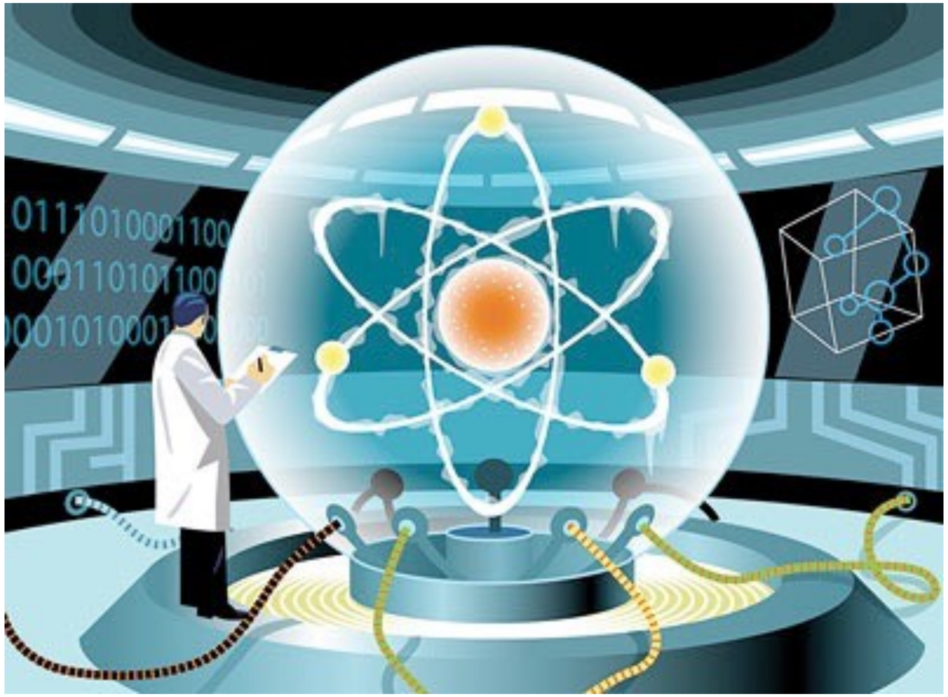
15-251
Great Ideas in Theoretical Computer Science

Lecture 28:
Quantum Computation: A gentle introduction



Dec 5th, 2017

Announcements

Please fill out the Faculty Course Evaluations (FCEs).

https://cmu.smartevals.com

Announcements

As a "thank you" for filling it out:

You can vote to eliminte 2 topics from the final exam:

Cake Cutting

Stable Matchings

Boolean Circuits

Social Choice

Approximation Algorithms

Announcements

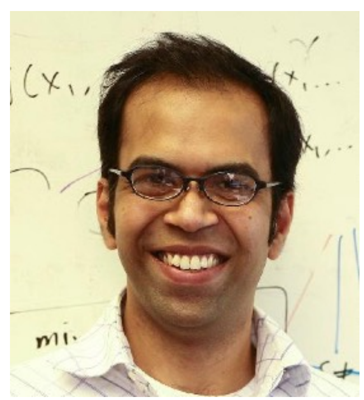
The Last Lecture on Thursday

Daniel Sleator

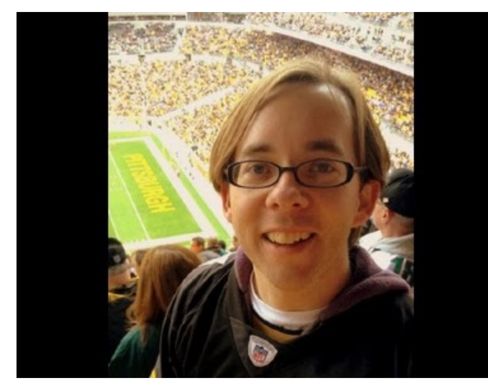
Mor Harchol-Balter

Nihar Shah

Rashmi Vinayak



Anupam Gupta



Ryan O'Donnell

<u>Announcements</u>

The Last Lecture on Thursday

Quantum Computation

The plan

Classical computers and classical theory of computation

Quantum physics (what the fuss is all about)

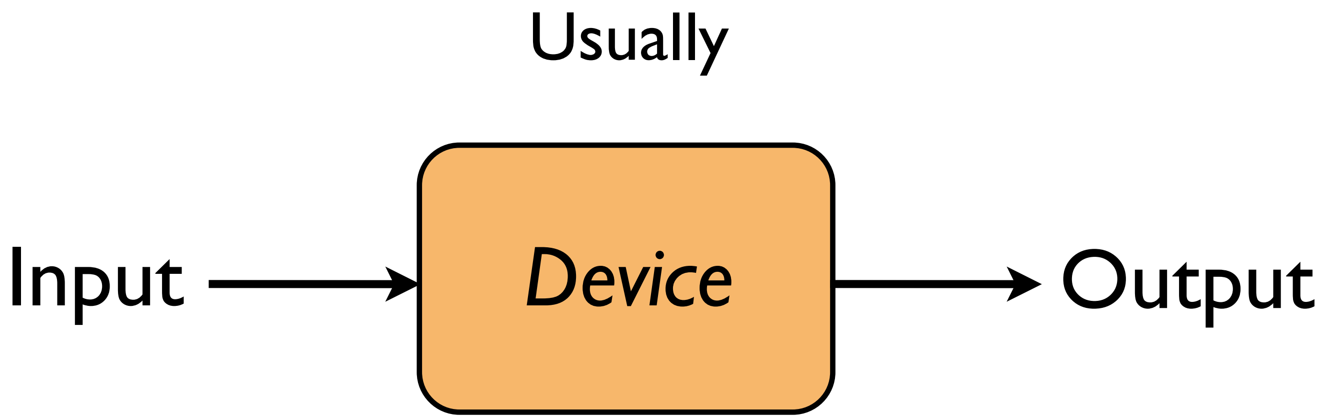
Quantum computers
(practical, scientific, and philosophical perspectives)

The plan

Classical computers and classical theory of computation

What is computer/computation?

A device that **manipulates** **data** (information)

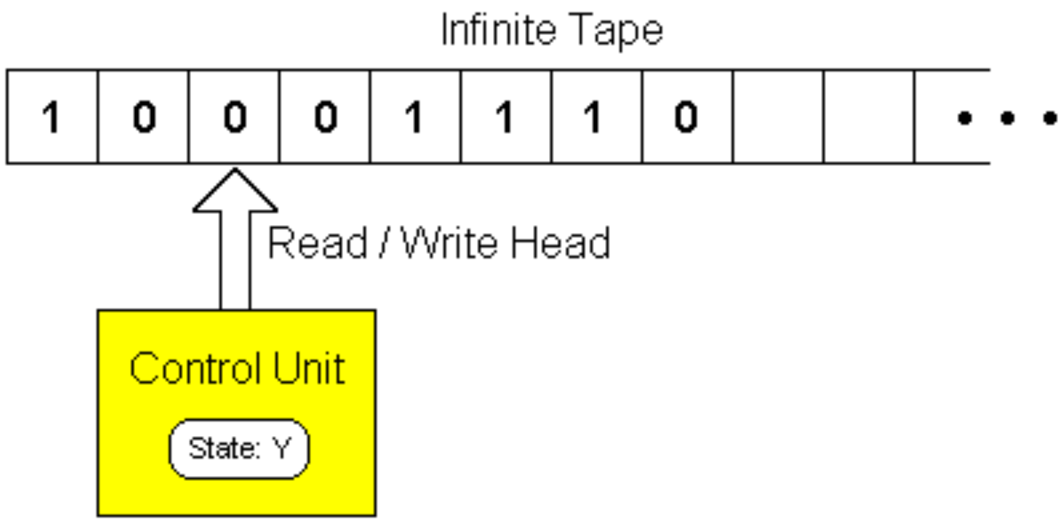


Theory of computation

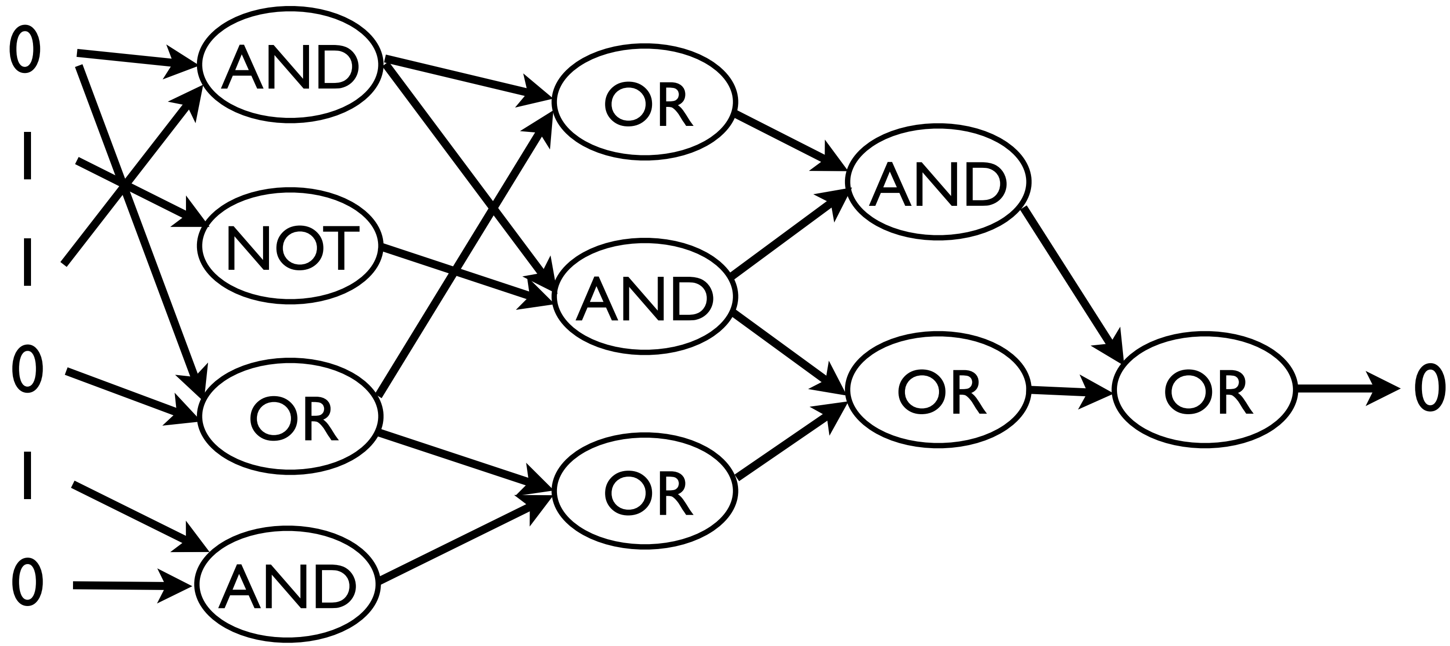
Mathematical model of a computer:

Turing Machines ~ **Boolean Circuits**

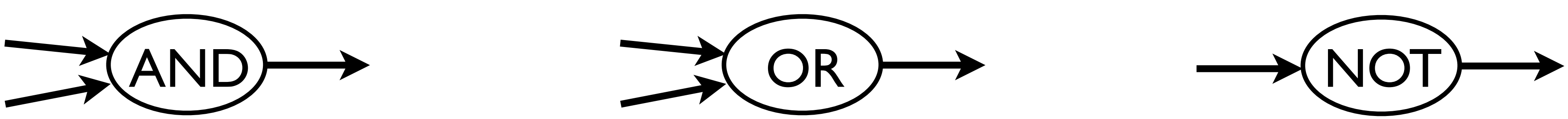
Turing Machines



Boolean Circuits

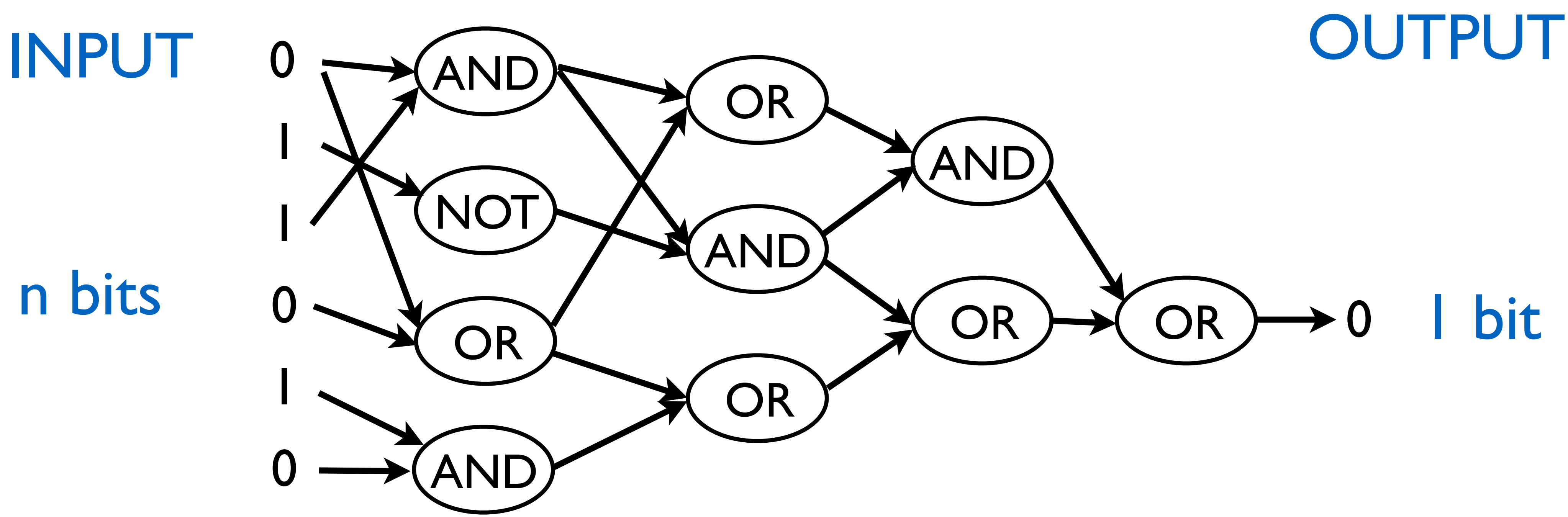


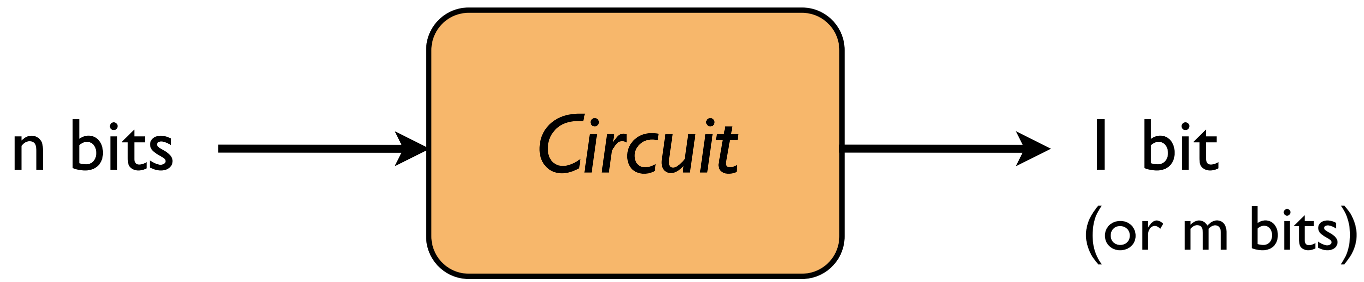
gates



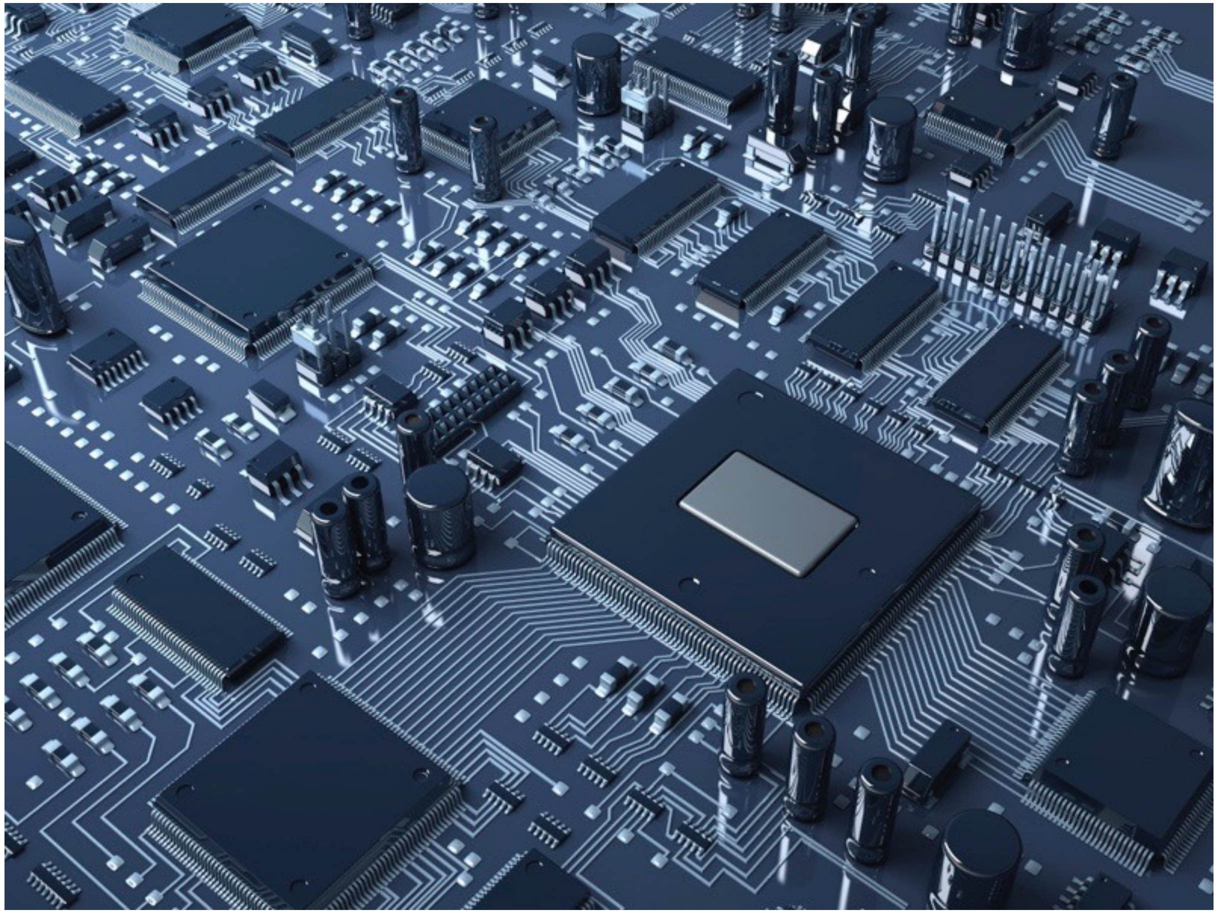
Theory of computation

Boolean Circuits





Physical Realization



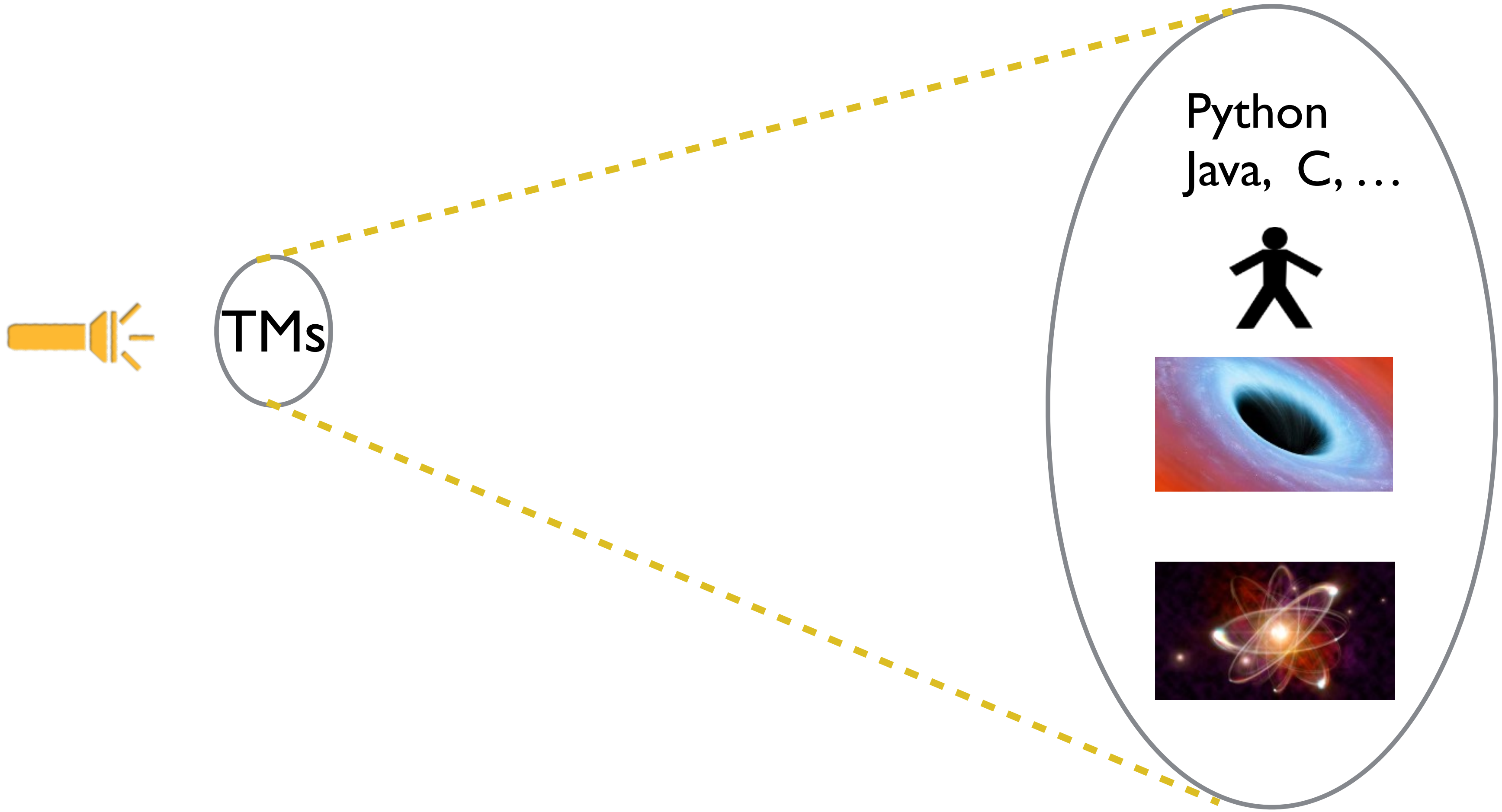
Circuits implement basic operations / instructions.

Everything follows classical laws of physics!

(Physical) Church-Turing Thesis

Turing Machines ~ (uniform) **Boolean Circuits**

universally capture all of computation.



All types of computation

(Physical) Church-Turing Thesis

Turing Machines ~ (uniform) **Boolean Circuits**

universally capture all of computation.

(Physical) Church Turing Thesis

Any computational problem that can be solved by a physical device, can be solved by a Turing Machine.

Strong version

Any computational problem that can be solved ***efficiently*** by a physical device, can be solved ***efficiently*** by a TM.

The plan

Classical computers and classical theory of computation

Quantum physics (what the fuss is all about)

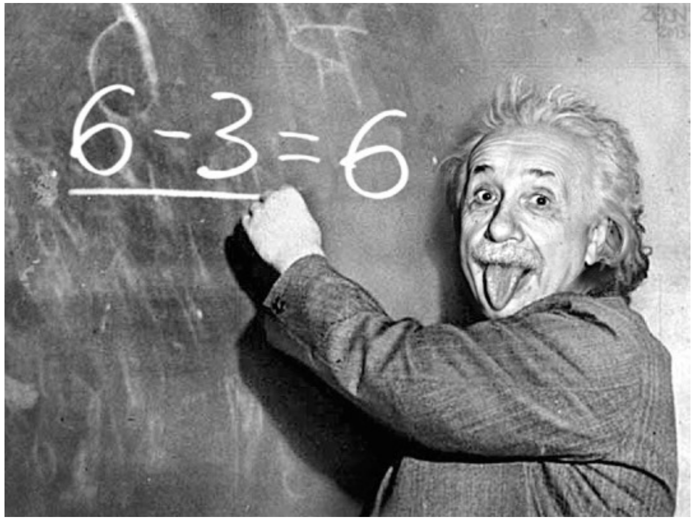
Quantum computers
(practical, scientific, and philosophical perspectives)

The plan

Quantum physics (what the fuss is all about)

One slide course on physics

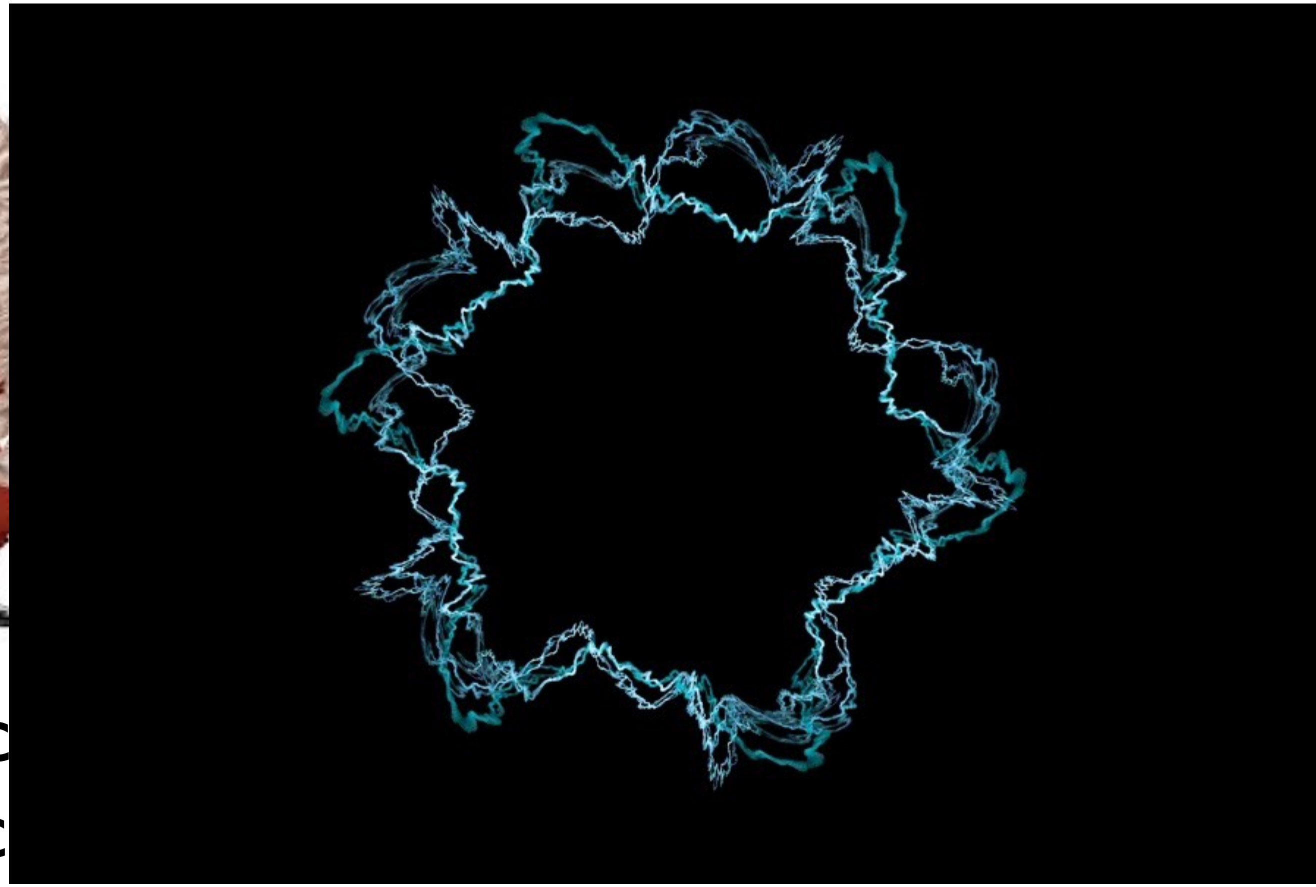
Classical
Physics

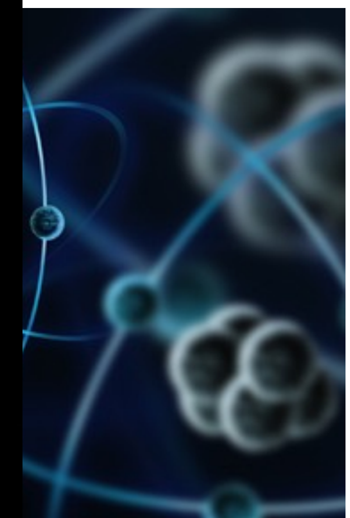


General Theory
of Relativity

Quantum
Physics

One slide course on physics





Classic
Physic

antum
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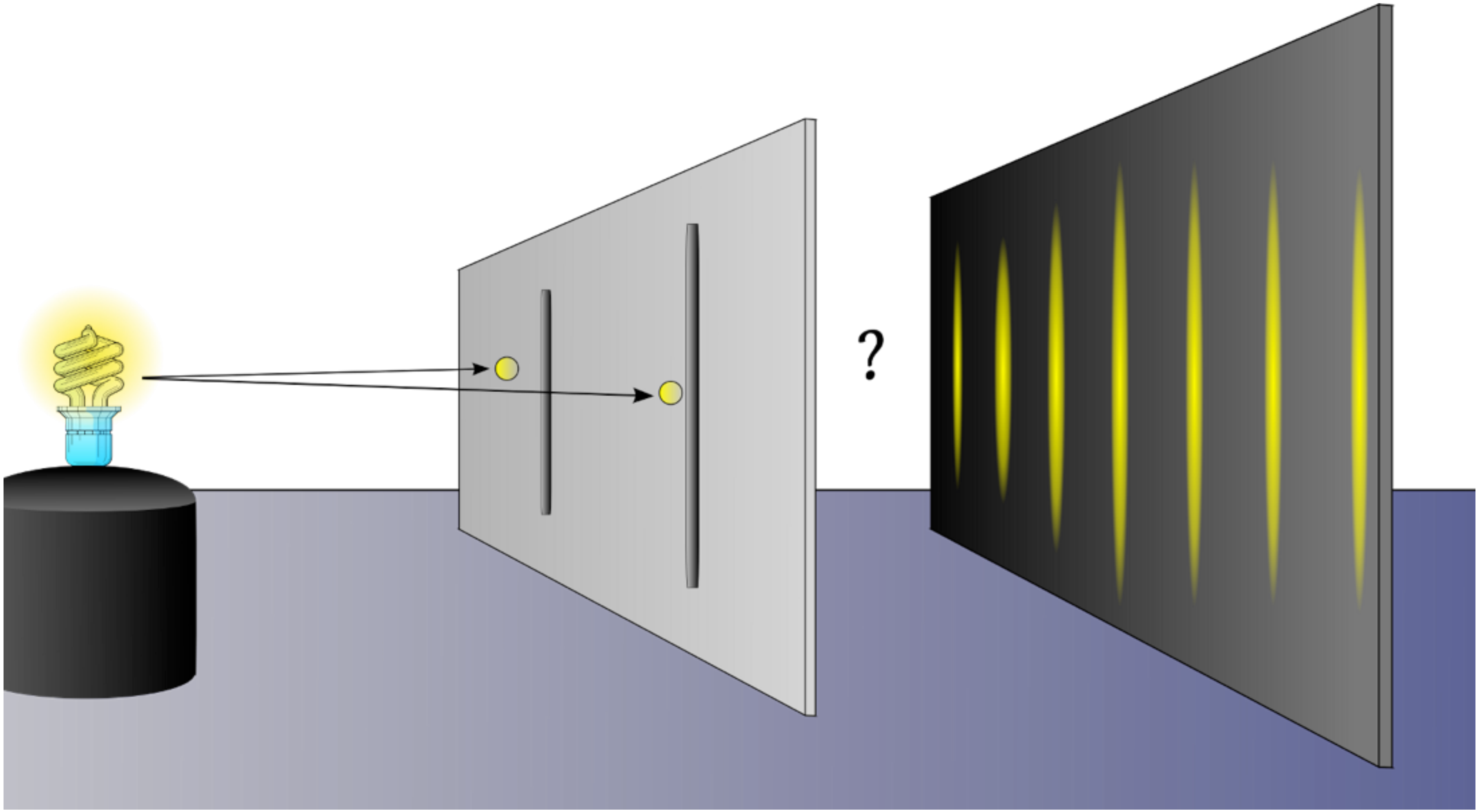
String Theory (?)

Video: Double slit experiment

http://www.youtube.com/watch?v=DfPeprQ7oGc



Nature has no obligation to conform to your intuitions.

Video: Double slit experiment

2 interesting aspects of quantum physics

1. Having multiple states "simultaneously"

e.g.: electrons can have states
spin "up" or spin "down": $|\mathrm{up}\rangle$ or $|\mathrm{down}\rangle$

In reality, they can be in a **superposition** of two states.

2. Measurement

Quantum property is **very** sensitive/fragile !

If you measure it (interfere with it), it "collapses".

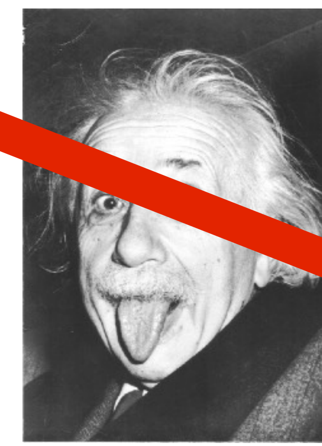
So you either see $|\mathrm{up}\rangle$ or $|\mathrm{down}\rangle$.

It must be just our ignorance

- ~~There is no such thing as *superposition*.~~
- ~~We don't know the state, so we say it is in a *superposition*.~~
- ~~In reality, it is always in one of the two states.~~
- ~~This is why when we measure/observe the state, we find it in one state.~~

~~God does not play dice with the world.~~

~~- *Albert Einstein*~~



Einstein, don't tell God what to do.

- *Niels Bohr*

How should we fix our intuitions
to put it in line with experimental results ?

Removing physics from quantum physics

mathematics underlying quantum physics

=

generalization/extension of probability theory

(allow "negative probabilities")

Probabilistic states and evolution

vs

Quantum states and evolution

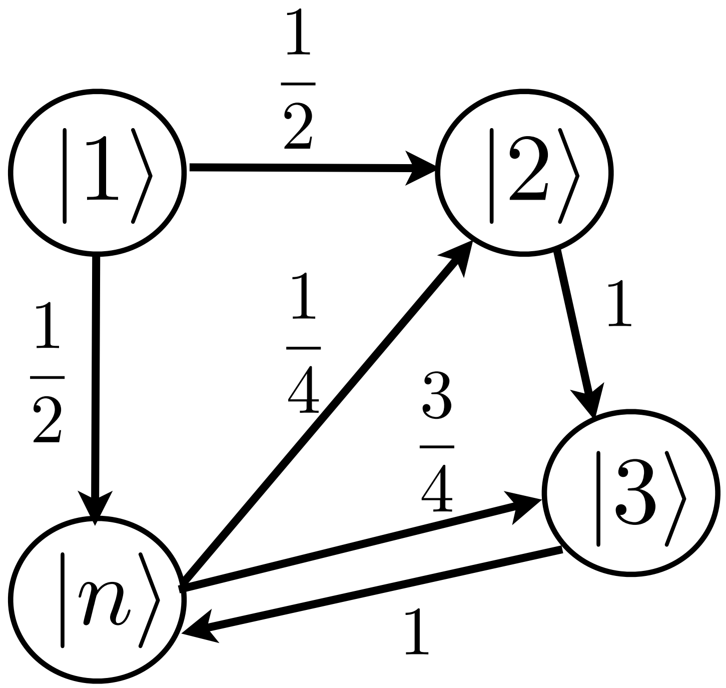
Probabilistic states

Suppose an object can have n possible states:

$$|1\rangle, |2\rangle, \cdots, |n\rangle$$

At each time step, the state can change probabilistically.

What happens if we start at state $|1\rangle$ and evolve?



Initial state:

$$\begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \\ \\ |n\rangle \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Probabilistic states

Suppose an object can have n possible states:

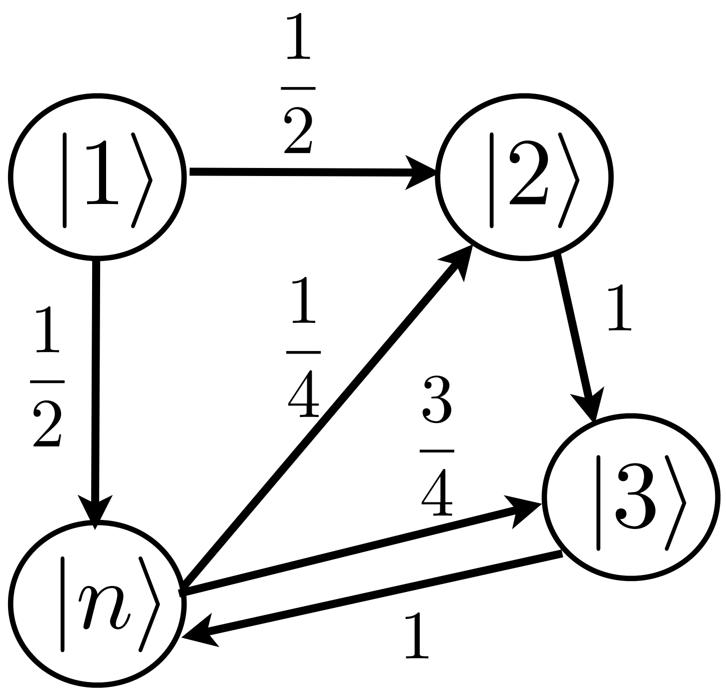
$$|1\rangle, |2\rangle, \cdots, |n\rangle$$

At each time step, the state can change probabilistically.

What happens if we start at state $|1\rangle$ and evolve?

After one time step:

$$\begin{bmatrix} \text{Transition} \\ \text{Matrix} \end{bmatrix} \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \\ \\ |n\rangle \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ \vdots \\ 1/2 \end{bmatrix}$$



Probabilistic states

$$\begin{bmatrix} \\ \text{Transition} \\ \text{Matrix} \\ \\ \end{bmatrix} \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \\ \\ |n\rangle \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ \vdots \\ 1/2 \end{bmatrix}$$

the new state (probabilistic)

A *general* probabilistic state:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

$p_i =$ the probability of being in state i

$p_1 + p_2 + \cdots + p_n = 1$

(ℓ_1 norm is 1)

Probabilistic states

$$\begin{bmatrix} \text{Transition} \\ \text{Matrix} \end{bmatrix} \begin{array}{c} |1\rangle \\ |2\rangle \\ |3\rangle \\ \\ |n\rangle \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ \vdots \\ 1/2 \end{bmatrix}$$

the new state
(probabilistic)

A *general* probabilistic state:

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = p_1 |1\rangle + p_2 |2\rangle + \cdots + p_n |n\rangle$$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Probabilistic states

Evolution of probabilistic states

$$\begin{bmatrix} \text{Transition} \\ \text{Matrix} \\ \\ \end{bmatrix}$$

Any matrix that maps probabilistic states to probabilistic states.

We won't restrict ourselves to just one transition matrix.

$$\pi_0 \xrightarrow{K_1} \pi_1 \xrightarrow{K_2} \pi_2 \xrightarrow{K_3} \cdots$$

Quantum states

$$\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

p_i's can be negative.

Quantum states

α_i's can be negative. (α_i's are called amplitudes.)

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \alpha_1|1\rangle + \alpha_2|2\rangle + \cdots + \alpha_n|n\rangle$$

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1 \qquad (\ell_2 \text{ norm is } 1)$$

(α_i can be a complex number)

$$\begin{bmatrix} \text{Unitary} \\ \text{Matrix} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \qquad \beta_1^2 + \beta_2^2 + \cdots + \beta_n^2 = 1$$

any matrix that preserves "quantumness"

Quantum states

Evolution of quantum states

$$\begin{bmatrix} \text{Unitary} \\ \text{Matrix} \\ \\ \end{bmatrix}$$

Any matrix that maps quantum states to quantum states.

We won't restrict ourselves to just one unitary matrix.

$$\psi_0 \xrightarrow{U_1} \psi_1 \xrightarrow{U_2} \psi_2 \xrightarrow{U_3} \cdots$$

Quantum states

Measuring quantum states

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \alpha_1|1\rangle + \alpha_2|2\rangle + \cdots + \alpha_n|n\rangle$$

$$\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1$$

When you measure the state,

you see state i with probability α_i^2 .

Probabilistic states vs Quantum states

Suppose we have just 2 possible states: $|0\rangle$ and $|1\rangle$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

randomize a random state
$\longrightarrow$ random state

$$|0\rangle \to \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$$

$$\frac{1}{2}\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \qquad \frac{1}{2}\left(\frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle\right)$$

$$\frac{1}{4}|0\rangle + \frac{1}{4}|1\rangle \quad + \quad \frac{1}{4}|0\rangle + \frac{1}{4}|1\rangle$$

Probabilistic states vs Quantum states

Suppose we have just 2 possible states: $|0\rangle$ and $|1\rangle$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \qquad \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$$

$$\cancel{\frac{1}{2}|0\rangle} + \frac{1}{2}|1\rangle \quad + \quad -\cancel{\frac{1}{2}|0\rangle} + \frac{1}{2}|1\rangle = |1\rangle$$

Probabilistic states vs Quantum states

Classical Probability

To find the probability of an event:

add the probabilities of every possible way it can happen

Probabilistic states vs Quantum states

Quantum

To find the probability of an event:

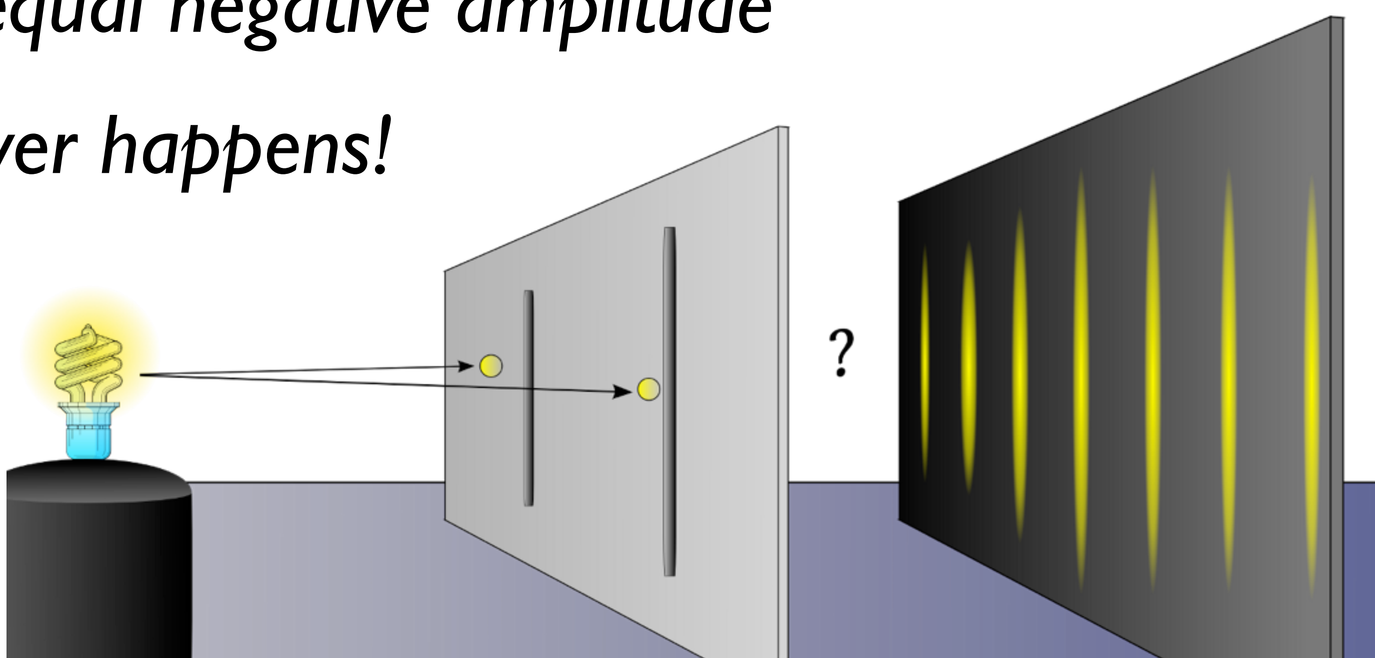
add the amplitudes of every possible way it can happen,

then square the value to get the probability.

one way has positive amplitude

the other way has equal negative amplitude

→ *event never happens!*



Probabilistic states vs Quantum states

A final remark

Quantum states are an **upgrade** to:

2-norm (Euclidean norm) and algebraically closed fields.

Nature seems to be choosing the mathematically more elegant option.

The plan

Classical computers and classical theory of computation

Quantum physics (what the fuss is all about)

Quantum computers
(practical, scientific, and philosophical perspectives)

The plan

Quantum computers
(practical, scientific, and philosophical perspectives)

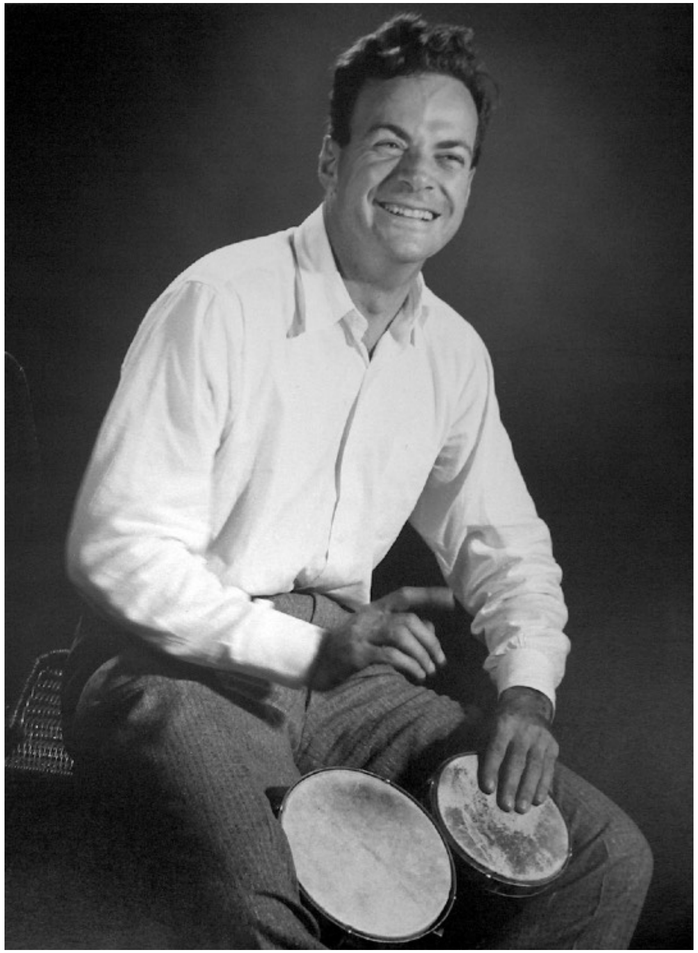
Two beautiful theories

Theory of computation

Quantum physics

Quantum Computation:

Information processing using laws of quantum physics.



Richard Feynman
(1918 - 1988)

It would be super nice to be able to simulate quantum systems.

With a classical computer this is extremely inefficient.

n state system $\longrightarrow$ complexity exponential in **n**

Why not view the quantum particles as a computer simulating themselves?

Why not do computation using quantum particles/physics?

Representing data/information

An electron can be in "spin up" or "spin down" state.

$|\mathrm{up}\rangle$ or $|\mathrm{down}\rangle$ ~ $|0\rangle$ or $|1\rangle$

A quantum bit: $\alpha_0|0\rangle + \alpha_1|1\rangle,$ $\alpha_0^2 + \alpha_1^2 = 1$
(qubit)

A *superposition* of $|0\rangle$ and $|1\rangle$.

When you measure:

With probability α_0^2 it is $|0\rangle$.

With probability α_1^2 it is $|1\rangle$.

Representing data/information

An electron can be in "spin up" or "spin down" state.

$|\mathrm{up}\rangle$ or $|\mathrm{down}\rangle$ ~ $|0\rangle$ or $|1\rangle$

<u>A quantum bit:</u> $\alpha_0|0\rangle + \alpha_1|1\rangle,\qquad \alpha_0^2 + \alpha_1^2 = 1$

(qubit)

---

2 qubits:

$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

$$\alpha_{00}^2 + \alpha_{01}^2 + \alpha_{10}^2 + \alpha_{11}^2 = 1$$

Representing data/information

An electron can be in "spin up" or "spin down" state.

$$|\mathrm{up}\rangle \quad \text{or} \quad |\mathrm{down}\rangle \qquad \sim \qquad |0\rangle \quad \text{or} \quad |1\rangle$$

<u>A quantum bit:</u> $\alpha_0|0\rangle + \alpha_1|1\rangle,$ $\qquad \alpha_0^2 + \alpha_1^2 = 1$
(qubit)

---

3 qubits:

$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle +$$
$$\alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

$$\alpha_{000}^2 + \alpha_{001}^2 + \alpha_{010}^2 + \alpha_{011}^2 + \alpha_{100}^2 + \alpha_{101}^2 + \alpha_{110}^2 + \alpha_{111}^2 = 1$$

Representing data/information

An electron can be in "spin up" or "spin down" state.

$|\mathrm{up}\rangle$ or $|\mathrm{down}\rangle$ ~ $|0\rangle$ or $|1\rangle$

<u>A quantum bit:</u> $\alpha_0|0\rangle + \alpha_1|1\rangle,\qquad \alpha_0^2 + \alpha_1^2 = 1$

(qubit)

\- -

For n qubits, how many amplitudes are there?

Processing data

What will be our model?

In the classical setting, we had:

- Turing Machines
- Boolean circuits

In the quantum setting,
more convenient to use the circuit model.

Processing data: quantum gates

One non-trivial classical gate for a single classical bit:

$$0 \longrightarrow \boxed{\text{NOT}} \longrightarrow 1$$

$$1 \longrightarrow \boxed{\text{NOT}} \longrightarrow 0$$

There are many non-trivial quantum gates for a single qubit.

<u>One famous example:</u> **Hadamard gate**

$$|0\rangle \longrightarrow \boxed{H} \longrightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

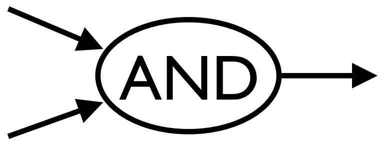
$$|1\rangle \longrightarrow \boxed{H} \longrightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

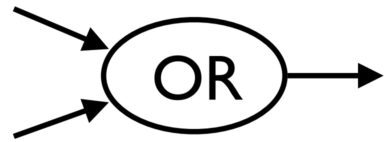
"transition" matrix:

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Processing data: quantum gates

Examples of classical gates on 2 classical bits:



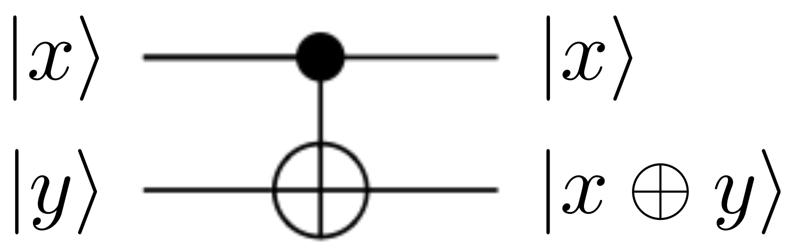


A famous example of a quantum gate on 2 qubits:

controlled NOT

For

$x, y \in \{0, 1\}$



"transition" matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

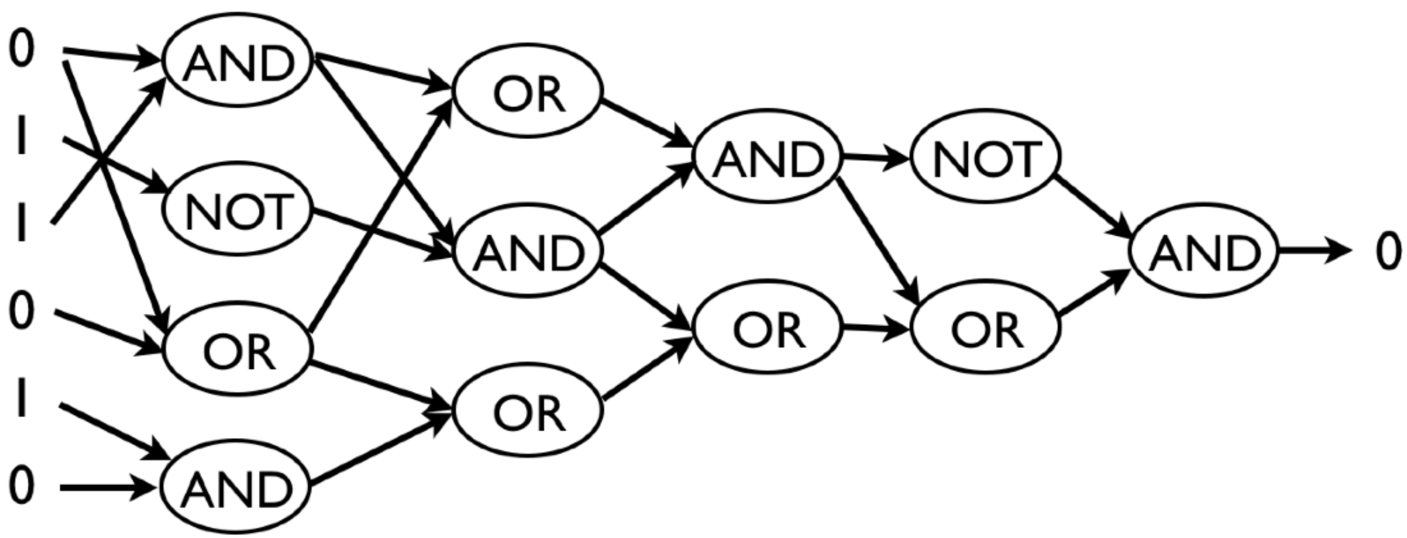
Processing data: quantum circuits

A classical circuit

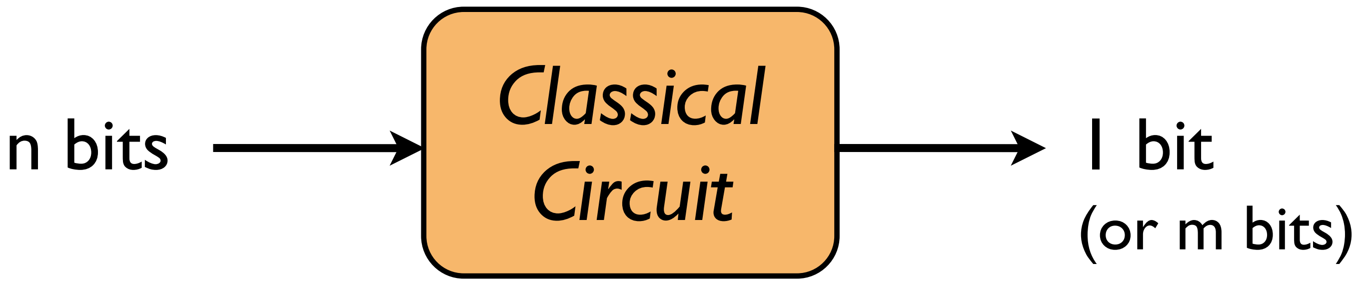
INPUT

OUTPUT

n bits



1 bit

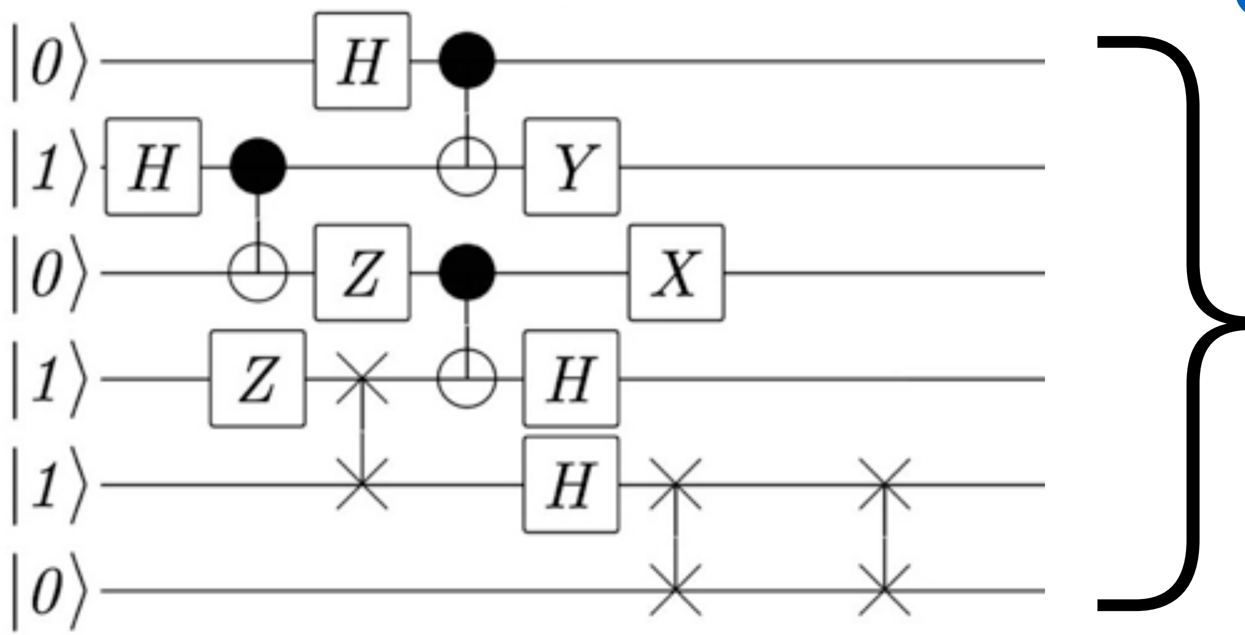


Processing data: quantum circuits

A quantum circuit

INPUT

n qubits

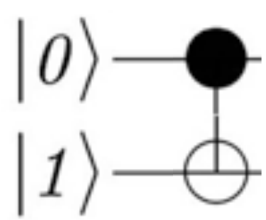


OUTPUT

n qubits

quantum gates

(acts on 1 qubit)



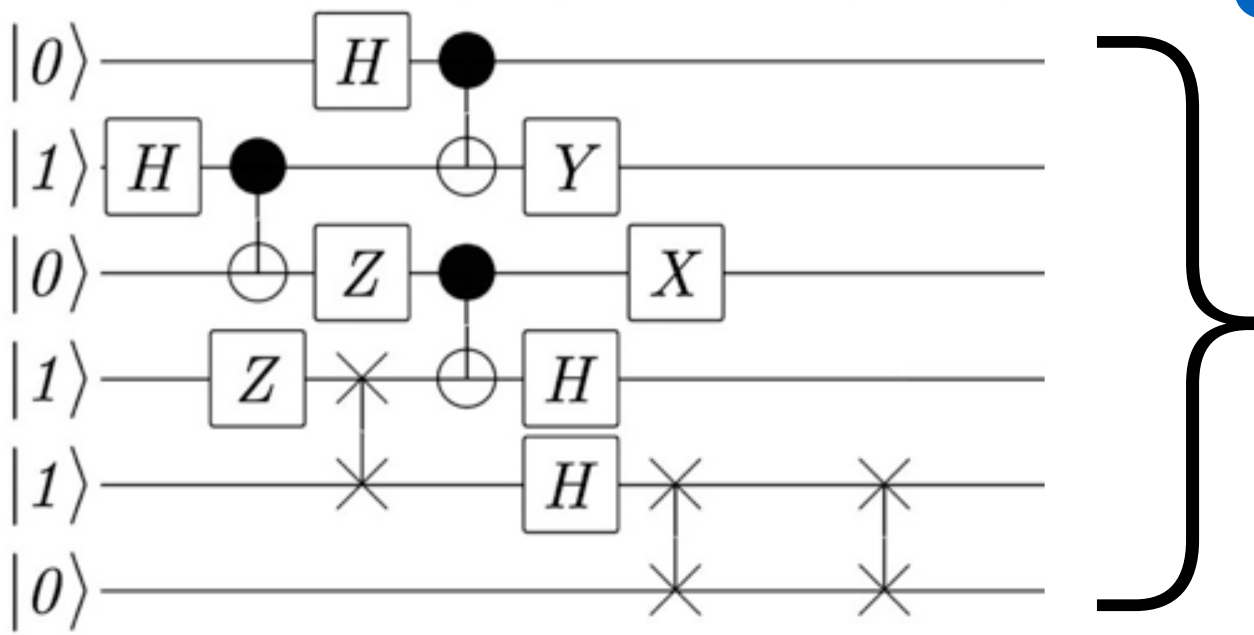
(acts on 2 qubits)

Processing data: quantum circuits

A quantum circuit

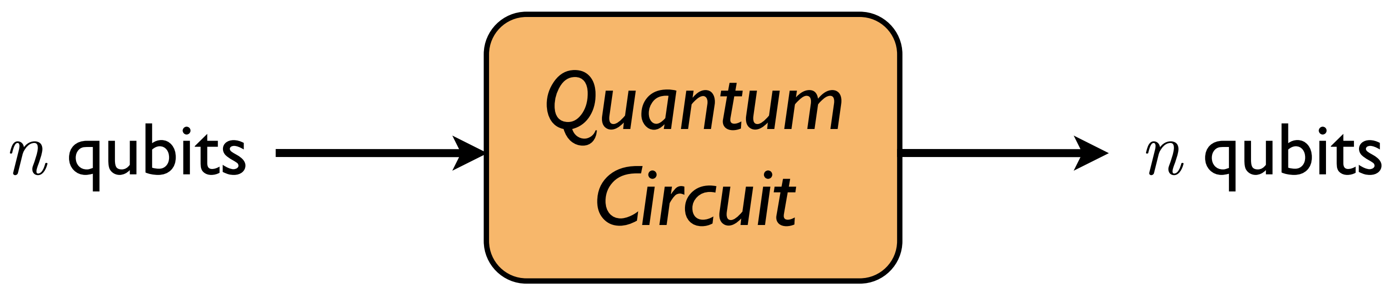
INPUT

n qubits



OUTPUT

n qubits

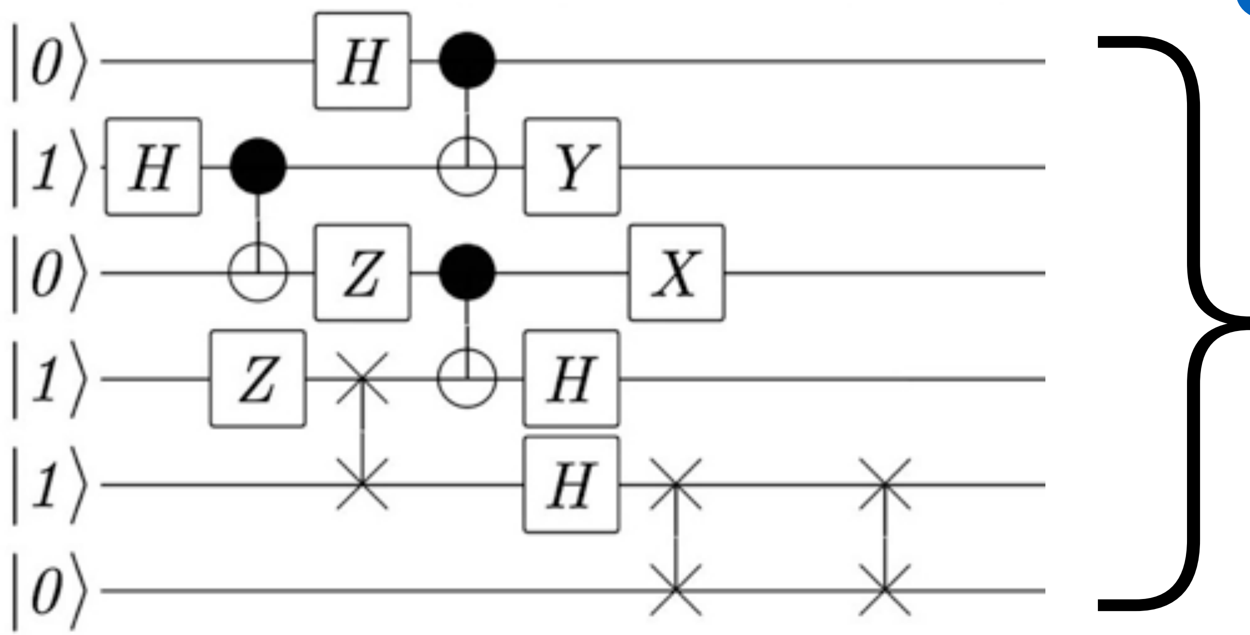


Processing data: quantum circuits

A quantum circuit

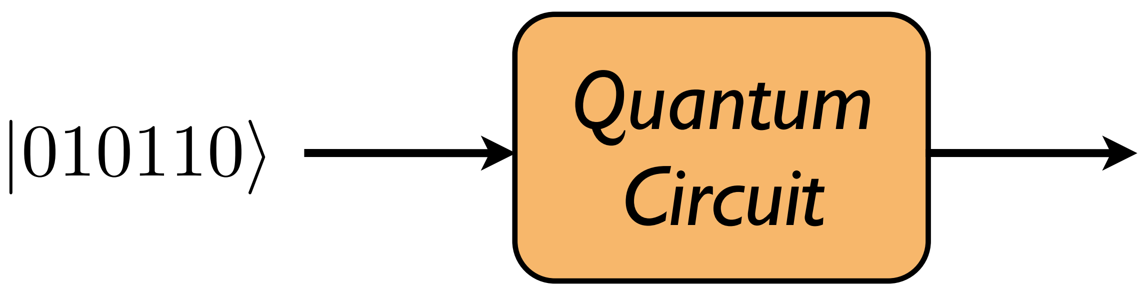
INPUT

n qubits



OUTPUT

n qubits

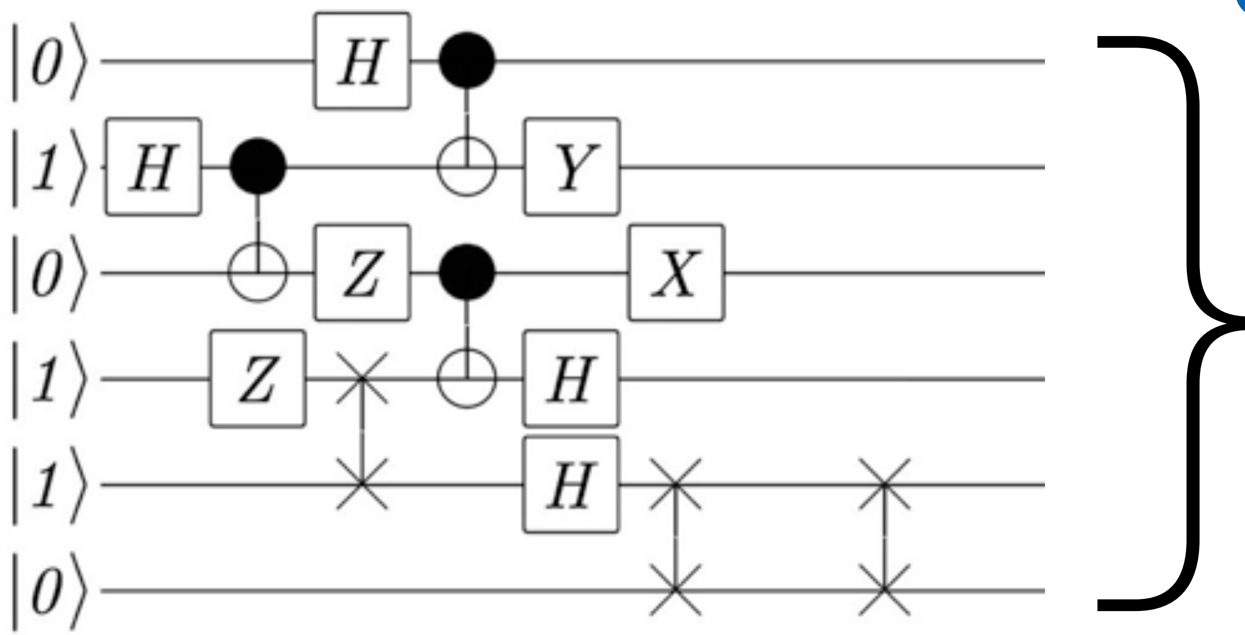


Processing data: quantum circuits

A quantum circuit

INPUT

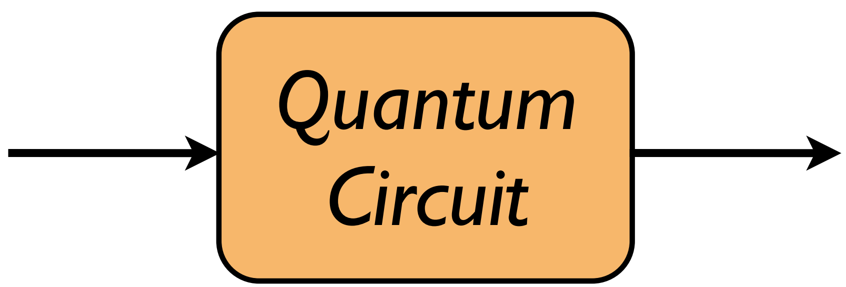
n qubits



OUTPUT

n qubits

$|010110\rangle$



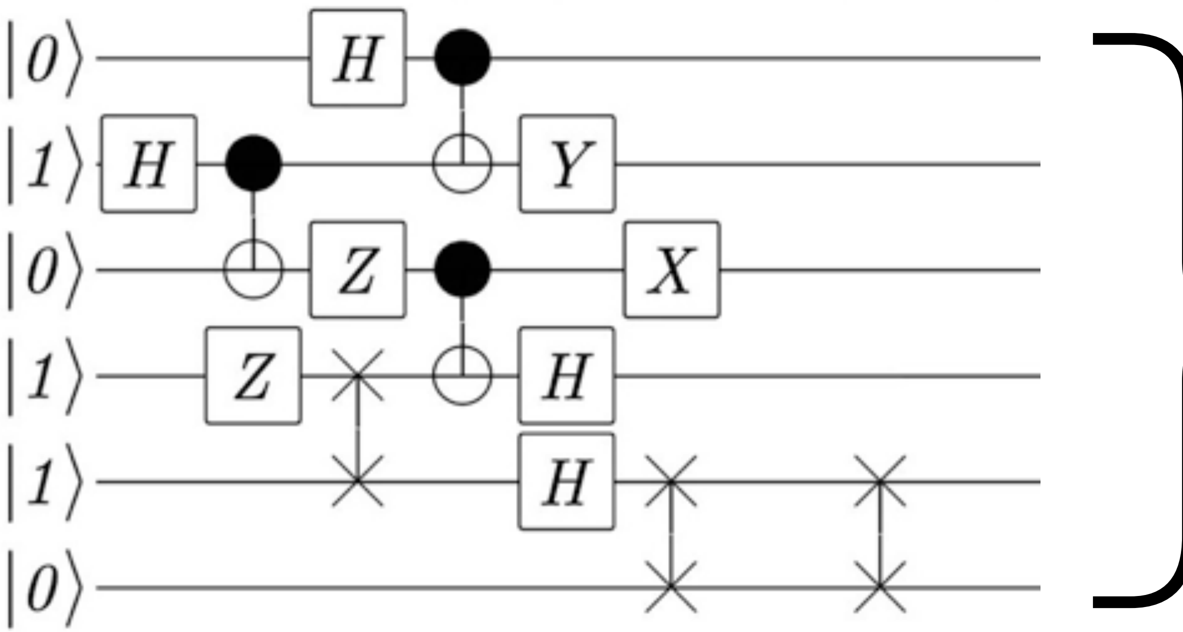
$$\alpha_{000000}|000000\rangle + \alpha_{000001}|000001\rangle + \alpha_{000010}|000010\rangle + \cdots \alpha_{111111}|111111\rangle$$

Processing data: quantum circuits

A quantum circuit

INPUT

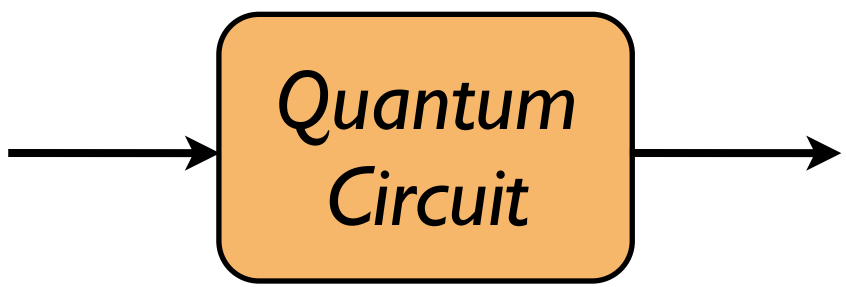
n qubits



OUTPUT

n qubits

n qubits



superposition of 2^n possible states.

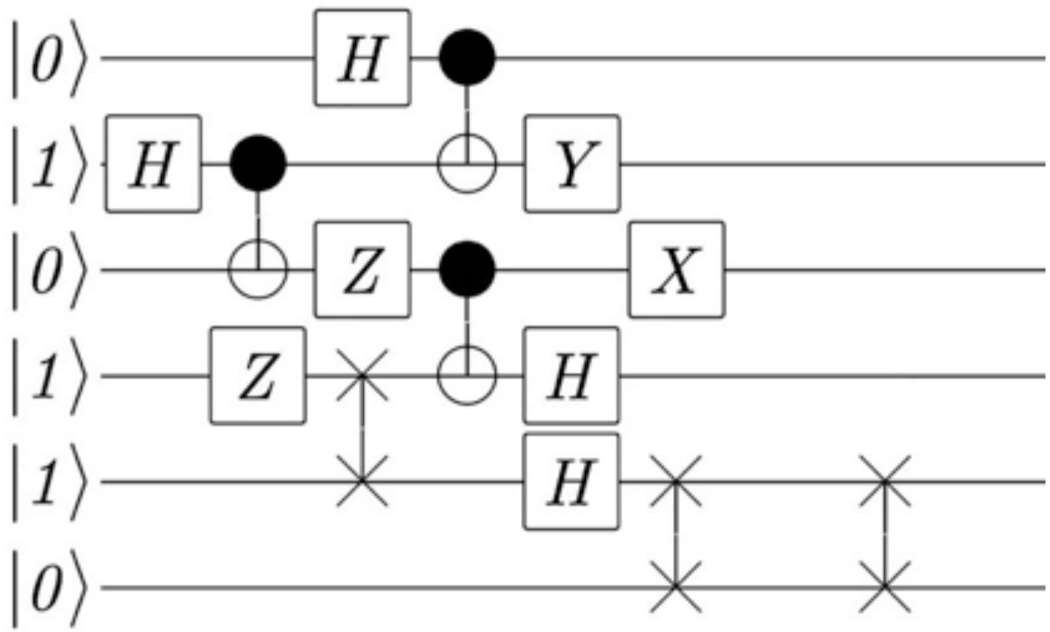
(2^n amplitudes)

Processing data: quantum circuits

A quantum circuit

INPUT

n qubits



OUTPUT

n qubits

How do we get "classical information" from the circuit?

We measure the output qubit(s).   e.g. we measure:

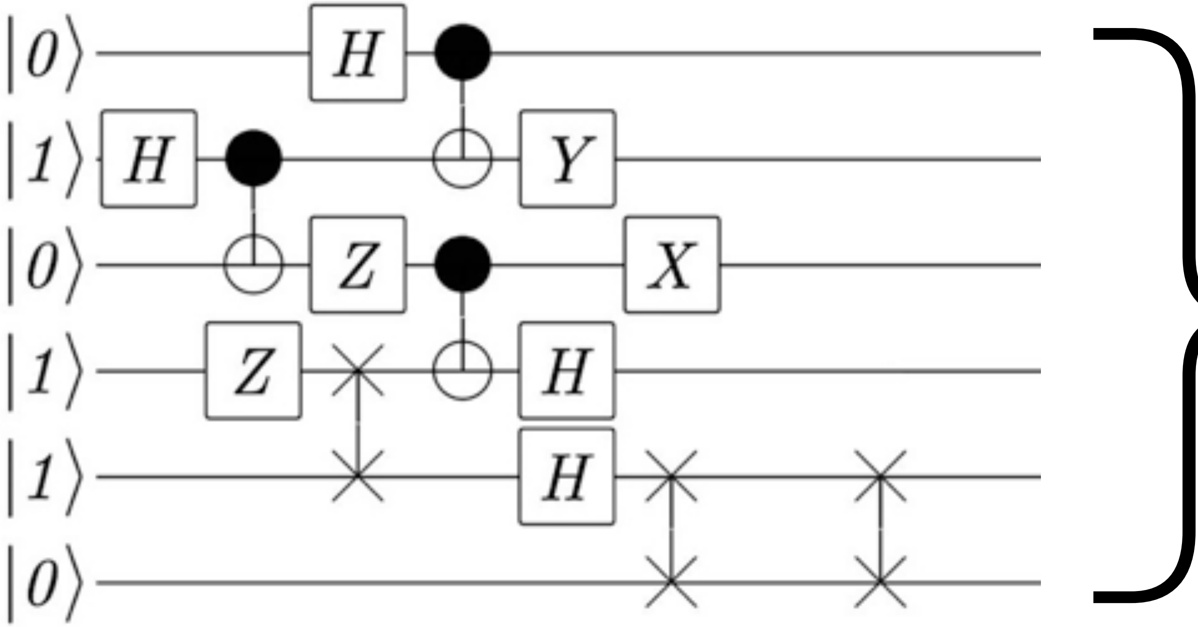
$$\alpha_{000000}|000000\rangle + \alpha_{000001}|000001\rangle + \cdots + \alpha_{111111}|111111\rangle$$

Processing data: quantum circuits

A quantum circuit

INPUT

n qubits



OUTPUT

n qubits

Complexity?

number of gates ~ computation time

Physical Realization

?

Practical, Scientific and Philosophical Perspectives

Practical perspective

What useful things can we do with a quantum computer?

We can factor large numbers efficiently!

2037035976334486086268445688409378161051468393665936250636140449354381299763336706183397
8445688409378161051468393665936250636140449354381299763336706183399283749287291091983417
9928347197479829827503487954789789527890241387943278904327367835537895078213785825498719

So what?

Can break RSA!

Can we solve every problem efficiently?

No !

Practical perspective

What useful things can we do with a quantum computer?

Can simulate quantum systems efficiently!

Better understand behavior of atoms and moleculues.

Applications:

- nanotechnology
- microbiology
- pharmaceuticals
- superconductors.

...

Scientific perspective

To know the limits of efficient computation:

Incorporate actual facts about physics.

Scientific perspective

(Physical) Church Turing Thesis

Any computational problem that can be solved by a physical device, can be solved by a Turing Machine.

Strong version

Any computational problem that can be solved ***efficiently*** by a physical device, can be solved ***efficiently*** by a TM.

Strong version doesn't seem to be true!

Philosophical perspective

Is the universe deterministic ?

How does nature keep track of all the numbers ?

$$1000 \text{ qubits } \rightarrow 2^{1000} \text{ amplitudes}$$

How should we interpret quantum measurement?
(the measurement problem)

Does quantum physics have anything to say about the human mind?

Quantum AI?

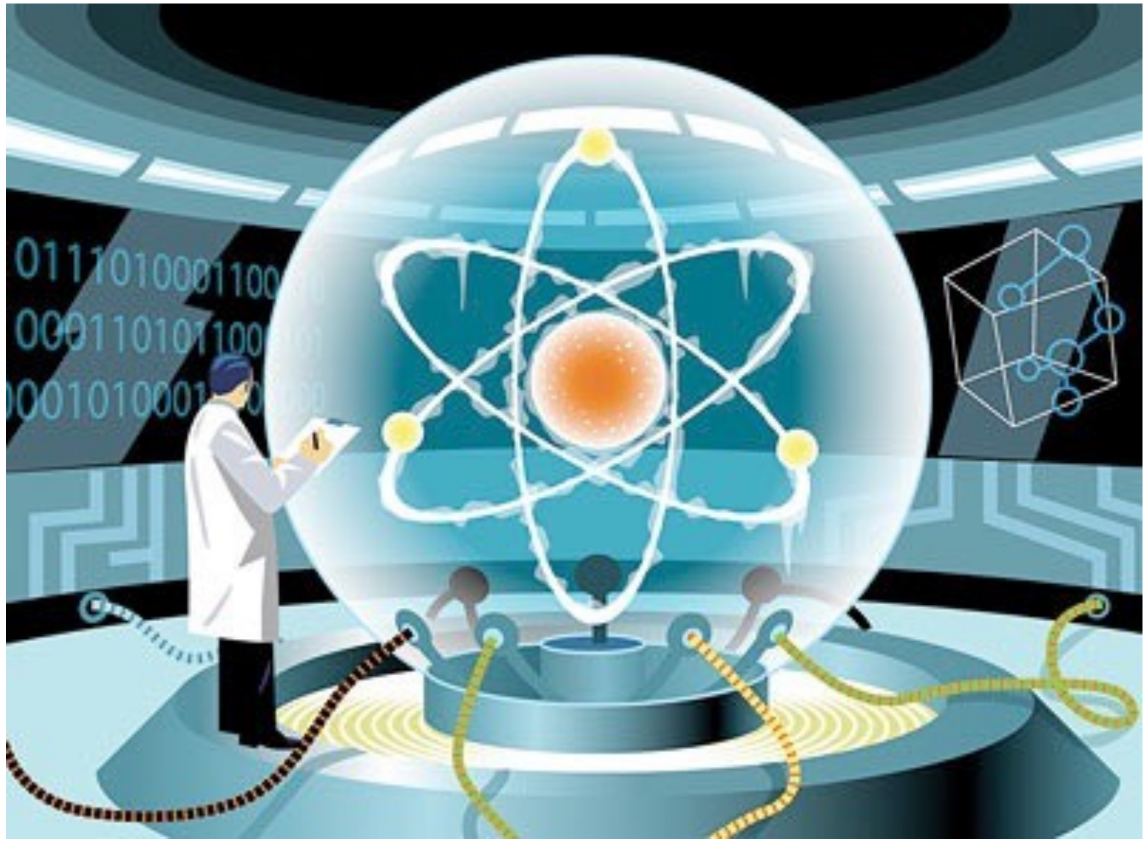
Where are we at building quantum computers?

When can I expect a quantum computer on my desk ?

After about 20 years and 1 billion dollars of funding :
Can factor 21 into 3 x 7.   (with high probability)

Challenge: Interference with the outside world.

"quantum decoherence"



A whole new exciting world of computation.

Potential to fundamentally change how we view computers and computation.