Great Ideas in Theoretical CS

Lecture 25:
Game Theory

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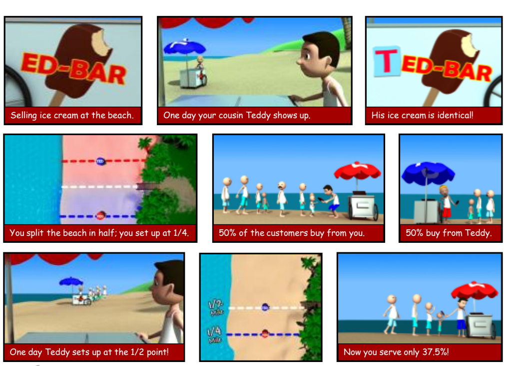
Normal-Form Game

- A game in normal form consists of:
  - Set of players $N = \{1, \ldots, n\}$
  - Strategy set S
  - For each $i \in N$, utility function $u_i : S^n \to \mathbb{R}$: if each $j \in N$ plays the strategy $s_j \in S$, the utility of player i is $u_i(s_1, \ldots, s_n)$
- Next example created by taking screenshots of http://youtu.be/jILgxeNBK_8



The Ice Cream Wars

- $N = \{1,2\}$
- $S = [0,1]$
- $u_i(s_i, s_j) = \begin{cases} \frac{s_i+s_j}{2}, & s_i < s_j \\ 1 - \frac{s_i+s_j}{2}, & s_i > s_j \\ \frac{1}{2}, & s_i = s_j \end{cases}$
- To be continued...

The prisoner's dilemma

- Two men are charged with a crime
- They are told that:
  - If one rats out and the other does not, the rat will be freed, other jailed for nine years
  - If both rat out, both will be jailed for six years
- They also know that if neither rats out, both will be jailed for one year

The prisoner's dilemma

	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

What would you do?

Understanding the dilemma

- Defection is a dominant strategy
- But the players can do much better by cooperating
- Related to the tragedy of the commons

In real life

- Presidential elections
  - Cooperate = positive ads
  - Defect = negative ads
- Nuclear arms race
  - Cooperate = destroy arsenal
  - Defect = build arsenal
- Climate change
  - Cooperate = curb CO_2 emissions
  - Defect = do not curb

On TV

http://youtu.be/S0qjK3TWZE8

The professor's dilemma

Class

Professor	Listen	Sleep
Make effort	$10^6, 10^6$	-10,0
Slack off	0,-10	0,0

Dominant strategies?

Nash equilibrium

- Each player's strategy is a best response to strategies of others
- Formally, a Nash equilibrium is a vector of strategies $s = (s_1 \dots, s_n) \in S^n$ such that $\forall i \in N, \forall s_i' \in S, u_i(s) \geq u_i(s_i', s_{-i})$

Nash equilibrium

- Poll 1: How many Nash equilibria does the Professor's Dilemma have?
  1. 0
  2. 1
  3. 2
  4. 3

	Listen	Sleep
Make effort	$10^6, 10^6$	-10,0
Slack off	0,-10	0,0

Nash equilibrium

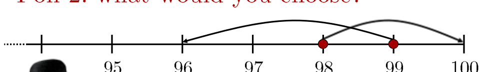
http://youtu.be/CemLiSI5ox8

Russel Crowe was wrong

End of the Ice Cream Wars

Does NE make sense?

- Two players, strategies are $\{2, \dots, 100\}$
- If both choose the same number, that is what they get
- If one chooses s, the other t, and $s < t$, the former player gets $s + 2$, and the latter gets $s - 2$
- Poll 2: what would you choose?



Back to prison

- The only Nash equilibrium in Prisoner's dilemma is bad; but how bad is it?
- Objective function: social cost = sum of costs
- NE is six times worse than the optimum

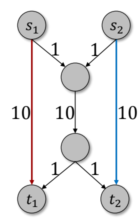
	Cooperate	Defect
Cooperate	-1,-1	-9,0
Defect	0,-9	-6,-6

Anarchy and stability

- Fix a class of games, an objective function, and an equilibrium concept
- The price of anarchy (stability) is the worst-case ratio between the worst (best) objective function value of an equilibrium of the game, and that of the optimal solution
- In this lecture:
  - Objective function = social cost (sum of costs)
  - Equilibrium concept = Nash equilibrium

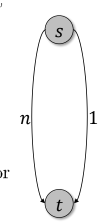
Example: Cost sharing

- n players in weighted directed graph G
- Player i wants to get from s_i to t_i; strategy space is $s_i \rightarrow t_i$ paths
- Each edge e has cost c_e
- Cost of edge is split between all players using edge
- Cost of player is sum of costs over edges on path



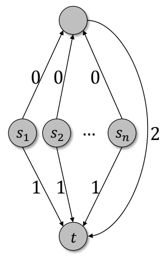
Example: Cost sharing

- With n players, the example on the right has an NE with social cost n
- Optimal social cost is 1
- $\Rightarrow$ Price of anarchy $\geq n$
- Price of anarchy is also $\leq n$
  - Each player can always deviate to his strategy at the optimal solution, and pay for it alone; the cost is at most OPT
  - At equilibrium, no player wants to deviate, so each player pays at most OPT



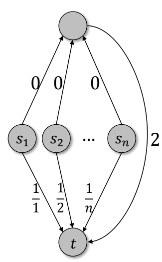
Example: Cost sharing

- Think of the 1 edges as cars, and the 2 edge as mass transit
- Bad Nash equilibrium with cost n
- Good Nash equilibrium with cost 2
- Now let's modify the example...



Example: Cost sharing

- OPT = 2
- Poll 3: What is the social cost at Nash equilibrium?
- ⇒ price of stability is at least this cost $f(n)/2$
- Theorem: The price of stability of cost sharing games is $\leq f(n)$



Cost sharing summary

- In every cost sharing game
  - $\forall$NE $\boldsymbol{s}$, $\text{cost}(\boldsymbol{s}) \leq n \cdot \text{OPT}$
  - $\exists$NE $\boldsymbol{s}$ such that $\text{cost}(\boldsymbol{s}) \leq f(n) \cdot \text{OPT}$
- There exist cost sharing games s.t.
  - $\exists$NE $\boldsymbol{s}$ such that $\text{cost}(\boldsymbol{s}) \geq n \cdot \text{OPT}$
  - $\forall$NE $\boldsymbol{s}$, $\text{cost}(\boldsymbol{s}) \geq \Omega(f(n)) \cdot \text{OPT}$

Summary

- Terminology:
  - Normal-form game
  - Nash equilibrium
  - Price of anarchy/stability
  - Cost sharing games
- Nobel-prize-winning ideas:
  - Nash equilibrium ☺