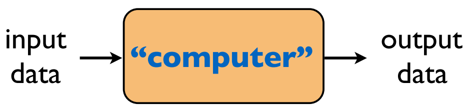
15-251
Great Ideas in
Theoretical Computer Science

Lecture 6:
Church-Turing Thesis + Decidability

September 14th, 2017

This Week



What is computation?

What is an algorithm?

How can we mathematically define them?

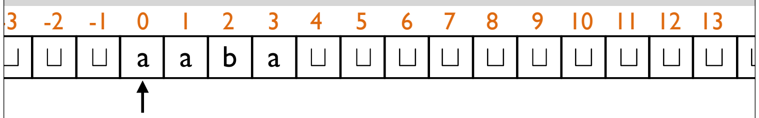
Last Time

A totally minimal (TM) programming language such that

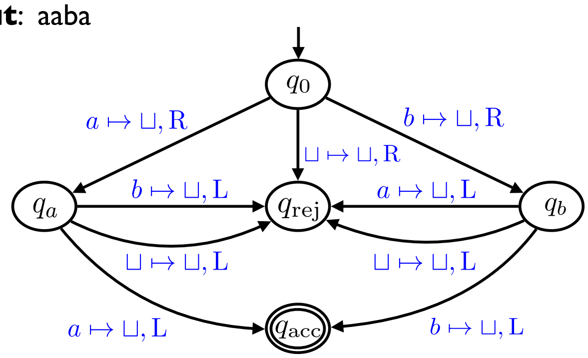
- it can simulate simple bytecode
  (and therefore Python, C, Java, SML, etc…)

- it is simple to define and reason about completely mathematically rigorously

Last Time



Input: aaba



Last Time

A Turing machine (TM) M is a 7-tuple

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\mathrm{acc}}, q_{rej})$$

where

- Q is a finite set (which we call the set of states);
- Σ is a finite set with $\sqcup \notin \Sigma$ (which we call the input alphabet);
- Γ is a finite set with $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$ (which we call the tape alphabet);
- δ is a function of the form $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ (which we call the transition function);
- $q_0 \in Q$ (which we call the start state);
- $q_{\mathrm{acc}} \in Q$ (which we call the accept state);
- $q_{\mathrm{rej}} \in Q$, $q_{\mathrm{rej}} \neq q_{\mathrm{acc}}$ (which we call the reject state);

Last Time

Definition: A TM is called a *decider* if it halts on all inputs.

Definition: A language L is called *decidable* (*computable*) if $L = L(M)$ for some <u>decider</u> TM M.

Theorem: Any language that can be computed in Python, C, Java, etc. can be decided by a TM.

QUESTIONS

2 of Hilbert's Problems

Hilbert's 10th problem (1900)

Is there a finitary procedure to determine if a given multivariate polynomial with integral coefficients has an integral solution?

e.g. $5x^2yz^3 + 2xy + y - 99xyz^4 = 0$

Entscheidungsproblem (1928)

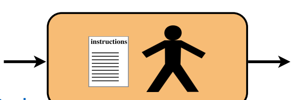
Is there a finitary procedure to determine the validity of a given logical expression?

e.g. $\neg\exists x, y, z, n \in \mathbb{N} : (n \geq 3) \wedge (x^n + y^n = z^n)$

(Mechanization of mathematics)

The quest for the right definition

input data (mathematical statement)



output data (True or False)

"Alright, let's define this thing mathematically."

Turing's thinking

- A (human) computer writes symbols on paper.
  (can view the paper as a sequence of squares)
- No upper bound on the number of squares.
- Humans can reliably distinguish finitely many shapes.
- Human observes one square at a time.
- Human has finitely many mental states.
- Human can change symbol,
  can change focus to neighboring square,
  based on its state and the symbol it observes
- Human acts deterministically.
- …

Turing's legacy

The beauty of his definition:

1. **simplicity**
2. **"clearly" captures what a human does given a set of instructions.**

Simplicity

1. **simplicity**

a reasonable definiton of computation

↓

strong enough to capture computation the way TMs do.

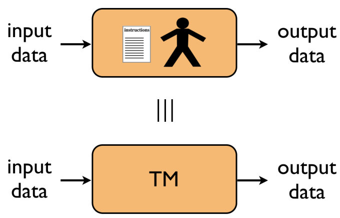
(anyone who attempted to define computation could accidentally hit a correct definition)

Generality

2. "clearly" captured what a human does given a set of instructions.

Church-Turing Thesis

The intuitive notion of "computable" is captured by functions computable by a Turing Machine.



What else did Turing do in his paper?

Entscheidungsproblem (1928)

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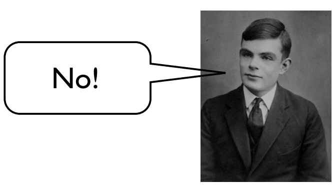
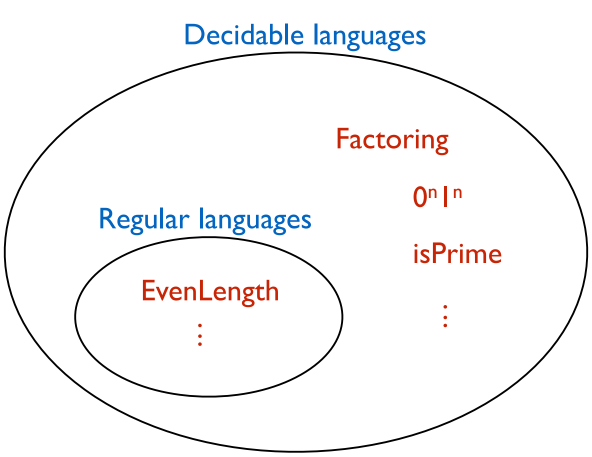


Entscheidungsproblem

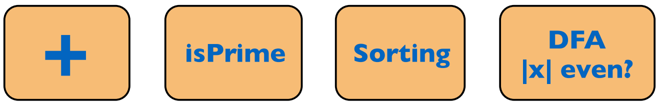
Are there others?



What else did Turing do in his paper?

Universal Machine

(one machine to rule them all)



Do we really need a separate machine for each task?

What else did Turing do in his paper?

Universal Machine

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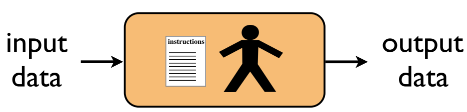
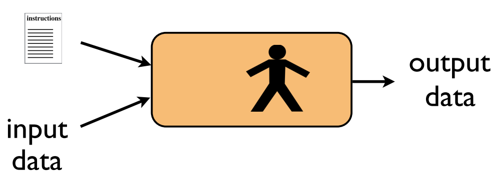


A human is a universal machine:



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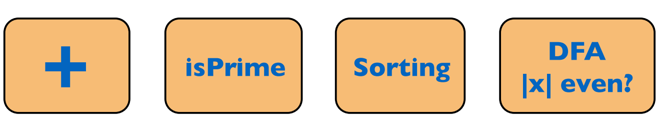


All can be encoded!

(e.g. think source code)

What else did Turing do in his paper?

Universal Machine

(one machine to rule them all)

We could use:

$$\langle M \rangle =$$

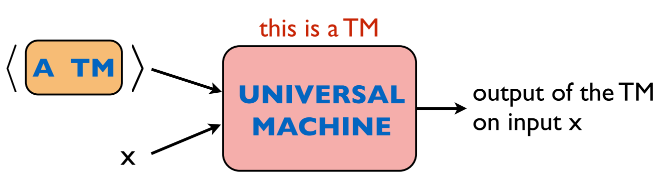
```
def foo(input):
  i = 0
  STATE 0:
    letter = input[i];
    switch(letter):
      case ‘a’: input[i] = ‘ ’; i++; go to STATE a;
      case ‘b’: input[i] = ‘ ’; i++; go to STATE b;
      case ‘ ’: input[i] = ‘ ’; i++; go to STATE rej;
  STATE a:
    letter = input[i];
    switch(letter):
      case ‘a’: input[i] = ‘ ’; i--; go to STATE acc;
      case ‘b’: input[i] = ‘ ’; i--; go to STATE rej;
      case ‘ ’: input[i] = ‘ ’; i--; go to STATE rej;
```

Code is data!

What else did Turing do in his paper?

Universal Machine

(one machine to rule them all)



Could you write a Python function that does this?

What else did Turing do in his paper?

Universal Machine

(one machine to rule them all)

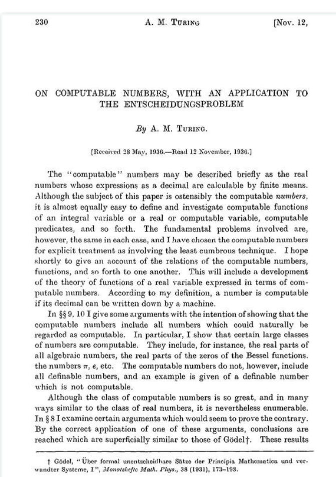
This is exactly what an **interpreter** does.

foo.py → **Python Interpreter** → output of the Python program on input x

x → **Python Interpreter**

Code is data!

The positive side	The negative side
Universal TM	Self-referencing
	(can feed a machine its own code as input.)
	↓
	Undecidability



230 A. M. Turing [Nov. 12,

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

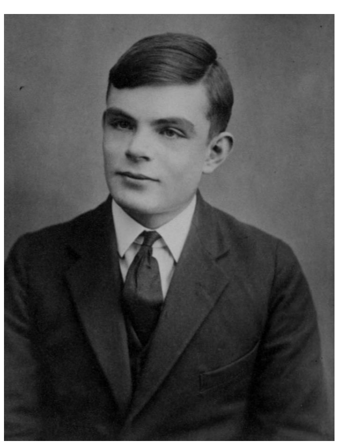
The "computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means. Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computable functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers. According to my definition, a number is computable if its decimal can be written down by a machine.

In §§ 9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers π, e, etc. The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable. In § 8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†. These results

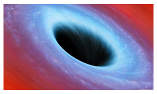
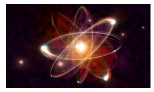
† Gödel, "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte Math. Phys.*, 38 (1931), 173–198.

1936



1912 - 1954

Perhaps Turing and others weren't ambitious enough!

 Solvable by any physical process 

||| **???**

Solvable by a TM

Physical Church-Turing Thesis

Physical Church-Turing Thesis

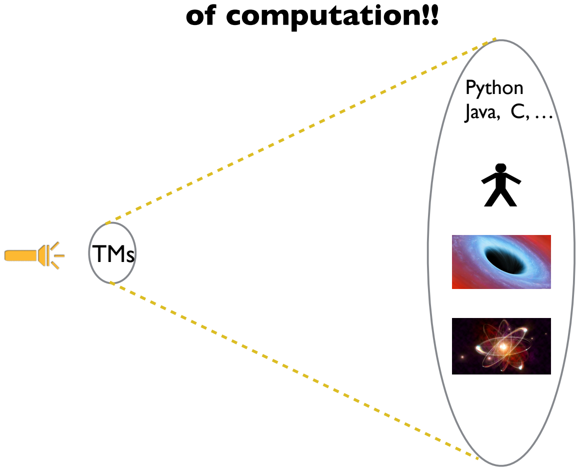
What can be computed in this universe, by any physical process or device, can by computed by a (rand.) TM.

Why should we expect this to be true?

Strong Physical Church-Turing Thesis

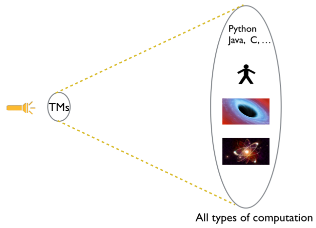
What can be computed efficiently in this universe, by any physical process or device, can by computed efficiently by a **Q**TM.

This is the grand unification/simplification of computation!!



All types of computation

This is the grand unification/simplification of computation!!



Complex things can be explained by simple rules.

- physics: try to find the simple rules that give rise to the universe
- evolution: complex life forms emerge from simple beginnings and rules
- math: complex proofs arise by combining very simple deductive rules
- programming: everything boils down to super simple instructions

Implications

1. Studying the power and limits of TMs

→ Studying the power and limits of our universe

(Can you come up with laws of physics that would allow it to compute any problem?)

2. Computation in its full generality is everywhere. Even in extremely simple systems!

(What is the simplest universe you can create that has the same computational capacity of our universe?)

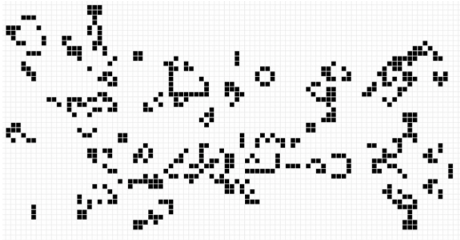
3. The universe may be a simulation. (a philosophical musing)

What is the simplest universe you can create that has the same computational capacity of our universe?

Conway's Game of Life

Imagine an infinite 2D grid.

Each cell can be dead or alive.



Laws of physics

Loneliness: live cell with fewer than 2 neigbors dies.

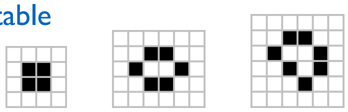
Overcrowding: live cell with more than 3 neighbors dies.

Procreation: dead cell with exactly 3 neighbors gets born.

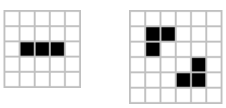
Conway's Game of Life

Some Patterns

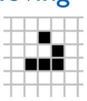
Stable



Periodic



Moving



Conway's Game of Life

Can a TM simulate any instance of Game of Life?

Can Game of Life simulate any TM?

Can Game of Life simulate Game of Life?

That's all for the Church-Turing Thesis.

Let's talk decidability.

Languages involving encodings of machines

Code is data!

There are many interesting problems
where the input data is code.

Working as a TA for 15-112

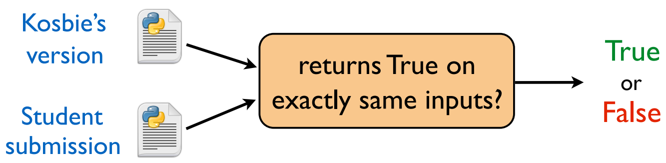
Autograder program

student submission
isPrime

the correct program
isPrime

Do they return True on exactly the same inputs?

Working as a TA for 15-112



Does such a program exist?

i.e., can we solve the following?

$\text{EQ} = \{\langle M_1, M_2 \rangle : M_1 \text{ and } M_2 \text{ are TMs s.t. } L(M_1) = L(M_2)\}$

Working as a TA for 15-112

Similar but simpler looking languages:

$\text{ACCEPTS} = \{\langle M, x \rangle : M \text{ is a TM and } x \in \Sigma^* \text{ s.t. } x \in L(M)\}$

$\text{EMPTY} = \{\langle M \rangle : M \text{ is a TM s.t. } L(M) = \emptyset\}$

Poll

Which ones do you think are decidable?

$\text{ACCEPTS}_{\text{DFA}} = \{\langle D, x \rangle : D \text{ is a DFA and } x \in \Sigma^* \text{ s.t. } x \in L(D)\}$

$\text{SELF-ACCEPTS}_{\text{DFA}} = \{\langle D \rangle : D \text{ is a DFA s.t. } \langle D \rangle \in L(D)\}$

$\text{EMPTY}_{\text{DFA}} = \{\langle D \rangle : D \text{ is a DFA s.t. } L(D) = \emptyset\}$

$\text{EQ}_{\text{DFA}} = \{\langle D_1, D_2 \rangle : D_1 \text{ and } D_2 \text{ are DFAs s.t. } L(D_1) = L(D_2)\}$

ACCEPTS$_{\text{DFA}}$

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$\mathrm{EQ}_{\mathrm{DFA}}$

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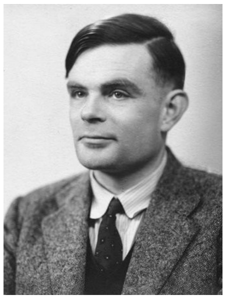
Reduction

NEXT WEEK

Turing's Legacy Continues



Undecidability