

Great Ideas in Theoretical CS

Lecture 25:
Game Theory

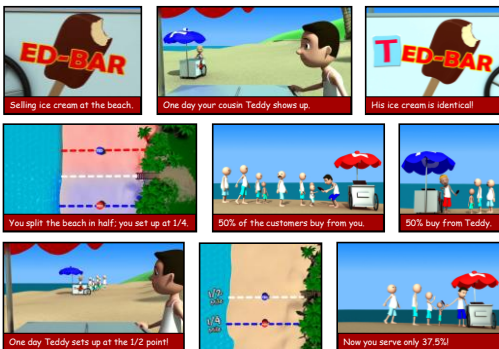
Anil Ada
Ariel Procaccia (this time)

NORMAL-FORM GAME

- A game in normal form consists of:
 - Set of players $N = \{1, \dots, n\}$
 - Strategy set S
 - For each $i \in N$, utility function $u_i: S^n \rightarrow \mathbb{R}$: if each $j \in N$ plays the strategy $s_j \in S$, the utility of player i is $u_i(s_1, \dots, s_n)$
- Next example created by taking screenshots of http://youtu.be/jILgxNBK_8



15251 Fall 2017: Lecture 25 Carnegie Mellon University 2





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THE ICE CREAM WARS

- $N = \{1,2\}$
- $S = [0,1]$
- $u_i(s_i, s_j) = \begin{cases} \frac{s_i+s_j}{2}, & s_i < s_j \\ 1 - \frac{s_i+s_j}{2}, & s_i > s_j \\ \frac{1}{2}, & s_i = s_j \end{cases}$
- To be continued...



THE PRISONER'S DILEMMA

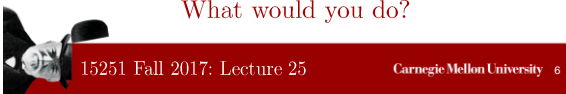
- Two men are charged with a crime
- They are told that:
 - If one rats out and the other does not, the rat will be freed, other jailed for nine years
 - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year



THE PRISONER'S DILEMMA

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

What would you do?



UNDERSTANDING THE DILEMMA

- Defection is a **dominant** strategy
- But the players can do much better by cooperating
- Related to the **tragedy of the commons**





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IN REAL LIFE

- Presidential elections
 - Cooperate = positive ads
 - Defect = negative ads
- Nuclear arms race
 - Cooperate = destroy arsenal
 - Defect = build arsenal
- Climate change
 - Cooperate = curb CO₂ emissions
 - Defect = do not curb





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ON TV



<http://youtu.be/S0qjk3TWZE8>



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THE PROFESSOR'S DILEMMA

		Class	
		Listen	Sleep
Professor	Make effort	10 ⁶ , 10 ⁶	-10, 0
	Slack off	0, -10	0, 0

Dominant strategies?



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NASH EQUILIBRIUM

- Each player's strategy is a **best response** to strategies of others
- Formally, a **Nash equilibrium** is a vector of strategies $s = (s_1, \dots, s_n) \in S^n$ such that $\forall i \in N, \forall s'_i \in S, u_i(s) \geq u_i(s'_i, s_{-i})$




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NASH EQUILIBRIUM

- Poll 1: How many Nash equilibria does the Professor's Dilemma have?

1. 0
2. 1
3. 2
4. 3

		Listen	Sleep
Make effort		10 ⁶ , 10 ⁶	-10, 0
	Slack off	0, -10	0, 0



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NASH EQUILIBRIUM



<http://youtu.be/CemLiSI5ox8>



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RUSSEL CROWE WAS WRONG



Turing's Invisible Hand
Computation, Economics, and Game Theory

HELLO NASH. I THINK THESE GALS EVERYWHERE ARE EXERCISING THIS LIKE YOUR NASH EQUILIBRIUM. RIGHT? ONE OF THEM IS HOT BUT WE SHOULD STAY FLAT WITH ONE OF THESE LESS DESIRABLE FRIENDS. OTHERWISE WE RISK COMING ON TOO STRONG TO THE HOT ONE AND JUST DOING THE GREAT OFF.

WELL THAT'S NOT REALLY THE SORT OF SITUATION I WERE ABOUT. ONCE WE'RE WITH THE HOT ONE THERE'S NO INCENTIVE FOR ONE OF US NOT TO TRY TO SWITCH TO THE HOT ONE IT'S NOT A STABLE EQUILIBRIUM.

OHAY. FORGET IT. IT LOOKS LIKE ALL THREE ARE LEAVING WITH ONE GUY. DARNIT, FERNANDEZ!



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END OF THE ICE CREAM WARS



The PLAN
75%

Day 3 of the ice cream wars...

Teddy sets up south of you!

You go south of Teddy.

Eventually...

Nash Equilibrium



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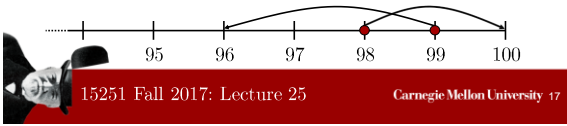





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DOES NE MAKE SENSE?

- Two players, strategies are $\{2, \dots, 100\}$
- If both choose the same number, that is what they get
- If one chooses s , the other t , and $s < t$, the former player gets $s + 2$, and the latter gets $s - 2$
- Poll 2: what would you choose?





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BACK TO PRISON

- The only Nash equilibrium in Prisoner's dilemma is bad; but how bad is it?
- Objective function: social cost = sum of costs
- NE is six times worse than the optimum

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6



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ANARCHY AND STABILITY

- Fix a class of games, an objective function, and an equilibrium concept
- The **price of anarchy (stability)** is the **worst-case ratio** between the **worst (best)** objective function value of an equilibrium of the game, and that of the optimal solution
- In this lecture:
 - Objective function = social cost (sum of costs)
 - Equilibrium concept = Nash equilibrium

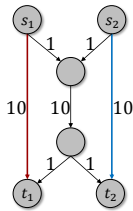


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EXAMPLE: COST SHARING

- n players in weighted directed graph G
- Player i wants to get from s_i to t_i ; strategy space is $s_i \rightarrow t_i$ paths
- Each edge e has cost c_e
- Cost of edge is split between all players using edge
- Cost of player is sum of costs over edges on path



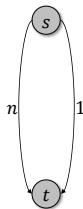


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EXAMPLE: COST SHARING

- With n players, the example on the right has an NE with social cost n
- Optimal social cost is 1
- \Rightarrow Price of anarchy $\geq n$
- Price of anarchy is also $\leq n$
 - Each player can always deviate to his strategy at the optimal solution, and pay for it alone; the cost is at most OPT
 - At equilibrium, no player wants to deviate, so each player pays at most OPT



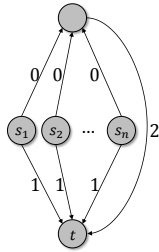


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EXAMPLE: COST SHARING

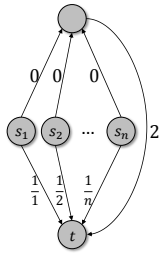
- Think of the 1 edges as cars, and the 2 edge as mass transit
- Bad Nash equilibrium with cost n
- Good Nash equilibrium with cost 2
- Now let's modify the example...





EXAMPLE: COST SHARING

- $OPT = 2$
- Poll 3: What is the social cost at Nash equilibrium?
- \Rightarrow price of stability is at least this cost $f(n)/2$
- Theorem: The price of stability of cost sharing games is $\leq f(n)$





COST SHARING SUMMARY

- In every cost sharing game
 - $\forall NE \mathbf{s}, \text{cost}(\mathbf{s}) \leq n \cdot OPT$
 - $\exists NE \mathbf{s}$ such that $\text{cost}(\mathbf{s}) \leq f(n) \cdot OPT$
- There exist cost sharing games s.t.
 - $\exists NE \mathbf{s}$ such that $\text{cost}(\mathbf{s}) \geq n \cdot OPT$
 - $\forall NE \mathbf{s}, \text{cost}(\mathbf{s}) \geq \Omega(f(n)) \cdot OPT$



SUMMARY

- Terminology:
 - Normal-form game
 - Nash equilibrium
 - Price of anarchy/stability
 - Cost sharing games
- Nobel-prize-winning ideas:
 - Nash equilibrium ☺



