CMU 15-251 **Fall 2017**

Lecture 26: Modular Arithmetic
Handout Notes

Interesting Things About Modular Arithmetic

State 3 of them:

The operations we will study in the modular world:

1.
2.
3.
4.
5.
6.
7.

1 Complexity of Operations in Integers

	Poly-time?	Algorithm	Additional notes
Addition			
Subtraction			
Multiplication			
Division			
Exponentiation			
Taking roots			
Taking logs			
Factorization	Don't know	Best one is exponential time	Want it to be computationally hard for cyrpto
isPrime	Yes	Miller-Rabin Monte Carlo alg.	A poly-time deterministic algorithm is also known
Generating n-bit prime	Yes	Random sampling + isPrime	No poly-time deterministic algorithm is known

2 Modular Arithmetic: Basic Definitions and Properties

Notation: "A is congruent to B modulo N":

Fact/Exercise: $A \equiv_N B$ if and only if N divides $A - B$.

Notation: $\mathbb{Z}_N =$

2.1 Addition

Definition ["plus" in $\mathbb{Z}_N$]:

Addition table for $\mathbb{Z}_5$

+	0	1	2	3	4
0					
1					
2					
3					
4					

What is the additive identity?

2.2 Subtraction

Definition ["additive inverse" in $\mathbb{Z}_N$]:

Definition ["minus" in $\mathbb{Z}_N$]:

For every $A \in \mathbb{Z}_N$, $-A$ exists (why?)

$$\Longrightarrow$$

Every row of the addition table of $\mathbb{Z}_N$ is a permutation of $\mathbb{Z}_N$.

2.3 Multiplication

Definition [“multiplication” in $\mathbb{Z}_N$]:

Multiplication table for $\mathbb{Z}_5$

•	0	1	2	3	4
0					
1					
2					
3					
4					

What is the multiplicative identity?

2.4 Division

Definition [“multiplicative inverse” in $\mathbb{Z}_N$]:

Definition [“division” in $\mathbb{Z}_N$]:

Is it true that for every $A \in \mathbb{Z}_N$, A^{-1} exists?

In $\mathbb{Z}_6$, which elements have a multiplicative inverse?

Fact: $A^{-1} \in \mathbb{Z}_N$ exists if and only if

Definition: $\mathbb{Z}_N^* =$

Definition: $\varphi(N) =$

Multiplication table for $\mathbb{Z}_8^*$

•	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

For every $A \in \mathbb{Z}_N^*$, A^{-1} exists

$$\Longrightarrow$$

Every row of the multiplication table of $\mathbb{Z}_N^*$ is a permutation of $\mathbb{Z}_N^*$.

2.5 Exponentiation (in particular in $\mathbb{Z}_N^*$)

Notation: For $A \in \mathbb{Z}_N$, $E \in \mathbb{N}$, $A^E =$

What is a **generator** in $\mathbb{Z}_N^*$?

Theorem [Euler's Theorem]:

What is Fermat's Little Theorem?

Using Euler's Theorem, compute $213^{248} \bmod 7$:

IMPORTANT NOTE:

When exponentiating elements in $\mathbb{Z}_N^*$,

3 Complexity of Operations Modulo N

	Poly-time?	Algorithm	Additional notes
Addition			
Subtraction			
Multiplication			
Division			
Exponentiation			
Taking roots			
Taking logs			

Additional notes for division (computing B^{-1}):