15-251: Great Theoretical Ideas In Computer Science

Recitation 4

Announcements

Be sure to take advantage of the following resources :

- Homework Solution Sessions - Saturday and Sunday 2:30-3:30 in GHC 4301
- Common Mistakes & Grading Rubrics - check Piazza!

These Decidable Definitions Have Undecidable Ends

- A **decider** is a TM that halts on all inputs.
- A language L is **undecidable** if there is no TM M that halts on all inputs such that $M(x)$ accepts if and only if $x \in L$.
- A language A **reduces** to B if it is possible to decide A using an algorithm that decides B as a subroutine. Denote this as $A \leq B$ (read: B is *at least* as hard as A)
- Countability cheat sheet : You are given a set A. Is it countable or uncountable?

$|A| \leq |\mathbb{N}|$ (A is countable)

- Show directly an injection from A to $\mathbb{N}$ ($A \hookrightarrow \mathbb{N}$) or a surjection from $\mathbb{N}$ onto A ($\mathbb{N} \twoheadrightarrow A$)
- Show $|A| \leq |B|$, where B is one of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Q}$, $\boxed{\Sigma^*}$[a], $\mathbb{Q}[x]$, etc.

[a] This one is important and very powerful

$|A| > |\mathbb{N}|$ (A is uncountable)

- Show directly using a diagonalization argument.
- Show that $|\{0,1\}^\infty| \leq |A|$, i.e. an injection from $\{0,1\}^\infty$ to A.

Counting sheep

For each set below, determine if it is countable or not. Prove your answers.

(a) $S = \{a_1a_2a_3 \ldots \in \{0,1\}^\infty \mid \forall n \geq 1$ the string $a_1 \ldots a_n$ contains more 1's than 0's.$\}$.

(b) Σ^*, where Σ is an alphabet that is allowed to be countably infinite (e.g., $\Sigma = \mathbb{N}$).

Doesn't Look Like Anything (Decidable) To Me

Prove that the following languages are undecidable (below, M, M_1, M_2 refer to TMs).

(a) $\textbf{REGULAR} = \{\langle M \rangle : L(M)$ is regular$\}$.

(b) $\textbf{TOTAL} = \{\langle M \rangle : M$ halts on all inputs$\}$.

(c) $\textbf{DOLORES} = \{\langle M_1, M_2 \rangle : \exists w \in \Sigma^*$ such that both $M_1(w)$ and $M_2(w)$ accept$\}$.

(Extra) Lose All Scripted Responses. Improvisation Only

Let $\mathbf{FINITE} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is finite}\}$.

Show that $\mathbf{TOTAL} \leq \mathbf{FINITE}$.

(Bonus) The Maize is not Meant For You

Josh Corn is trying to write a program P such that given a natural number n, $P(n)$ is the most number of steps a TM on n states can take before halting. Show that this is not possible.