xi = R\int_0^s [cos(\kappa\xi) - cos(\kappa s)]sin(\kappa\xi)d\xi$$

$$S_{ys} = -Rcos(\kappa s)\int_0^s sin(\kappa\xi)d\xi + R\int_0^s cos(\kappa\xi)sin(\kappa\xi)d\xi$$

$$S_{ys} = -Rcos(\kappa s)S_s(s) + RS_{sc}(s)$$

$$S_{ys} = -Rcos(\kappa s)R(1 - cos(\kappa s)) + R^2 sin(\kappa s)^2/2$$

$$S_{ys} = R^2\left\{-cos(\kappa s) + cos(\kappa s)^2 + \frac{sin(\kappa s)^2}{2}\right\} \tag{176}$$

$$S_{ys} = \frac{R^2}{2}\{-2cos(\kappa s) + 2cos(\kappa s)^2 + sin(\kappa s)^2\}$$

$$S_{ys} = \frac{R^2}{2}\{cos(\kappa s)^2 - 2cos(\kappa s) + 1\}$$

$$S_{ys} = \frac{R^2}{2}\{cos(\kappa s) - 1\}^2 = \frac{y^2}{2}$$

The hybrid rotation moments are:

$$\Theta_{xc} = \kappa S_{xc} = R\frac{sin(\kappa s)^2}{2} = \frac{x^2}{2R}$$

$$\Theta_{xs} = \kappa S_{xs} = R\left\{-\frac{\theta}{2} + sin(\theta) - \frac{sin2\theta}{4}\right\} = R\left(-\frac{s}{2} + x\right) - \frac{x}{2}(R - y)$$

$$\Theta_{yc} = \kappa S_{yc} = R\left\{\frac{\theta}{2} - \frac{sin(2\theta)}{4}\right\} = \frac{x}{2}(R - y) + R\left(\frac{s}{2}\right) \tag{177}$$

$$\Theta_{ys} = \kappa S_{ys} = \frac{R}{2}\{cos(\kappa s) - 1\}^2 = \frac{y^2}{2R}$$

The hybrid duration moments are:

$$T_{xc} = \frac{S_{xc}}{V} = \frac{TR\,sin(\omega t)^2}{2} = \frac{x^2}{2V}$$

$$T_{xs} = \frac{S_{xs}}{V} = TR\left\{-\frac{\omega t}{2} + sin(\omega t) - \frac{sin 2\omega t}{4}\right\} = T\left(-\frac{s}{2} + x\right) - \frac{x}{2V}(R - y)$$

$$T_{yc} = \frac{S_{yc}}{V} = TR\left\{\frac{\omega t}{2} - \frac{sin(2\omega t)}{4}\right\} = \frac{x}{2V}(R - y) + T\left(\frac{s}{2}\right) \tag{178}$$

$$T_{ys} = \frac{S_{ys}}{V} = \frac{TR}{2}\{cos(\omega t) - 1\}^2 = \frac{y^2}{2V}$$

14. References

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