Great Ideas in Theoretical CS

Lecture 20:
Approximation Algorithms

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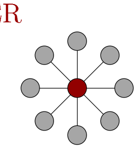
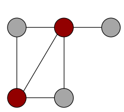
Computational hardness

- We saw that NP-hardness can be a force for good (preventing manipulation)
- But typically it just gets in the way of solving problems we want to solve!
- What can we do?
  - In practice: Heuristics often work well
  - In theory: Run in polynomial time and provide formal guarantees wrt the quality of the solution

Vertex cover

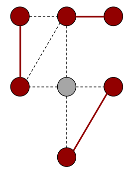
- Vertex Cover: Given a graph $G = (V, E)$ find the smallest $S \subseteq V$ such that every edge in E is incident on a vertex in S
- Decision version of the problem is NP-complete





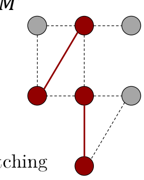
Vertex cover

- We don't know the size of the optimal vertex cover, but…
- Lemma: Let M be a matching in G, and S be a vertex cover. Then $|S| \geq |M|$
- Proof: S must cover at least one vertex for each edge in M; this covers no other edges in M ■



Vertex cover

- Reminder: A matching M is maximal if $\nexists$ matching $M' \neq M$ such that $M \subseteq M'$
- Poll 1: Which of the following algs would find a maximal matching:
  1. Greedily add edges that are disjoint from the edges added so far, while such edges exist
  2. Compute a maximum cardinality matching
  3. Both
  4. Neither



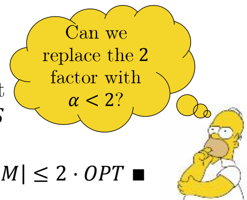
Vertex cover

Approx-VC(G)
$M \leftarrow$ maximal matching on G
$S \leftarrow$ all vertices incident on M
Return S

- Theorem: Given a graph G, let $OPT(G)$ be the size of the optimal vertex cover and $S = \text{Approx-VC}(G)$; S is a valid cover with $|S| \leq 2 \cdot OPT(G)$

Vertex cover

- Theorem: Given a graph G, let $OPT(G)$ be the size of the optimal vertex cover and $S =$ Approx-VC(G); S is a valid cover with $|S| \leq 2 \cdot OPT(G)$
- Proof:
  - For each $e \in E$, at least one vertex is in M, so S is a valid vertex cover
  - By the lemma, $|S| = 2|M| \leq 2 \cdot OPT$ ■

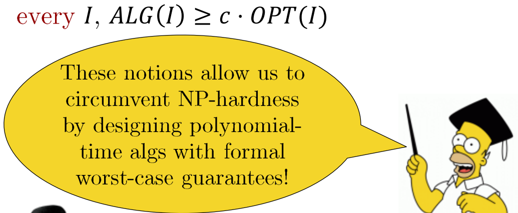


Approximation

- For a minimization problem instance I and algorithm ALG, let $ALG(I)$ be the quality of the algorithm's output and $OPT(I)$ be the quality of the optimal solution
- For $c > 1$, ALG is a c-approximation alg if for every I, $ALG(I) \leq c \cdot OPT(I)$
- Approx-VC is a polytime 2-approximation algorithm for Vertex-Cover

Approximation

- For a maximization problem and $c < 1$, ALG is a c-approximation algorithm if for every I, $ALG(I) \geq c \cdot OPT(I)$



Approximation

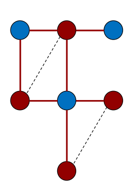
- Algorithm STUPID-APPROX(G): Return all vertices of G (assume G is not empty)
- Poll 2: What is the smallest value of α for which STUPID-APPROX is an α-approx algorithm for VERTEX COVER?
  1. $\alpha = 3$
  2. $\alpha = \log n$
  3. $\alpha = \lceil n/2 \rceil$
  4. $\alpha = n$

Max Cut

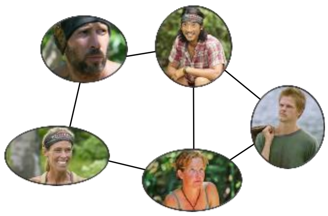
Ryan's favorite problem!

Max Cut

- Given a coloring of vertices in red and blue, an edge is a cut edge if and only if its endpoints have different colors
- MAX CUT: Given a graph $G = (V, E)$, find a coloring of V in red and blue that maximizes the number of cut edges



Max Cut



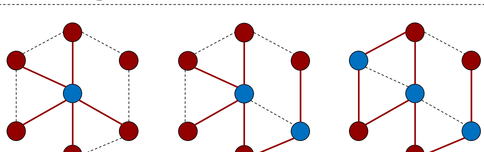
Partition into two tribes to break as many friendships as possible (to maximize drama)

Max Cut

More natural if the social network recorded "enemyships" instead of friendships

Max Cut

APPROX-MC(G)
Start from arbitrary coloring
While $\exists$vertex v such that changing its color increases the number of cut edges
Change the color of v



Max Cut

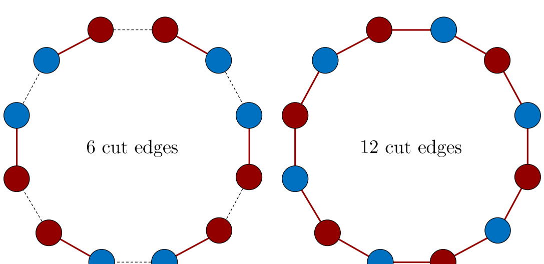
APPROX-MC(G)
Start from arbitrary coloring
Loop
    If $\exists$vertex such that changing its color increases the number of cut edges, change its color

- Poll 3: What is the maximum number of iterations in the worst case?
  1. $\Theta(m)$
  2. $\Theta(mn)$
  3. $\Theta(m^2)$
  4. $\Theta(m^2 n)$

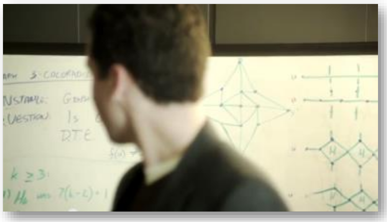
Max Cut

- Theorem: APPROX-MC is a $\frac{1}{2}$-approximation algorithm for MAX CUT
- Proof:
  - When the algorithm returns, each $v \in V$ has at least $\deg(v)/2$ of its edges cut (why?)
  - Therefore, the solution is guaranteed to have at least $m/2$ cut edges (exercise)
  - $OPT \leq m$ ∎

Max Cut



Interlude



https://youtu.be/6ybd5rbQ5rU

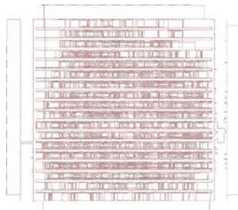
Traveling Salesman

- Traveling Salesman (TSP): Given a graph $G = (V, E)$ with edge costs $c: E \to \mathbb{N}$, find a minimum cost tour that visits each vertex exactly once
- NP-complete by reduction from Hamiltonian cycle: Given an instance, assign $c(e) = 1$ for each $e \in E$ and ask whether there is a tour of cost n
- Metric TSP: can visit vertices multiple times (also NP-complete)

Traveling Salesman

Shortest traveling salesman route going through all 13,509 cities in the United States with a population of at least 500 (as of 1998)

Traveling Salesman



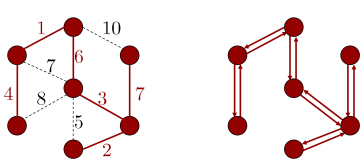
An 85,900-vertex route. The graph corresponds to the design of a customized computer chip created at Bell Laboratories, and the solution exhibits the shortest path for a laser to follow as it sculpts the chip.

Traveling Salesman

Anderson et al., PNAS 2015

Traveling Salesman

Approx-TSP(G)
$T \leftarrow$ Minimum spanning tree of G
$2T \leftarrow$ double edges of T
Return Eulerian tour of $2T$



Traveling Salesman

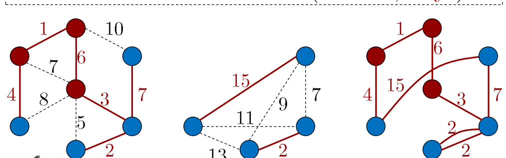
- Theorem: Approx-TSP is a 2-approximation algorithm for Metric TSP
- Proof:
  - A TSP tour can be converted into a lower cost spanning tree (how?), therefore

$$c(T) = \sum_{e \in E(T)} c(e) \le OPT$$

  - Clearly $c(2T) = 2c(T)$
  - It follows that $c(2T) \le 2OPT$ ■

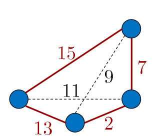
Traveling Salesman*

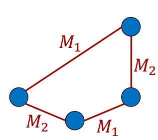
Christofides(G)
$T \leftarrow$ Minimum spanning tree of G
S $\leftarrow$ Vertices of odd degree in T ($|S|$ is even, why?)
$M \leftarrow$ Min cost perfect matching on S in G
Return Eulerian tour of $T \cup M$ (it exists, why?)



Traveling Salesman*

- Lemma: $C(M) \le \frac{1}{2}OPT$
- Proof:
  - $\exists$ tour of S of cost at most OPT (because $S \subseteq V$)
  - Decompose into two matchings M_1 and M_2
  - $c(M_1) + c(M_2) \le OPT$, but $c(M) \le c(M_1)$ and $c(M) \le c(M_2) \Rightarrow c(M) \le \frac{1}{2}OPT$ ■





Traveling Salesman*

- Theorem: Christofides is a $\frac{3}{2}$-approximation algorithm for Metric TSP
- Proof: Using the lemma,

$$\begin{aligned} ALG &= c(M) + c(T) \\ &\leq \frac{1}{2} OPT + \mathrm{OPT} \\ &= \frac{3}{2} \mathrm{OPT} \quad \blacksquare \end{aligned}$$

* Just for fun

Summary

- Definitions
  - Approximation algorithm
  - Vertex Cover, Max Cut, traveling Salesman
- Algorithms
  - 2-approximation for Vertex Cover
  - $\frac{1}{2}$-approximation for Max Cut
  - 2-approximation for Metric TSP