CMU 15-896

Fair division 4: Indivisible goods

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Indivisible goods

- Set G of m goods
- Each good is indivisible
- Players $N = \{1, \dots, n\}$ have arbitrary valuations V_i for bundles of goods
- Envy-freeness and proportionality are infeasible!

Minimizing envy

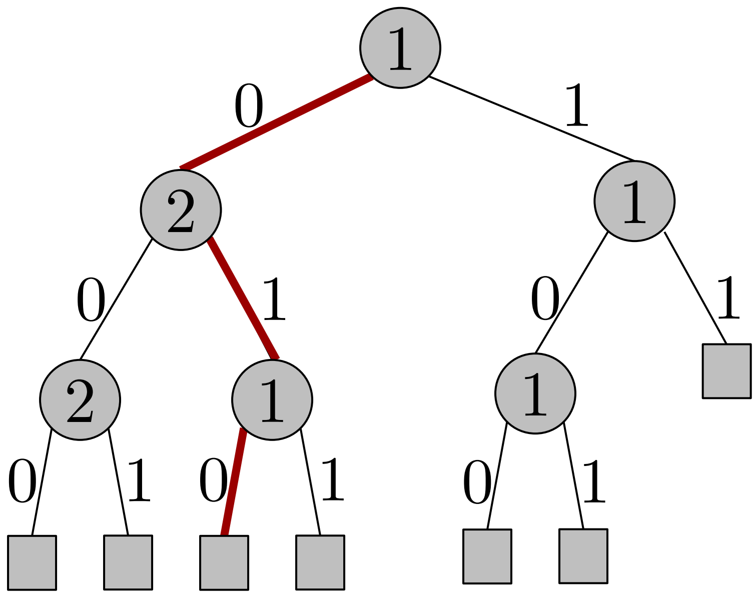
- Given allocation $\boldsymbol{A}$, denote

$$e_{ij}(\boldsymbol{A}) = \max\{0, V_i(A_j) - V_i(A_i)\}$$
$$e(\boldsymbol{A}) = \max\{e_{ij}(\boldsymbol{A}) : i, j \in N\}$$

- Theorem [Nisan and Segal 2002]: Every protocol that finds an allocation minimizing $e(\boldsymbol{A})$ must use an exponential number of bits of communication in the worst case

Communication complexity

- Protocol defined by a binary tree
- Complexity is the height of the tree
- Complexity of a problem is the height of the shortest tree



Proof of theorem

- Let $m = 2k$
- $\mathcal{F}$ is a set of functions s.t. for all $V \in \mathcal{F}$, $S \subseteq G$,

$$V(S) = \begin{cases} 1 & |S| > k \\ 0 & |S| < k \\ 1 - V(G \setminus S) & |S| = k \end{cases}$$

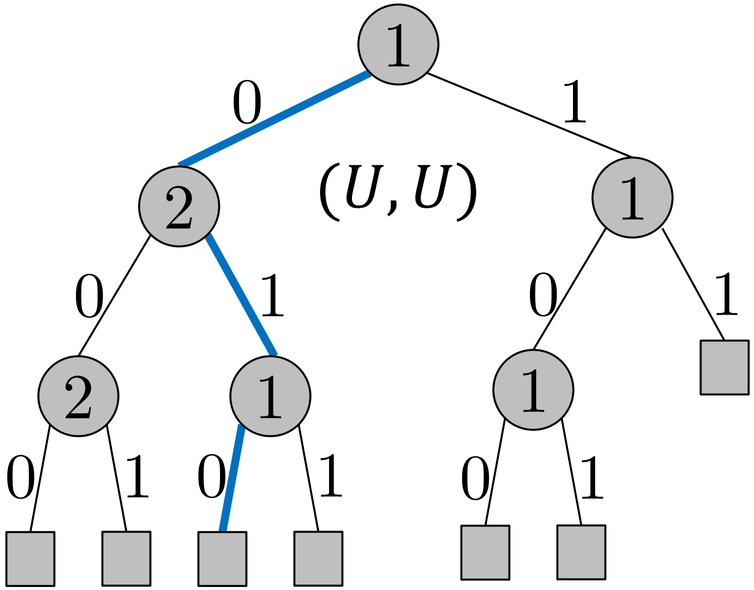
- $|\mathcal{F}| = 2^{\frac{\binom{m}{k}}{2}}$

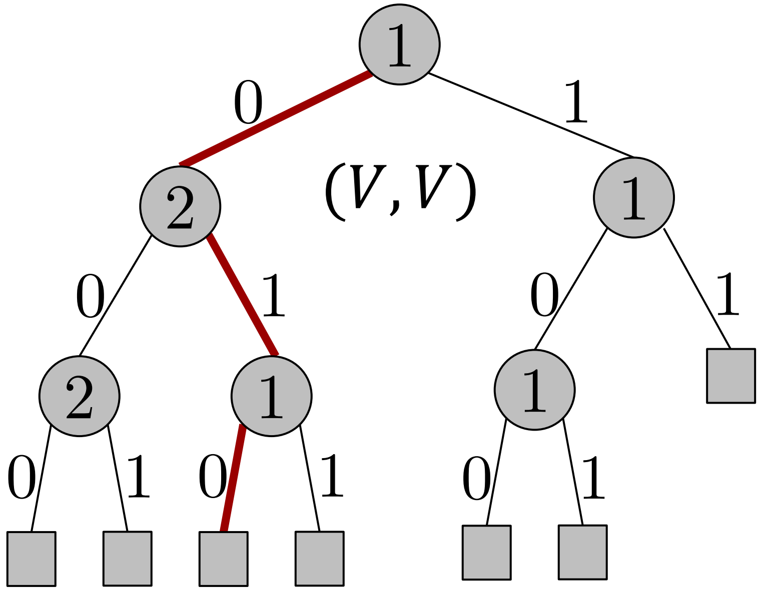
Proof of theorem

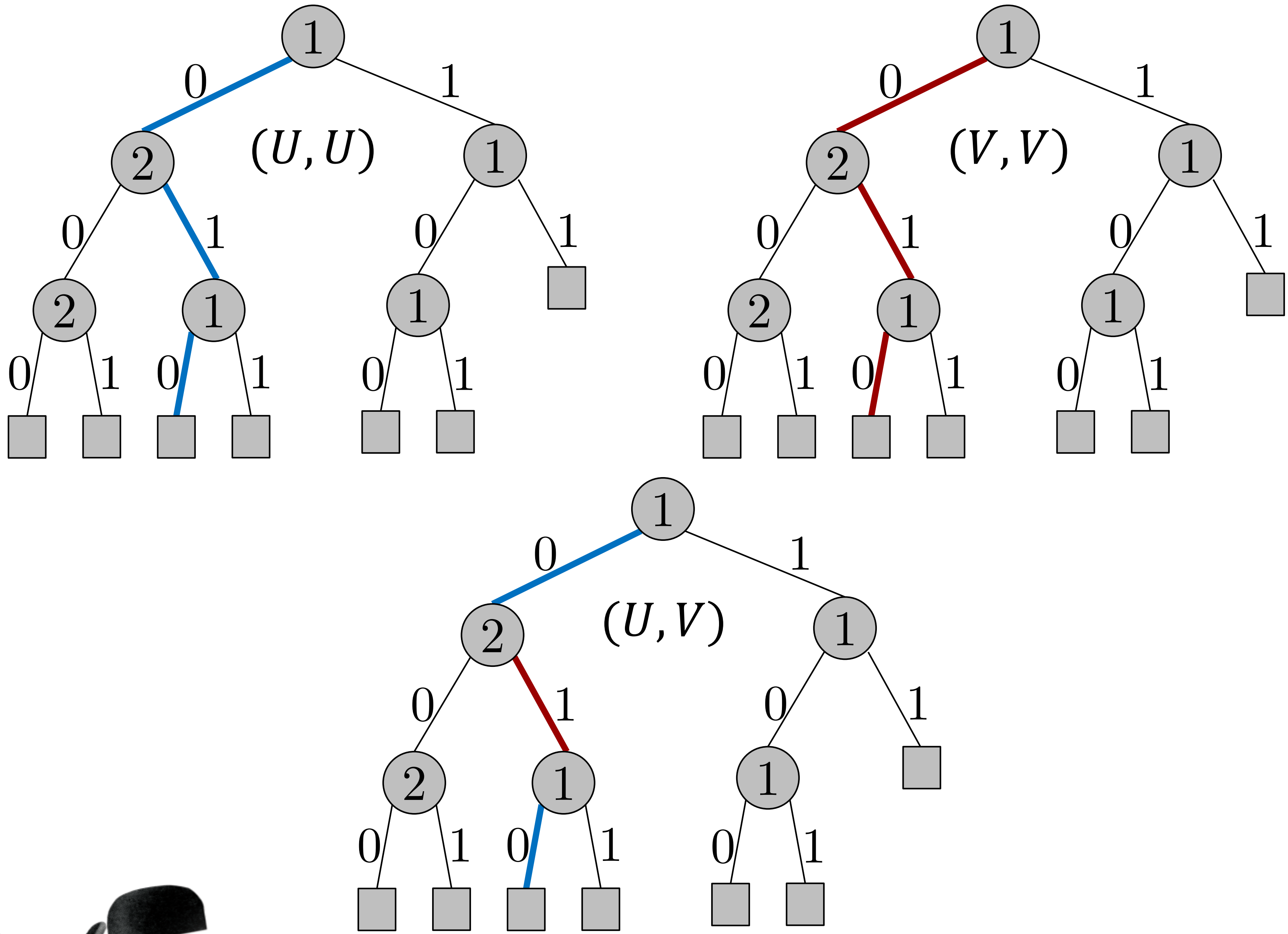
- Suppose $n = 2$, and denote a valuation profile by $(U, V) \in \mathcal{F}^2$
- Lemma: Suppose $U \in \mathcal{F}, V \in \mathcal{F} \setminus \{U\}$, then the sequence of bits transmitted on input (U, U) is different from the sequence transmitted on (V, V)
- Assume the lemma is true, then there must be at least $|\mathcal{F}|$ sequences, and the height of the tree must be at least $\log |\mathcal{F}| = \binom{m}{k}/2$ ■

Proof of lemma

- Assume not; then (U, V) and (V, U) generate the same sequence






(U, U)
(V, V)
(U, V)

Proof of lemma

- If $U \neq V$, $\exists T \subset G$ such that $U(T) = 1$, $V(T) = 0$
- The allocation $(T, G \setminus T)$ is EF for (U, V), $(G \setminus \mathrm{T}, T)$ is EF for (V, U)
- Given (U, V), protocol produces an EF $(S, G \setminus S) \Rightarrow U(S) = 1, V(G \setminus S) = 1$
- $(S, G \setminus S)$ is also returned on (V, U), but is not EF ■

Approximate EF

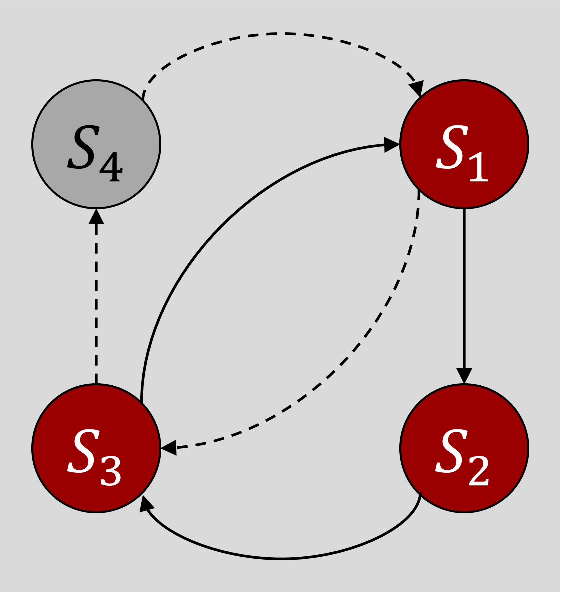
- Define the maximum marginal utility
  $$\alpha = \max\{V_i(S \cup \{x\}) - V_i(S)\colon\ i, x, S\}$$
- Theorem [Lipton et al. 2004]: An allocation with $e(A) \leq \alpha$ can be found in polynomial time
- Note: we are still not assuming anything about the valuation functions!

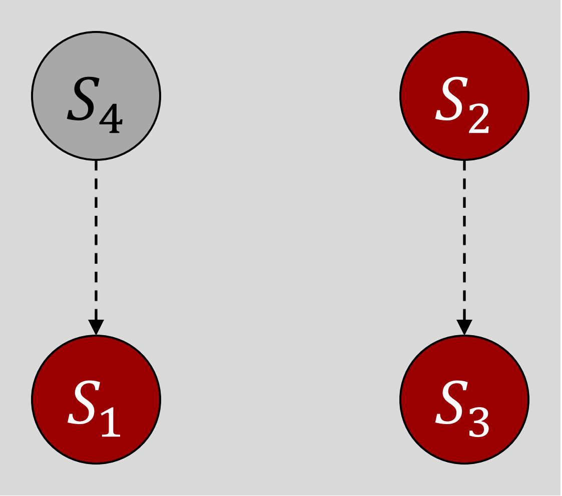
Proof of Theorem

- Given allocation $\boldsymbol{A}$, we have an edge (i, j) in its envy graph if i envies j
- Lemma: Given partial allocation $\boldsymbol{A}$ with envy graph G, can find allocation $\boldsymbol{B}$ with acyclic envy graph H s.t. $e(\boldsymbol{B}) \leq e(\boldsymbol{A})$

Proof of lemma

- If G has a cycle C, shift allocations along C to obtain $\boldsymbol{A}'$; clearly $e(\boldsymbol{A}') \le e(\boldsymbol{A})$
- #edges in envy graph of $\boldsymbol{A}'$ decreased:
    - Same edges between $N \setminus C$
    - Edges from $N \setminus C$ to C shifted
    - Edges from C to $N \setminus C$ can only decrease
    - Edges inside C decreased
- Iteratively remove cycles ■





Proof of theorem

- Maintain envy $\leq \alpha$ and acyclic graph
- In round 1, allocate good g_1 to arbitrary agent
- $g_1, \dots, g_{k-1}$ are allocated in acyclic $\boldsymbol{A}$
- Derive $\boldsymbol{B}$ by allocating g_k to source $\boldsymbol{i}$
- $e_{ji}(B) \leq e_{ji}(A) + \alpha = \alpha$
- Use lemma to eliminate cycles ■

EF cake cutting, revisited

- Want to get ϵ-EF cake division
- Agent i makes $1/\epsilon$ marks $x_1^i, \ldots, x_{1/\epsilon}^i$ such that for every k, $V_i\left(\left[x_k^i, x_{k+1}^i\right]\right) = \epsilon$
- If intervals between consecutive marks are indivisible goods then $\alpha \leq \epsilon$
- Now we can apply the theorem
- Need n/ϵ cut queries and n^2/ϵ eval queries

An even simpler solution

- Relies on additive valuations
- Create the "indivisible goods" like before
- Agents choose pieces in a round-robin fashion: $1, \ldots, n, 1, \ldots, n, \ldots$
- Each good chosen by agent i is preferred to the next good chosen by agent j
- This may not account for the first good g chosen by j, but $V_i(\{g\}) \leq \epsilon$

Maximin share guarantee

- Let us focus on indivisible goods and additive valuations
- Communication complexity is not an issue
- But computational complexity is
- Observation: Deciding whether there exists an EF allocation is NP-hard, even for two players with identical valuations

Maximin share guarantee

- Maximin share (MMS) guarantee [Budish, 2011] of player i:

$$\max_{X_1,\ldots,X_n} \min_j V_i(X_j)$$

- Theorem [P & Wang, 2014]: $\forall n \geq 3$ there exist additive valuation functions that do not admit an MMS allocation

Counterexample for $n = 3$

17	25	12	1
2	22	3	28
11	0	21	23

Counterexample for $n = 3$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \times 10^6 + \begin{array}{|c|c|c|c|}\hline 17 & 25 & 12 & 1 \\ \hline 2 & 22 & 3 & 28 \\ \hline 11 & 0 & 21 & 23 \\ \hline \end{array} \times 10^3 +$$

3	-1	-1	-1
0	0	0	0
0	0	0	0

Player 1

3	-1	0	0
-1	0	0	0
-1	0	0	0

Player 2

3	0	-1	0
0	0	-1	0
0	0	0	-1

Player 3

- Maximin share (MMS) guarantee [Budish, 2011] of player i:

$$\max_{X_1,\ldots,X_n} \min_j V_i(X_j)$$

- Theorem [P & Wang, 2014]: $\forall n \geq 3$ there exist additive valuation functions that do not admit an MMS allocation
- Theorem [P & Wang, 2014]: It is always possible to guarantee each player $2/3$ of his MMS guarantee