15-251: Great Theoretical Ideas In Computer Science

Recitation 7

- A **matching** in G is a subset of G's edges which share no vertices.
  A **maximal** matching is one which isn't a subset of any other matching.
  A **maximum** matching is a matching which is at least as large as any possible matching.
  A **perfect** matching is a matching such that every vertex is contained in one of its edges.
- An **alternating path** (with respect to some matching M) is one which alternates between edges in M and edges not in M.
  An **augmenting path** is an alternating path which begins and ends with vertices not matched in M.
- An **unstable pair** is a pair who prefer each other to their assigned partners.
- A **stable matching** is a perfect matching (includes all vertices) which contains no unstable pairs.
- Gale Shapley algorithm on sets A (men) and B (women):
  While there is a man, $m \in A$ who is not matched
  (a) Let $w \in B$ be the highest ranked woman in m's list whom he hasn't proposed to yet.
  (b) If w is unmatched: match w and m.
  (c) If w prefers m to her current match, match w and m

A Theorem about Corridors

Recall from lecture, Hall's Theorem:
For any bipartite graph $G = (X, Y, E)$, where G has a matching covering all the vertices of X iff for every $S \subseteq X$, $|S| \leq |N(S)|$ (where $N(S) = \{y \in Y \mid \exists\, x \in S. \{x, y\} \in E\}$). Prove Hall's Theorem.

A Misogynist Algorithm

(a) Prove that the Gale-Shapley algorithm always matches every guy with his best valid partner. That is, show that every guy prefers the girl he is paired with by the Gale-Shapley algorithm at least as much as any girl he is paired with in any other stable matching.

(b) Prove that the Gale-Shapley algorithm always matches every girl with her worst valid partner. That is, show that in any other stable matching, each girl is paired with a guy she likes at least as much as the one she is paired with by Gale Shapley.

(Extra) Soulmates

Call a man m and a woman w "soulmates" if they are paired with each other in every stable matching.

(a) Given a man m and a woman w, design a polynomial-time algorithm to determine if they are soulmates.

(b) Give a polynomial time algorithm to determine if an instance of the stable matching problem has a *unique* stable matching.

(Extra) Counting Couples

(a) Find, with proof, the maximum possible number of perfect matchings in a graph on n vertices.

(b) Find, with proof, the maximum possible number of perfect matchings in a *bipartite* graph on n vertices.

(c) Find a way to construct an instance of the stable marriage problem with n men and n women which has at least n stable matchings (Tight bounds on the number of stable matchings for n pairs of men and women are not known).

(Bonus) A Theorem About Egyptian Kings

Prove the following theorem: A (not-necessarily bipartite) graph $G = (V, E)$ has a perfect matching if and only if for every $S \subseteq V$, the number of connected components of $G \setminus S$ with an odd number of vertices is at most $|S|$. ($G \setminus S$ is G with all the vertices of S and all edges incident to them removed)