# 15-251: Great Theoretical Ideas In Computer Science

## Recitation 10

## Probability Review

- The sample space $\Omega$ is the set of all outcomes, each of which has some nonnegative probability, and the sum of these probabilities is equal to 1.
- An event is a subset of outcomes.
- A conditional probability $Pr(A \mid B) = \dfrac{Pr(A \cap B)}{Pr(B)}$
- The Law of Total Probability states that given an event $A$ and a partition of the sample space $B_1, \cdots B_k$, $Pr(A) = \sum_{i=1}^{k} Pr(A \mid B_i)Pr(B_i)$
- Two events are independent if $Pr(A \cap B) = Pr(A)Pr(B)$, or if $Pr(A \mid B) = Pr(A)$, or if $Pr(B \mid A) = Pr(B)$. The latter two definitions require nonzero probability of what you condition on.
- A random variable $X$ is a function from $\Omega \to \mathbb{R}$.
- Random variables $X, Y$ are independent if for all $x, y \in \mathbb{R}$, events $X = x$ and $Y = y$ are independent.
- An indicator random variable for an event $A$ is 1 when $A$ happens and 0 otherwise.
- The expected value of a random variable $X$ is $\sum_{l \in \Omega} Pr(l)X(l)$
- If $X = \sum_{i=1}^{k} X_i$ for random variables $X_i$, linearity of expectation states that $\mathbb{E}[X] = \sum_{i=1}^{k} \mathbb{E}[X_i]$

## Expected Cost

Suppose the numbers from 1 to $n$ are given to you in some order. You need to keep track of the minimum of the numbers you've seen so far. If the minimum changes, it costs \$1.

(a) What is the best possible cost? Worst?

(b) If the permutation of 1 to $n$ is chosen uniformly at random, what is the expected cost of keeping track of the minimum.

## A tournament

$64$ teams compete in a single-elimination tournament. You are in a betting pool, wherein you need to pick the winners of each of the $63$ games before the tournament begins. You get $32$ points for picking the overall winner correctly, $16$ points for each correctly-picked finalist, ... etc..., and $1$ point for each first-round game picked correctly. Since you know absolutely nothing about basketball, you make all your picks by tossing a fair coin. What is the expected number of points you'll get?

## Geometric Distributions

Let $X \sim$ Geometric($p$) and $Y \sim$ Geometric($q$) be independent (for $0 < p, q < 1$).

(a) Compute $Pr[X = Y]$.

(b) Compute $Pr[\min(X, Y) = k]$ (for $k \in \mathbb{N}^+$).

(c) Compute $\mathbb{E}[\max(X, Y)]$.

## Monte Carlo Algorithm

(a) We are interested in the answer to a certain Yes or No question. Herman the Wise knows the correct answer, but is a little mischievous. We ask him the question $6n$ times, and each time he gives the correct answer with probability $q$, where $q$ is some probability at least $3/4$. (Furthermore, he does this independently for each of the $6n$ questions.) Show that if we pick the more common answer he gave out of the $6n$ trials, then this will be the correct answer except with probability at most $O(2^{-n})$.
Hint: All you need is basic arithmetic and the fact that the number of subsets of a set of size $6n$ is $2^{6n}$. You do not need anything sophisticated about binomial coefficients.

(b) Let $A$ be a polynomial time Monte Carlo algorithm that solves some decision problem with error probability at most $1/4$. That is, for every input, there's at most a $1/4$ chance that the algorithm will output the wrong answer. Design a new polynomial time algorithm $A'$ that solves the same decision problem but has error probability at most $O(2^{-n})$.