# Graduate AI

## Lecture 2:
## Search I

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# Search Problems

- A search problem has:
  - States (configurations)
  - Start state and goal states
  - Successor function: maps states to (action,state,cost) triples

# Example: Pancakes

Discrete Mathematics 27 (1979) 47–57.


**BOUNDS FOR SORTING BY PREFIX REVERSAL**

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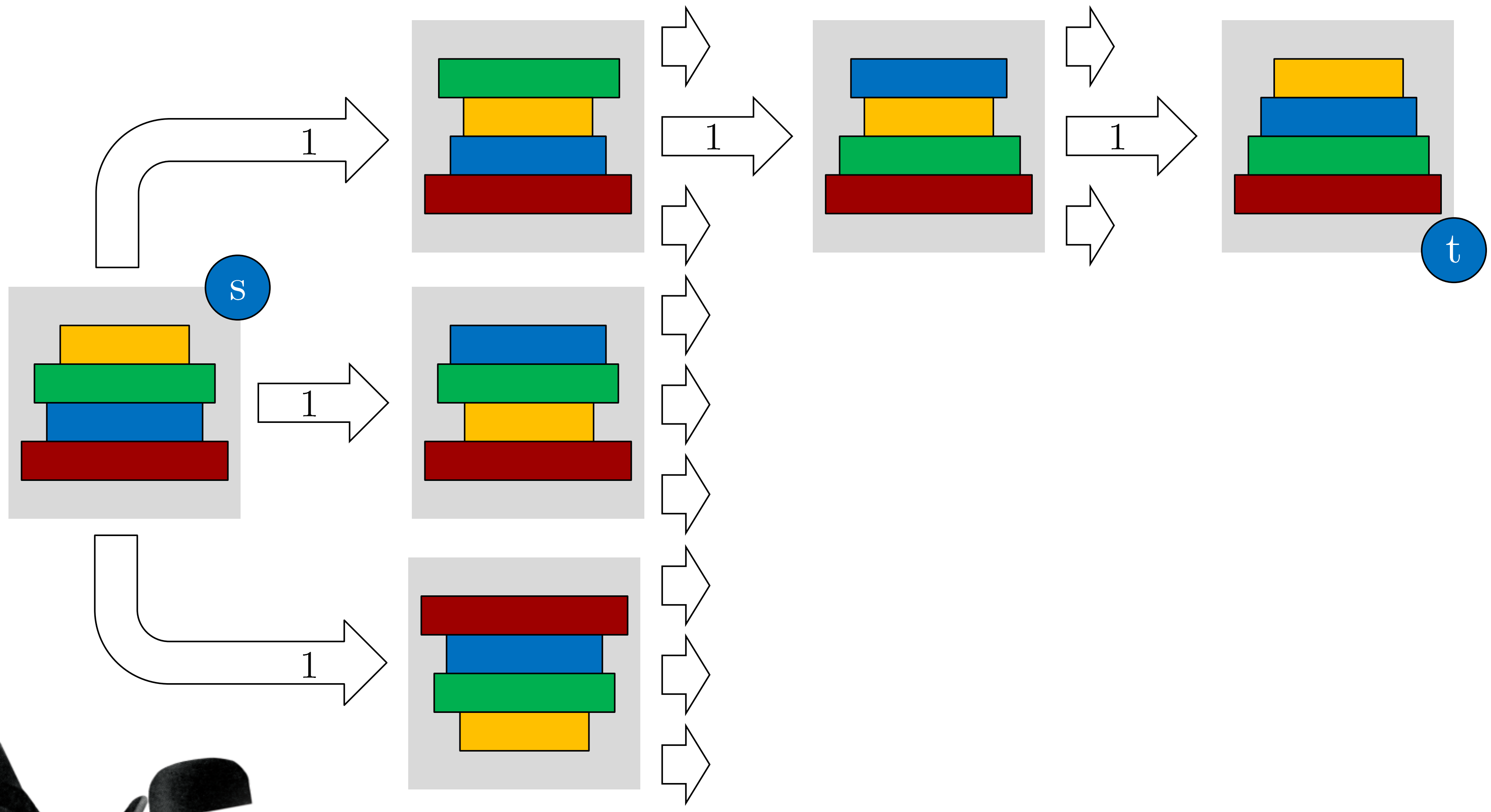
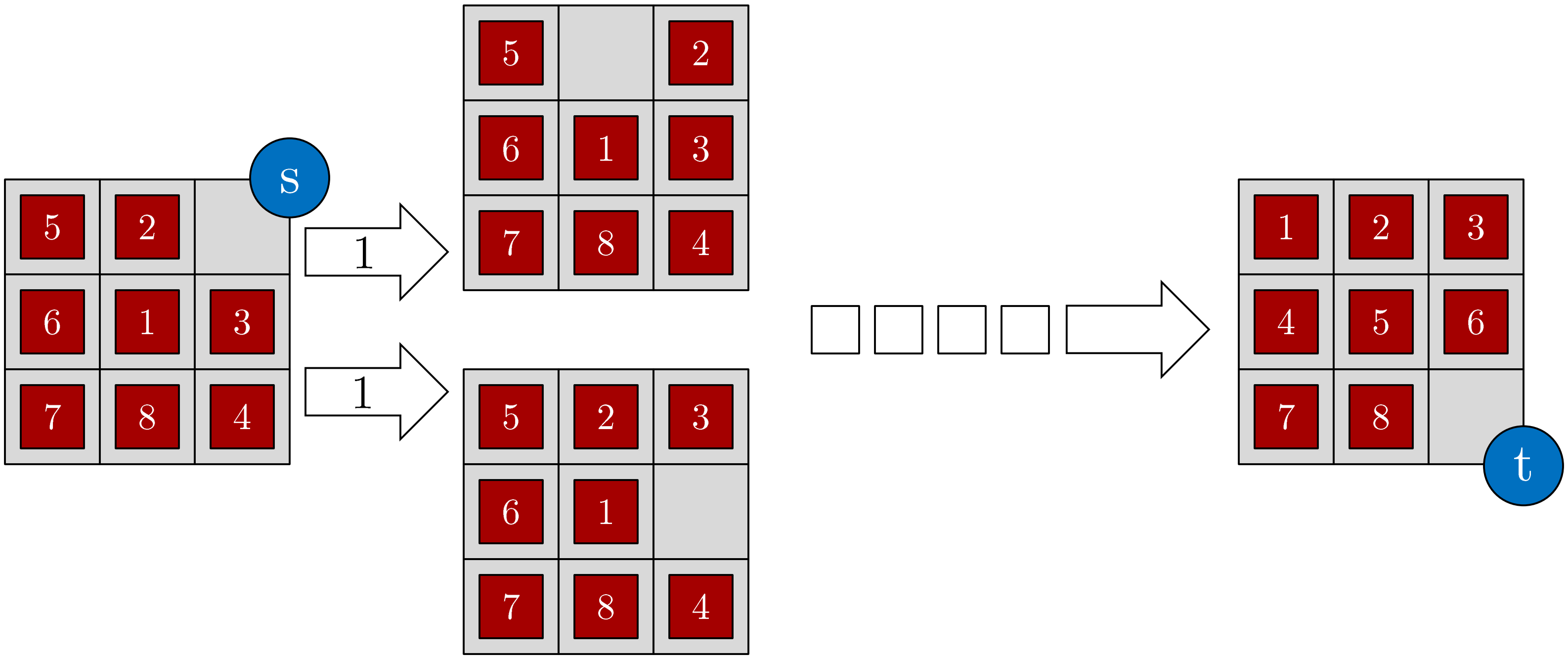
For a permutation $\sigma$ of the integers from 1 to $n$, let $f(\sigma)$ be the smallest number of prefix reversals that will transform $\sigma$ to the identity permutation, and let $f(n)$ be the largest such $f(\sigma)$ for all $\sigma$ in (the symmetric group) $S_n$. We show that $f(n) \leqslant (5n+5)/3$, and that $f(n) \geqslant 17n/16$ for $n$ a multiple of 16. If, furthermore, each integer is required to participate in an even number of reversed prefixes, the corresponding function $g(n)$ is shown to obey $3n/2-1 \leqslant g(n) \leqslant 2n+3$.

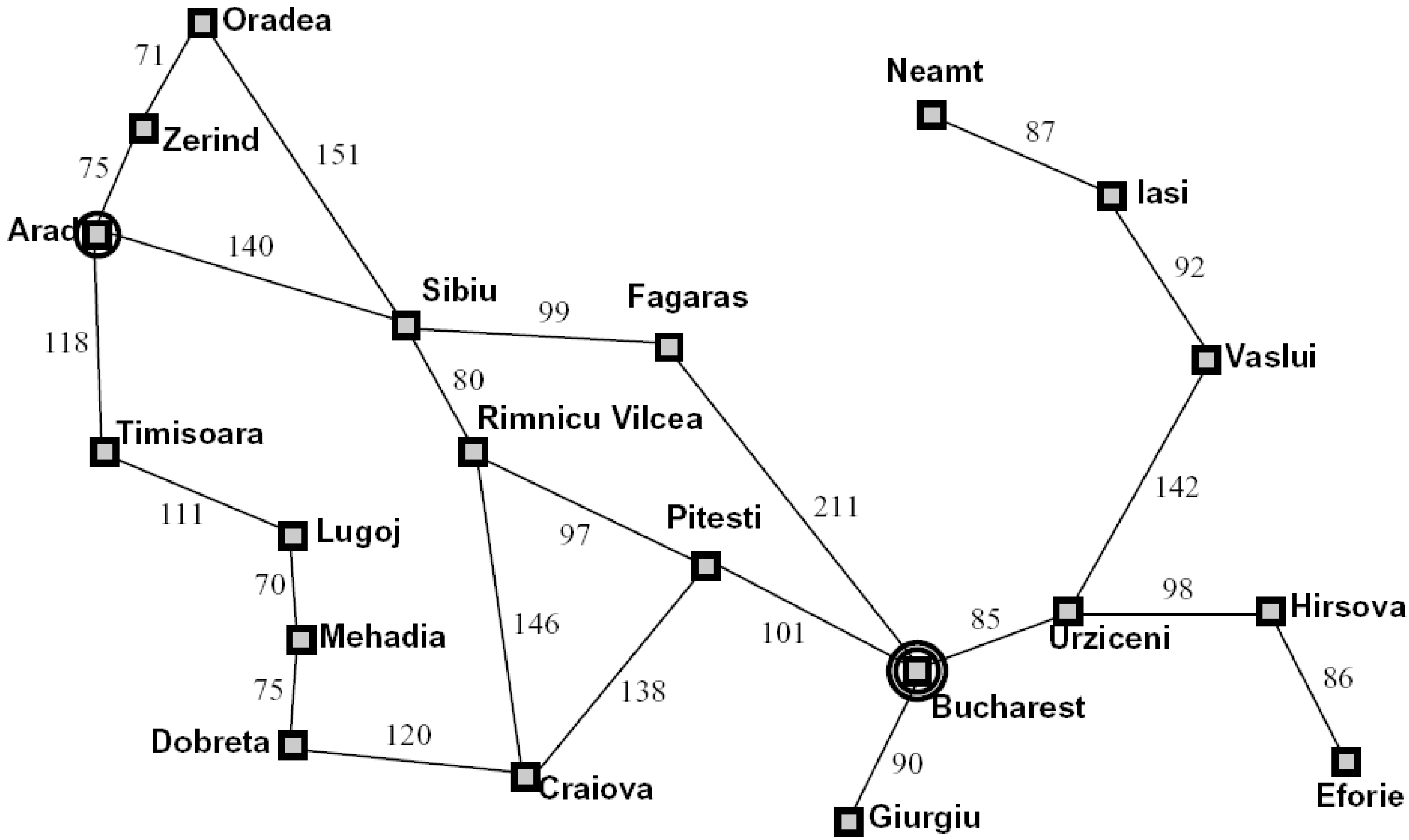
# Example: Pancakes



# Example: 8-Puzzle



# Example: Pathfinding



# TREE SEARCH

function TREE-SEARCH(problem, strategy)

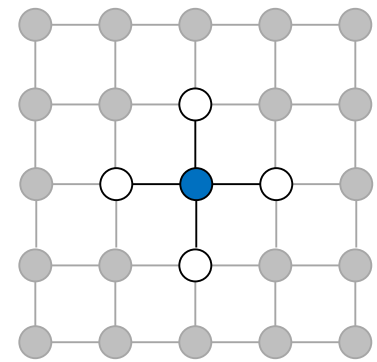
set of frontier nodes contains the start state of problem

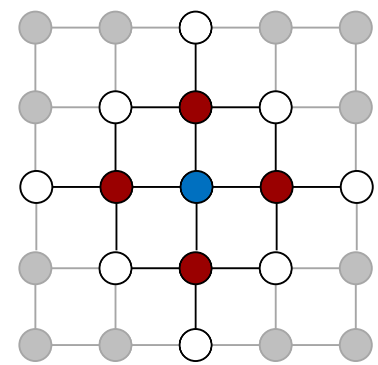
loop

- if there are no frontier nodes then return failure
- choose a frontier node for expansion using strategy
- if the node contains a goal then return the corresponding solution
- else expand the node and add the resulting nodes to the set of frontier nodes

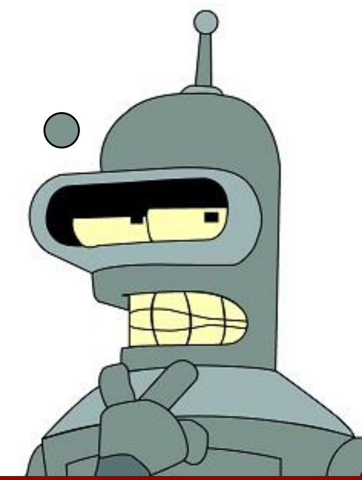
# Tree Search

- Tree search can expand many nodes corresponding to the same state
- In a rectangular grid:
  - Search tree of depth $d$ has $4^d$ leaves
  - There are only $4d$ states at Manhattan distance exactly $d$ from any given state





Algorithms that forget their history are doomed to repeat it!



# Graph Search

function Graph-Search(problem, strategy)

set of frontier nodes contains the start state of problem

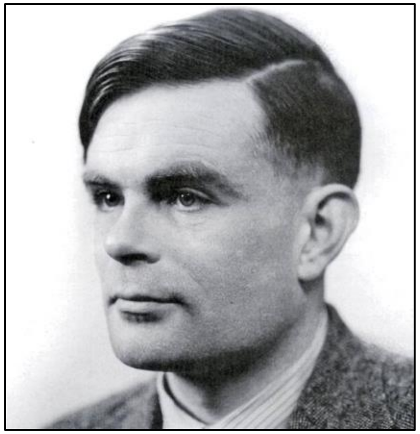
loop

- if there are no unexpanded frontier nodes then return failure
- choose an unexpanded frontier node for expansion using strategy, and add it to the expanded set
- if the node contains a goal then return the corresponding solution
- else expand the node and add the resulting nodes to the set of frontier nodes, only if not in the expanded set

# Uninformed vs. Informed

## Uninformed

Can only generate successors and distinguish goals from non-goals

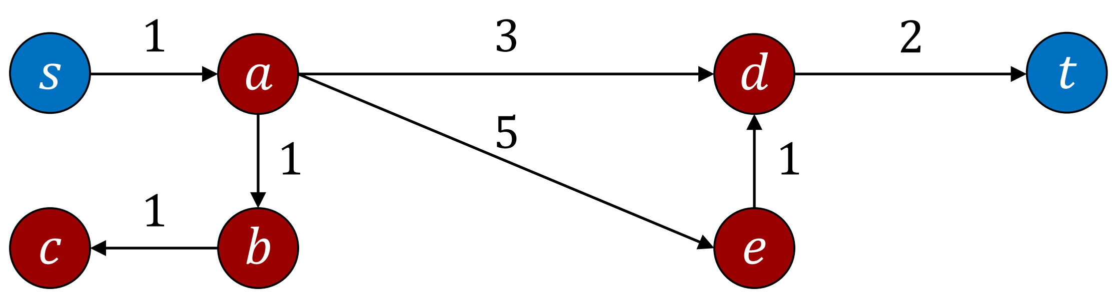


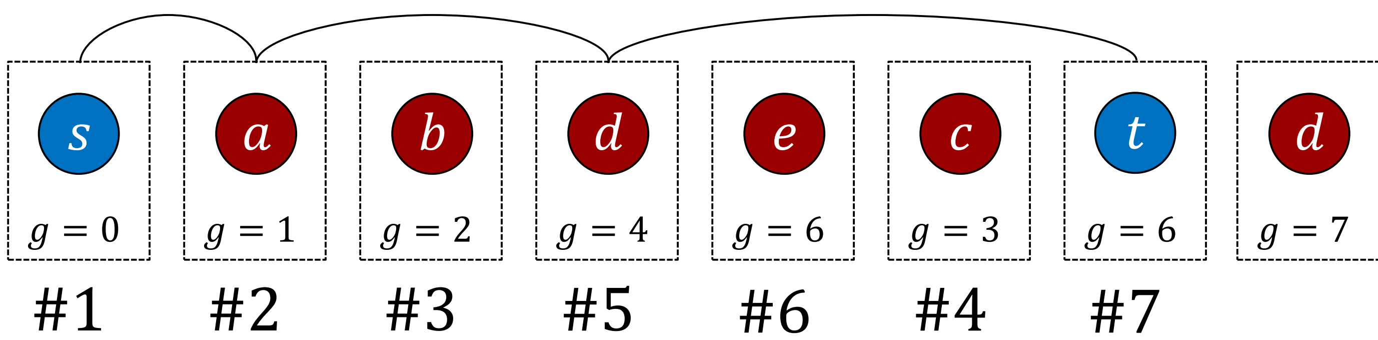
## Informed

Strategies that know whether one non-goal is more promising than another

# Uniform Cost Search

- Strategy: Expand by $g(x)$ = work done so far



# Uninformed vs. Informed

## Uninformed

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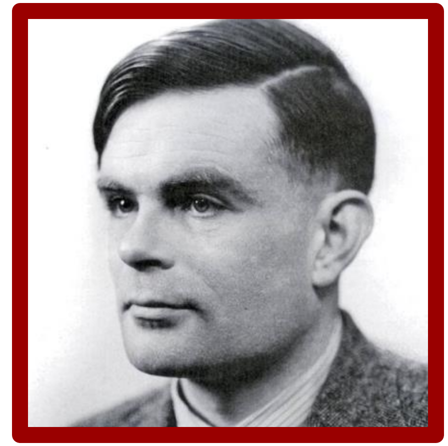
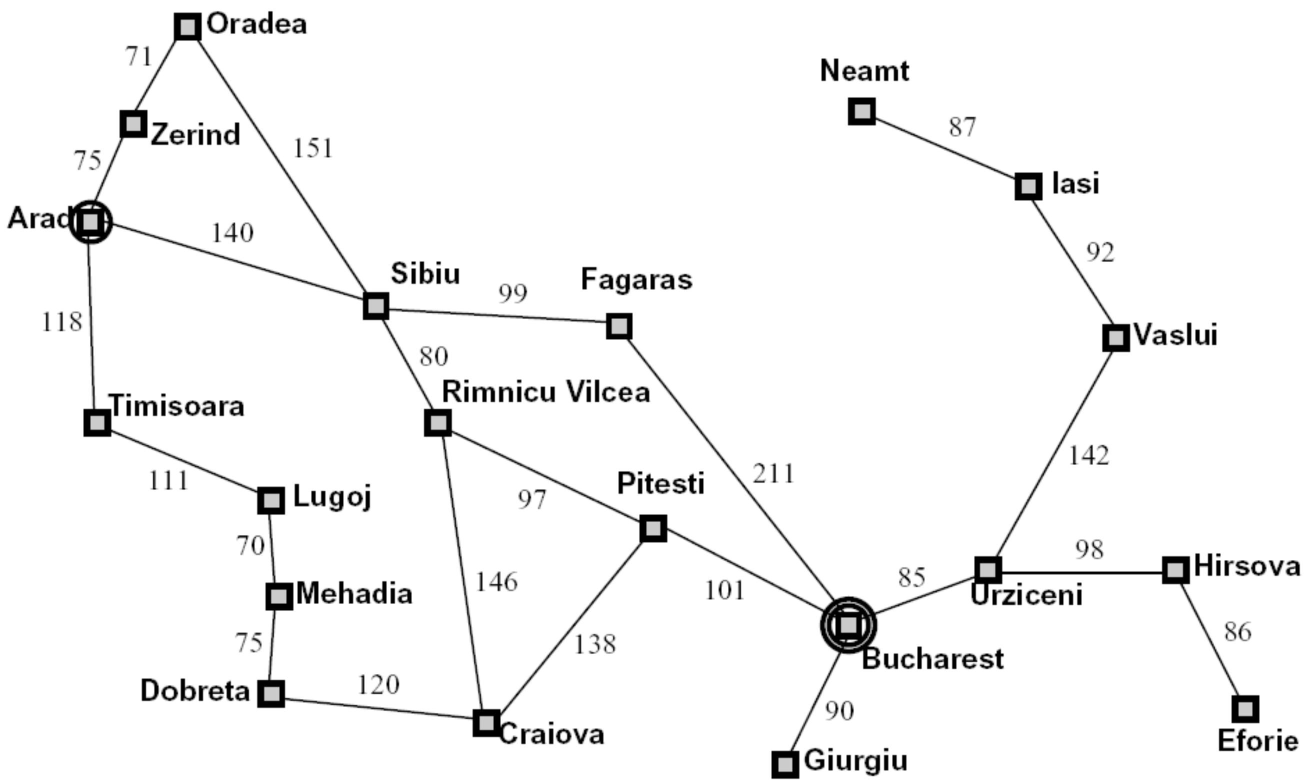


## Informed

Strategies that know whether one non-goal is more promising than another

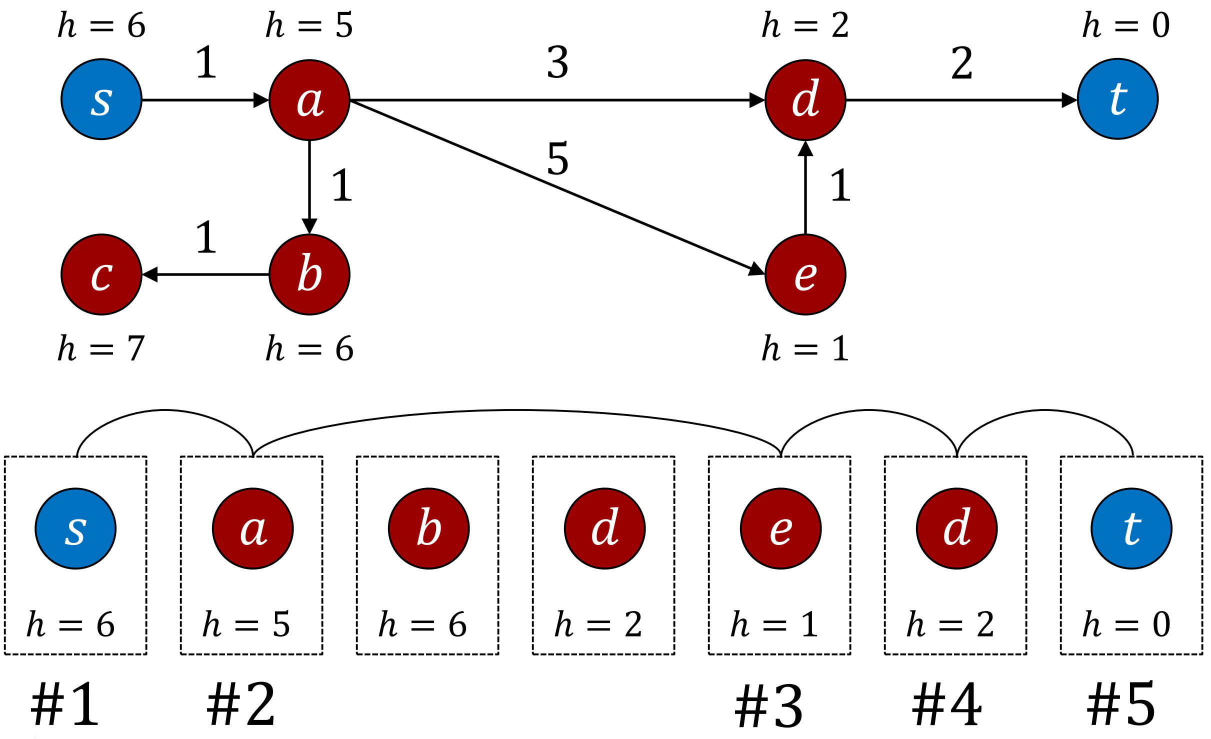
# Example: Heuristic

| City | Aerial dist |
|---|---|
| Arad | 366 |
| Sibiu | 253 |
| Rimnicu Vilcea | 193 |
| Fagaras | 176 |
| Pitesti | 100 |



# Greedy Search

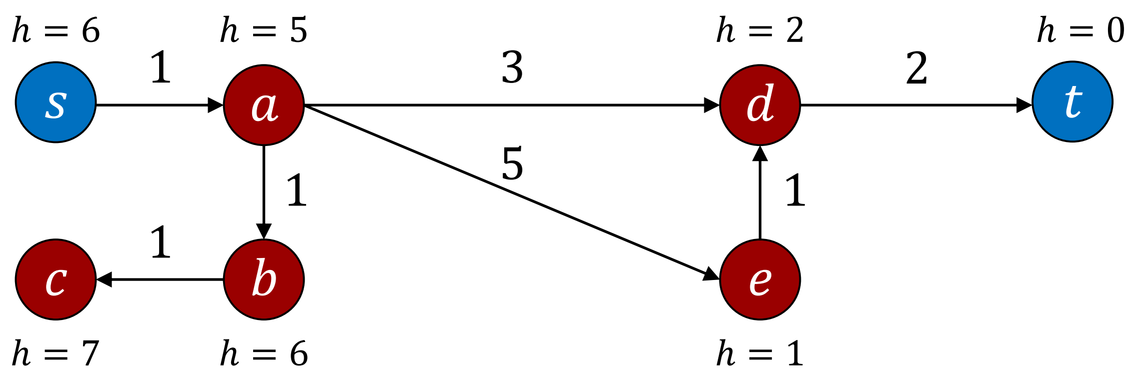
- Strategy: Expand by $h(x)$ = heuristic evaluation of cost from $x$ to goal



# A* Search

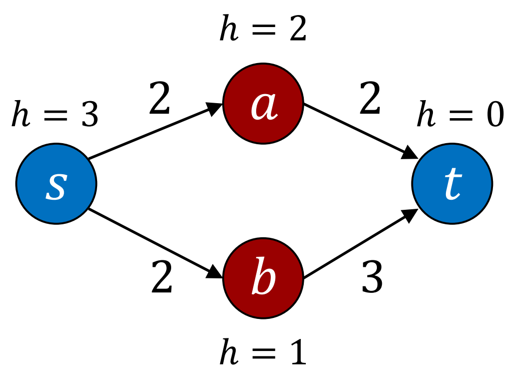
- **Strategy:** Expand by $f(x) = h(x) + g(x)$
- **Poll 1:** Which node is expanded fourth?

1. $d$
2. $e$
3. $t$
4. $c$



# A* Search

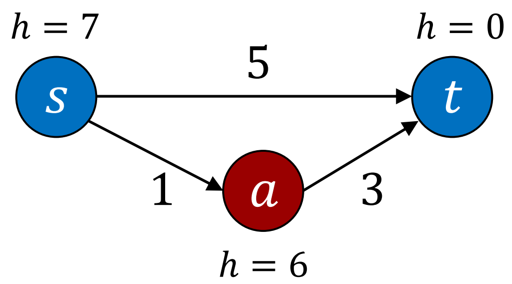
- Should we stop when we discover a goal?




- No: Only stop when we expand a goal

# A* Search

- Is A* optimal?



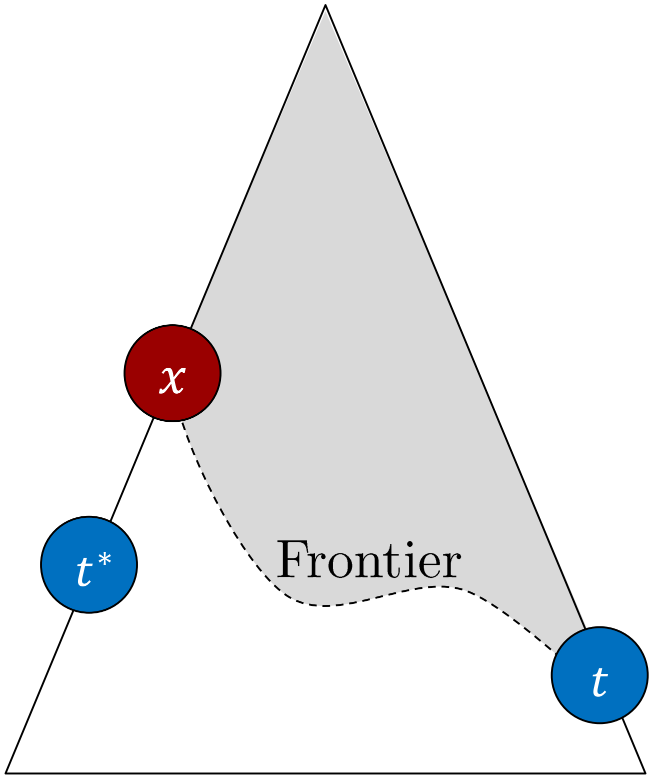

- Good path has pessimistic estimate
- Circumvent this issue by being optimistic!

# Admissible Heuristics

- $h$ is admissible if for all nodes $x$,
$$h(x) \leq h^*(x),$$
where $h^*$ is the cost of the optimal path to a goal
- Example: Aerial distance in the pathfinding example
- Example: $h \equiv 0$

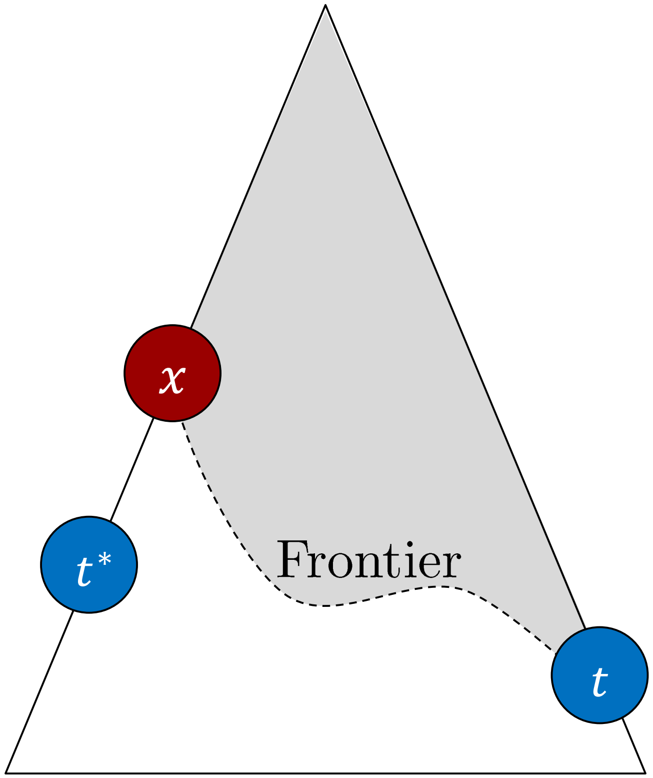
# Optimality of A*

- Theorem: A* tree search with an admissible heuristic returns an optimal solution
- Proof:
  - Assume suboptimal goal $t$ is expanded before optimal goal $t^*$



# Optimality of A*

- Proof (cont.):
  - There is a node $x$ on the optimal path to $t^*$ that has been discovered but not expanded
  - $f(x) = g(x) + h(x)$
    $\leq g(x) + h^*(x)$
    $= g(t^*) < g(t) = f(t)$
  - $x$ should have been expanded before $t$! ■




# 8-PUZZLE HEURISTICS

- $h_1$: #tiles in wrong position
- $h_2$: sum of Manhattan distances of tiles from goal
- Poll 2: Which heuristic is admissible?
  1. Only $h_1$
  2. Only $h_2$
  3. Both $h_1$ and $h_2$
  4. Neither one

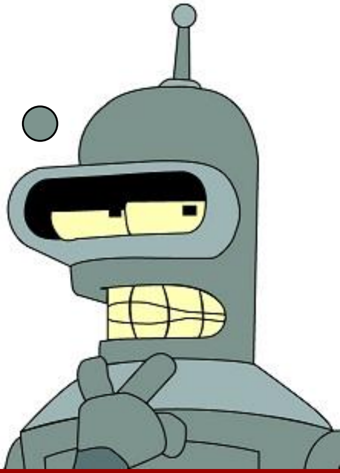
| | | |
|---|---|---|
| 5 | 2 | |
| 6 | 1 | 3 |
| 7 | 8 | 4 |

Example state

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | |

Goal state

Heuristic for designing admissible heuristics: relax the problem!



# 8-puzzle Heuristics

- $h_1$: #tiles in wrong position
- $h_2$: sum of Manhattan distances of tiles from goal
- $h$ dominates $h'$ iff $\forall x, h(x) \geq h'(x)$
- Poll 3: What is the dominance relation between $h_1$ and $h_2$?
  1. $h_1$ dominates $h_2$
  2. $h_2$ dominates $h_1$
  3. $h_1$ and $h_2$ are incomparable

| | | |
|---|---|---|
| 5 | 2 | |
| 6 | 1 | 3 |
| 7 | 8 | 4 |

Example state

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | |

Goal state

# 8-PUZZLE HEURISTICS

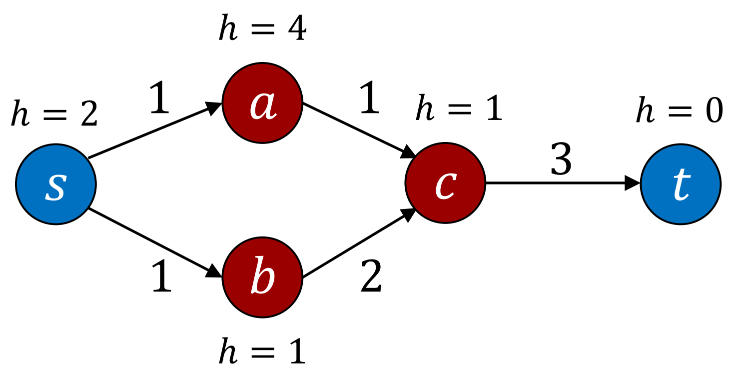
- The following table gives the number of nodes expanded by A* with the two heuristics, averaged over random 8-puzzles, for various solution lengths

| Length | $A^*(h_1)$ | $A^*(h_2)$ |
|---|---|---|
| 16 | 1301 | 211 |
| 18 | 3056 | 363 |
| 20 | 7276 | 676 |
| 22 | 18094 | 1219 |
| 24 | 39135 | 1641 |

- Moral: Good heuristics are crucial!

# A* Graph Search

- Recall: Graph search is the same as tree search, but never expand a node twice
- Is optimality of A* under admissible heuristics preserved? No!

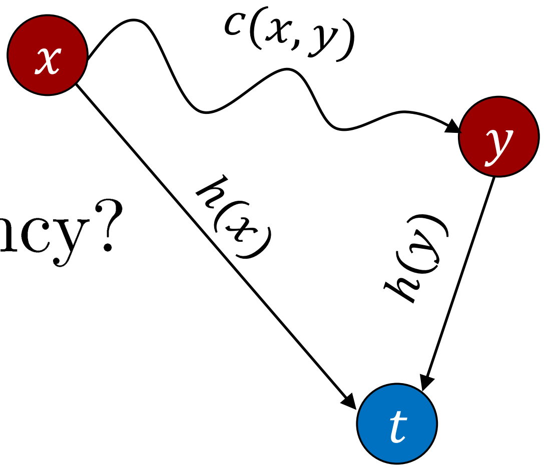


# Consistent Heuristics

- $c(x, y)$ = cost of cheapest path between $x$ and $y$
- $h$ is consistent if for every two nodes $x, y$,

$$h(x) \leq c(x, y) + h(y)$$

- Assume $h(t) = 0$ for each goal $t$
- Poll 4: What is the relation between admissibility and consistency?
  1. Admissible ⇒ consistent
  2. Consistent ⇒ admissible
  3. They are equivalent
  4. They are incomparable




# 8-puzzle Heuristics, Revisited

- $h_1$: #tiles in wrong position
- $h_2$: sum of Manhattan distances of tiles from goal
- Poll 5: Which heuristic is consistent?
  1. Only $h_1$
  2. Only $h_2$
  3. Both $h_1$ and $h_2$
  4. Neither one

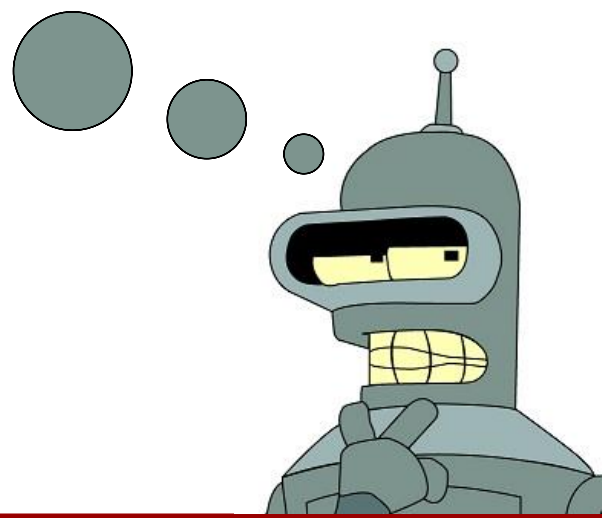
| | | |
|---|---|---|
| 5 | 2 | |
| 6 | 1 | 3 |
| 7 | 8 | 4 |

Example state

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | |

Goal state

Relaxed problem heuristics are always consistent!



# Optimality of A*, Revisited

- Theorem: A* graph search with a consistent heuristic returns an optimal solution
- Proof sketch:*
  - Assume $h(x) \leq c(x, y) + h(y)$
  - Values of $f(x)$ on a path are nondecreasing: if $y$ is the successor of $x$, $f(x) = g(x) + h(x) \leq g(x) + c(x, y) + h(y) = g(y) + h(y) = f(y)$
  - When A* selects $x$ for expansion, the optimal path to $x$ has been found: otherwise there is a frontier node $y$ on optimal path to $x$ that should be expanded first
  - Nodes expanded in nondecreasing $f(x)$
  - First goal state that is expanded must be optimal ■

* Just for fun

# SUMMARY

- Terminology and algorithms:
  - Search problems
  - Tree search, graph search, uniform cost search, greedy, A*
  - Admissible and consistent heuristics
- Theorems:
  - A* tree search is optimal with admissible $h$
  - A* graph search is optimal with consistent $h$
- Big ideas:
  - Don't be too pessimistic!