Great Ideas in Theoretical CS

Lecture 12:
Graphs II: Basic Algorithms

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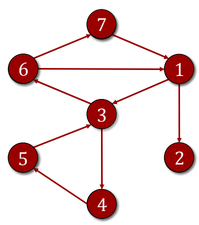
Depth-First Search

- For each unexplored $u \in V$
  - DFS(G,u)

DFS(graph G, $u \in V$)

- mark u as explored
- for each $\{u, v\} \in E$
  - if v is unexplored then DFS(G,v)



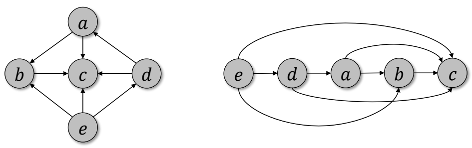
Running time
$O(m+n)$

Graph Search Problems

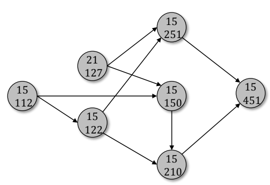
- Given a graph G
  - Check if there is a path between two given vertices s and t
  - Decide if G is connected
  - Identify the connected components of G
- All these problems can be solved directly using any kind of vertex traversal, including DFS

Topological Sorting

- A topological order of a directed graph G is a bijection $f: V \to \{1, \dots, n\}$ such that if $(u, v) \in E$ then $f(u) < f(v)$



Topological Sorting



Topological Sorting

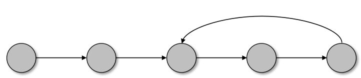
- An undirected graph is a clique iff for all distinct $u, v \in V$, $\{u, v\} \in E$
- Poll 1: Which of the following undirected graphs can have an orientation that does not admit a topological sorting?
  1. Tree
  2. Clique
  3. Both
  4. Neither

Topological Sorting

- Clearly if a graph has a cycle then it does not have a topological order
- We will give an algorithm that finds a topological order given any directed acyclic graph
- A sink vertex is a vertex with no outgoing edges
- Lemma: Every directed acyclic graph has a sink vertex

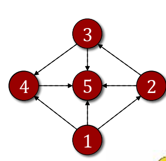
Proof of Lemma

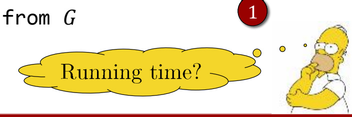
- Suppose for contradiction that every vertex has an outgoing edge
- By following the outgoing edges, after at most n steps we must revisit an vertex we've already seen, leading to a cycle! ■



Naïve Algorithm

- $p \leftarrow n$
- while $p \geq 1$
  - If the graph doesn't have a sink then return "not acyclic"
  - else find a sink v and remove it from G
  - $f(v) \leftarrow p$
  - $p \leftarrow p - 1$



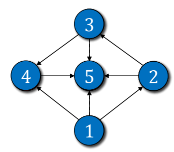


Better Algorithm Via DFS

- $p \leftarrow n$
- For each unexplored $u \in V$, DFS(G,u)

DFS(graph G, $u \in V$)

- mark u as explored
- for each $\{u, v\} \in E$, if v is unexplored then DFS(G,v)
- $f(u) \leftarrow p$
- $p \leftarrow p - 1$



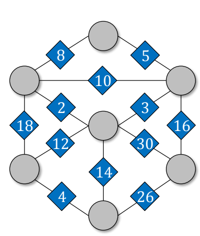
Correctness

- Theorem: If G is acyclic and $(u, v) \in E$ then $f(u) < f(v)$
- Proof: We consider two cases
  - Case 1: u is discovered before v, then because $(u, v) \in E$, v will be explored before DFS(G, u) returns
  - Case 2: v is discovered before u, then we cannot discover u from DFS(G, v) because that would imply a cycle, so DFS(G, u) is run after DFS(G, v) terminates ■

Weighted Graphs

- It is often useful to consider graphs with
  - weights
  - lengths
  - distances
  - costs

  associated to their edges
- Model as a cost function $c: E \to \mathbb{R}^+$



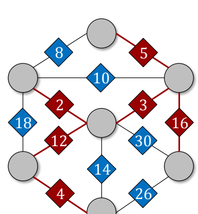
Minimum Spanning Tree

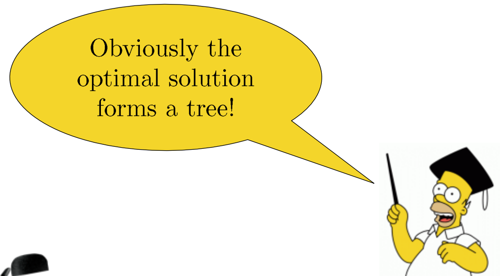
The year: 1926
The place: Brno, Moravia
Our hero: Otakar Borůvka

Borůvka's had a pal called Jindřich Saxel who worked for Západomoravské elektrárny (the West Moravian Power Plant company). Saxel asked him how to figure out the most efficient way to electrify southwest Moravia.

Minimum Spanning Tree

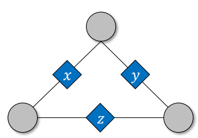
- MST problem:
  - Input: Graph G, cost function $c: E \to \mathbb{R}^+$
  - Output: $E' \subseteq E$ such that (V, E') is connected and $\sum_{e \in E'} c(e)$ is minimized
- Example: The MST has cost 42

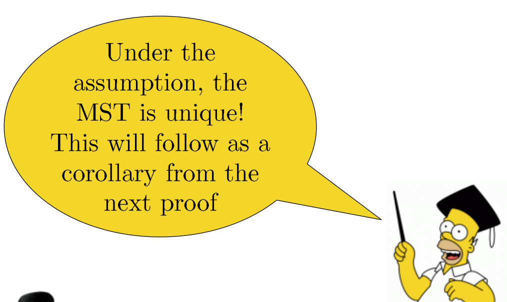




Number of MSTs

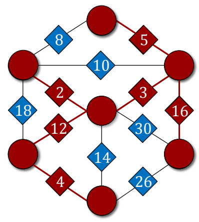
- Assumption (for convenience): Edges have distinct weights
- Poll 2: What is the max #MSTs that a 3-clique can have?
    1. 1
    2. 2
    3. 3
    4. 4





Prim's Algorithm

- $V' \leftarrow$arbitrary $\{u\}$, $E' \leftarrow \emptyset$
- While $V' \neq V$
  - Let (u, v) be a minimum cost edge such that $u \in V'$, $v \notin V'$
  - $E' \leftarrow E \cup \{\{u, v\}\}$
  - $V' \leftarrow V' \cup \{v\}$



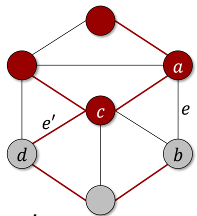
Running time? It's clearly polynomial, and that's surprising!

Proof of Correctness

- Fix an MST T; we will show that for every $0 \leq k \leq n$, the first k edges added by the alg are in T
- The proof is by induction
- Base case ($k = 0$) is vacuously true
- Induction step: Suppose the algorithm has added k edges so far that are in the MST; show that next edge is also in the MST

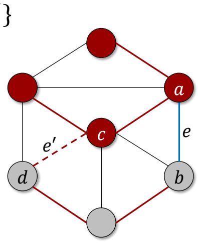
Proof of Correctness

- Consider the current V'
- Let $e = \{a, b\}$ be the next edge added by the alg
- Suppose e is not in the MST T (shown in red)
- T has a path $a \to b$
- Let $e' = (c, d)$ be the first edge on the path that exits V'

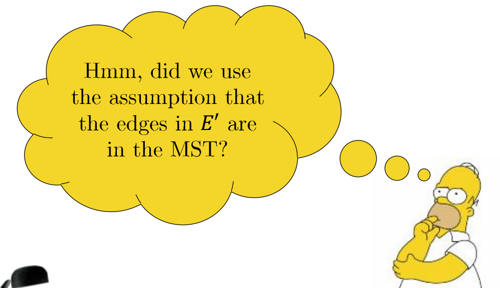


Proof of Correctness

- Consider $T' = T \cup \{e\} \setminus \{e'\}$
  - Its cost is lower than T
  - It has $n - 1$ edges
- T' is connected because any path $u \to c \to d \to v$ that uses e' is replaced by $u \to c \to a \to b \to d \to v$
- So T is not an MST! ■



Why does the proof imply that the MST is unique?



The MST Cut Property

- A similar proof shows: Let G and $V' \subseteq V$, and let e be the cheapest edge between V' and $V \setminus V'$, then e is in the MST
- Using this it is not hard to show that any natural greedy algorithm works, e.g.,
- Kruskal's Algorithm:
  - Go through edges from cheapest to most expensive
  - Add the next edge if it doesn't create a cycle

Run-Time Race for MST

- A naïve implementation of Kruskal and Prim runs in time $O(m^2)$
- A better implementation runs in time $O(m \log m)$
- That's very good!
- In practice, these algorithms are great
- Nevertheless, algorithms and data structures wizards tried to do better

Run Time Race for MST

1984: Fredman and Tarjan invent the Fibonacci heap data structure

$O(m \log m) \rightarrow O(m \log^* m)$

Tarjan | Not Fredman | Also not Fredman

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Run Time Race for MST

1986: Gabow, Galil, Spencer, and Tarjan improved the algorithm

$$O(m \log^* m) \rightarrow O(m \log(\log^* m))$$

Gabow

Galil

Tarjan & Not-Spencer

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Run Time Race for MST

2000: Chazelle invents the soft heap data structure

$O(m \log(\log^* m))$
$\rightarrow O(m \cdot \alpha(m))$

What is $\alpha(\cdot)$?

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Detour: $\alpha(\cdot)$

- $\log^*(m)$ = #times you need to do $\log$ to get down to 1
- $\log^{**}(m)$ = #times you need to do $\log^*$ to get down to 1
- $\log^{***}(m)$ = #times you need to do $\log^{**}$ to get down to 1
- …
- $\alpha(m)$ = #stars you need to do so that $\log^{*\cdots*}(m) \leq 2$

It is incomprehensibly slow growing!

Run Time Race for MST

- 2002: Pettie and Ramachandran give an optimal MST algorithm
- But… nobody knows what its running time is!

Pettie

Ramachandran

Summary

- Terminology:
  - Topological order
  - Weighted graph
  - Minimum spanning tree
- Algorithms:
  - DFS
  - Topological sort via DFS
  - Prim's Algorithm
- Theorems:
  - MST Cut property