Great Ideas in Theoretical CS

Lecture 16:
NP I: Poly-Time Reductions

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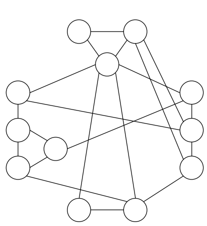
k-COLORING

- Reminder: a k-coloring of a graph satisfies:
  - Each node has a color
  - There are at most k different colors
  - Every two nodes connected by an edge have different colors
- A graph is k-colorable iff it has a k-coloring

2-COLORING

- Is this graph 2-colorable?



2-COLORING

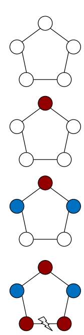
- Given a graph G, how can we decide if it is 2-colorable?
- Enumerate all possible 2^n colorings to look for a valid one...
- OK, but how can we efficiently decide if G is 2-colorable?
  - In polynomial time in the number of vertices n

2-COLORING

- Poll 1: $G = (V, E)$ is 2-colorable iff:
  1. G has a Hamiltonian cycle
  2. $|E| \leq |V| - 1$
  3. Every vertex in G has even degree
  4. G has no odd cycles

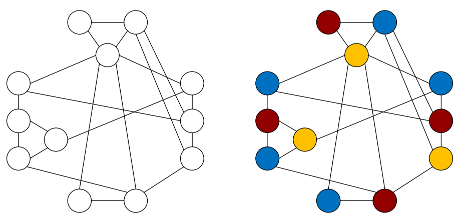
2-COLORING

- Algorithm (reminder):
  - Choose an arbitrary node, color it red and its neighbors blue
  - Color the uncolored neighbors of the blue vertices red, etc.
  - If G is not connected, repeat for every component



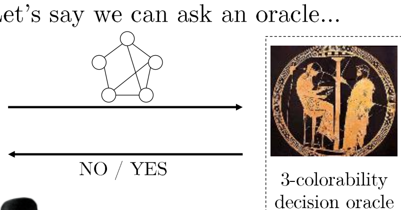
3-COLORING

- Is this graph 3-colorable?



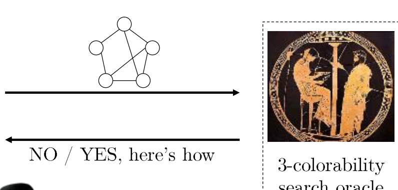
3-COLORABILITY ORACLES

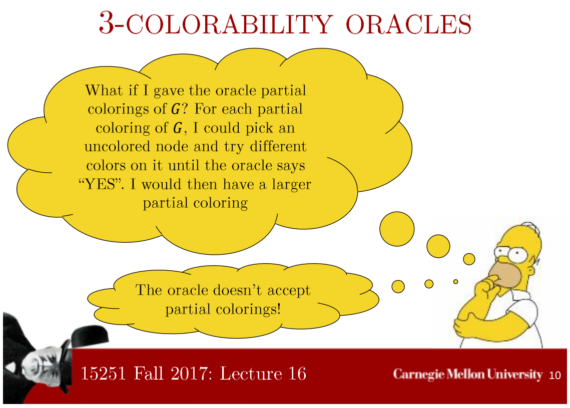
- We can decide 3-colorability by trying all 3^n possible colorings
- Let's say we can ask an oracle…



3-COLORABILITY ORACLES

- How do we turn a decision oracle into a search oracle?




3-COLORABILITY ORACLES
What if I gave the oracle partial colorings of G? For each partial coloring of G, I could pick an uncolored node and try different colors on it until the oracle says "YES". I would then have a larger partial coloring
The oracle doesn't accept partial colorings!
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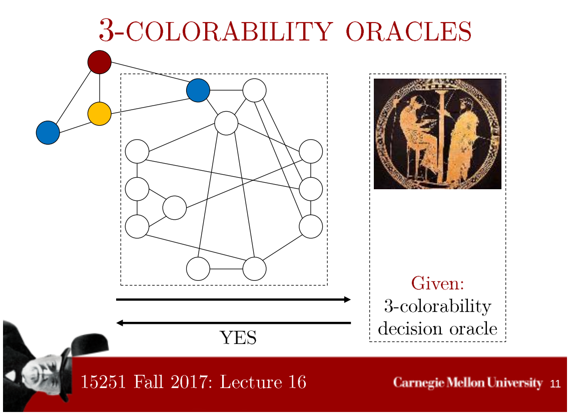
3-COLORABILITY ORACLES
Given:
3-colorability decision oracle
YES
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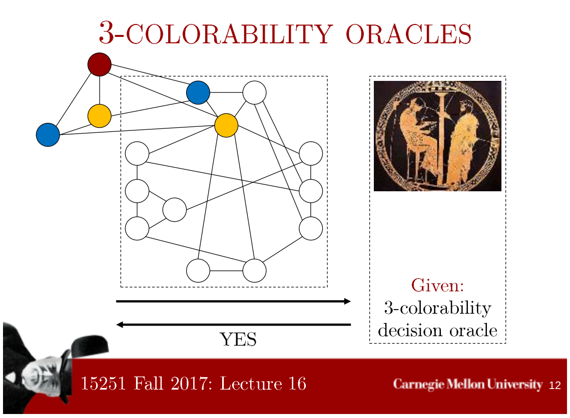
3-COLORABILITY ORACLES
Given:
3-colorability decision oracle
YES
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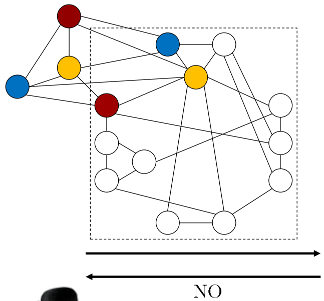
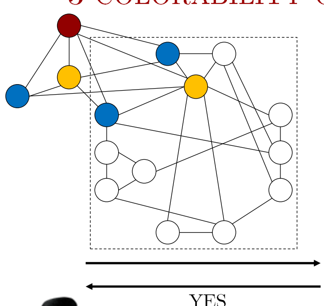
3-COLORABILITY ORACLES



Given:
3-colorability
decision oracle

3-COLORABILITY ORACLES



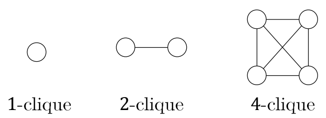
Given:
3-colorability
decision oracle

3-COLORABILITY ORACLES

A 3-colorability search oracle can be simulated using a linear number of calls to a decision oracle!

Clique

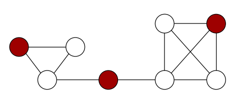
- Reminder: A *k-clique* is a set of k nodes with all possible edges between them



- Clique: Given a graph G and $k \in \mathbb{N}$, does G contain a k-clique?

Independent set

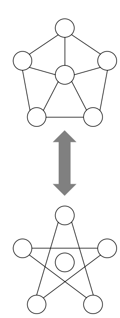
- A *k-independent set* is a set of k nodes with no edges between them



- Independent-Set: Given a graph G and $k \in \mathbb{N}$, does G contain a k-independent set?

Clique vs. IS

- Let $G^* = (V, E^*)$ be the complement of $G = (V, E)$
  $(u, v) \in E \Leftrightarrow (u, v) \notin E^*$
- Poll 2: G has a k-clique for $k \geq 2$ iff:
  1. G^* has an IS of size k
  2. G^* has an IS of size $2k$
  3. G^* has an IS of size k^2
  4. G^* has an IS of size $n = |V|$



Clique vs. IS

$\langle G, k\rangle$ →

← YES/NO

Construct:
Indep.-Set
decision oracle

$\langle G^*, k\rangle$ →

← YES/NO

Given:
Clique
decision oracle

Clique vs. IS

$\langle G, k\rangle$ →

← YES/NO

Construct:
Clique
oracle

$\langle G^*, k\rangle$ →

← YES/NO

Given:
Indep.-Set
oracle

Clique vs. IS

- We can quickly reduce an instance of Clique to an instance of Independet-Set, and vice versa
- There is a fast method for one iff there is a fast method for the other

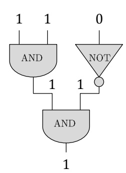
Clique and Independent-Set are cosmetically different but essentially the same!

Poly-Time Reductions

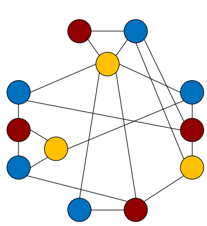
- L has a polynomial-time reduction to L', denoted $L \leq_T^P L'$, if and only if it is possible to solve L in polynomial time using a polynomial-time algorithm for L'
- If $L \leq_T^P L'$ then:
  1. $L' \in \mathbf{P} \Rightarrow L \in \mathbf{P}$
  2. $L \notin \mathbf{P} \Rightarrow L' \notin \mathbf{P}$

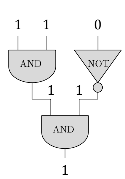
Circuit-Sat

- AND, OR, NOT gates wired together
- Circuit-Satisfiability: Given a circuit with n inputs and one output, is there a way to assign $0/1$ values to the input wires so that the output value is 1 (true)?



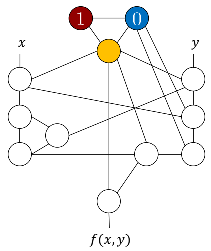
3-Colorability vs. Circuit-Sat





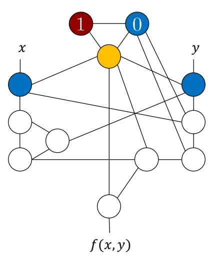
Fundamentally different problems?

3-COLORABILITY VS. CIRCUIT-SAT



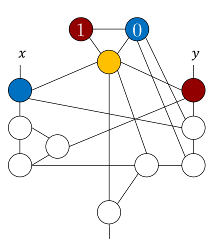
x	y	$f(x,y)$

3-COLORABILITY VS. CIRCUIT-SAT



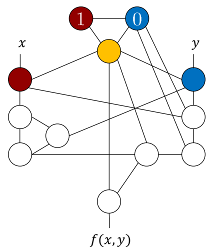
x	y	$f(x,y)$
0	0	

3-COLORABILITY VS. CIRCUIT-SAT



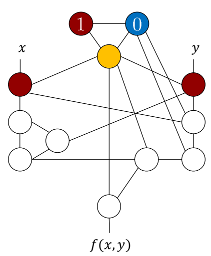
x	y	$f(x,y)$
0	0	0
0	1	

3-COLORABILITY VS. CIRCUIT-SAT



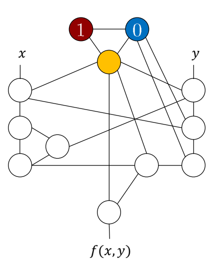
x	y	$f(x,y)$
0	0	0
0	1	1
1	0	

3-COLORABILITY VS. CIRCUIT-SAT



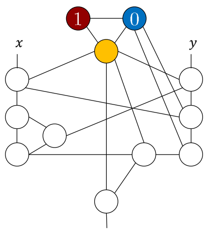
x	y	$f(x,y)$
0	0	0
0	1	1
1	0	1
1	1	

3-COLORABILITY VS. CIRCUIT-SAT



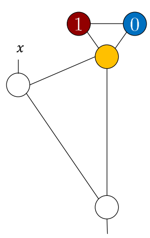
x	y	$f(x,y)$
0	0	0
0	1	1
1	0	1
1	1	1

3-COLORABILITY VS. CIRCUIT-SAT



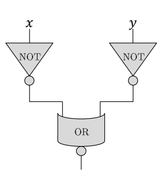
x	y	OR
0	0	0
0	1	1
1	0	1
1	1	1

3-COLORABILITY VS. CIRCUIT-SAT



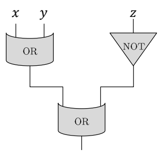
x	NOT
0	1
1	0

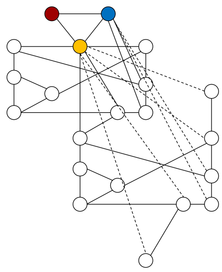
3-COLORABILITY VS. CIRCUIT-SAT



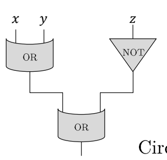
AND Gate from OR and NOT

3-COLORABILITY VS. CIRCUIT-SAT

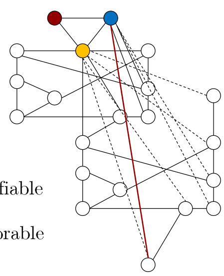




3-COLORABILITY VS. CIRCUIT-SAT



Circuit is satisfiable
⇕
Graph is 3-colorable



3-COLORABILITY VS. CIRCUIT-SAT

Circuit →
← YES/NO

Construct:
CIRCUIT-SAT
decision oracle

Graph with subgraphs corresponding to gates and edge between output and false →
← YES/NO

Given:
3-COLORAB.
decision oracle

3-COLORABILITY VS. CIRCUIT-SAT

- There is a polynomial-time reduction from CIRCUIT-SAT to 3-COLORABILITY
- Fact: Any of the four problems we discussed polynomial-time reduces to any of the others

But nobody knows how to efficiently solve any of these four problems in the worst case!

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SUMMARY

- Terminology:
  - k-COLORING, CLIQUE, INDEPENDENT-SET, CIRCUIT-SAT
  - Polynomial-time reduction
- Principles:
  - Computationally efficient reductions between problems!

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