Stable Matching

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Recap: matching

- **Have**: graph $G = (V,E)$
- **Want**: a matching M (maximizes some objective)
- **Matching**: set of edges such that each vertex is included at most once

Last time!

Online bipartite matching

Wanted: max cardinality

Proved: $1 - 1/e$ worst case

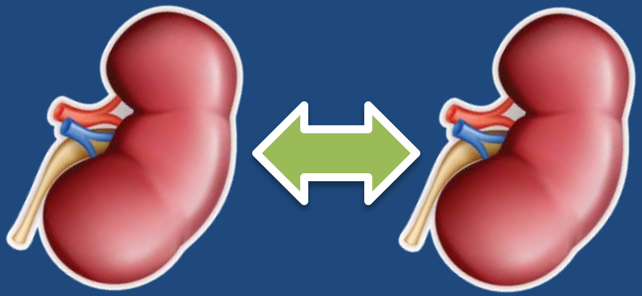
Overview of today's lecture

- Stable marriage problem
  - Bipartite, one vertex to one vertex

- Hospitals/Residents problem
  - Bipartite, one vertex to many vertices

- Stable roommates problem
  - Not bipartite, one vertex to one vertex

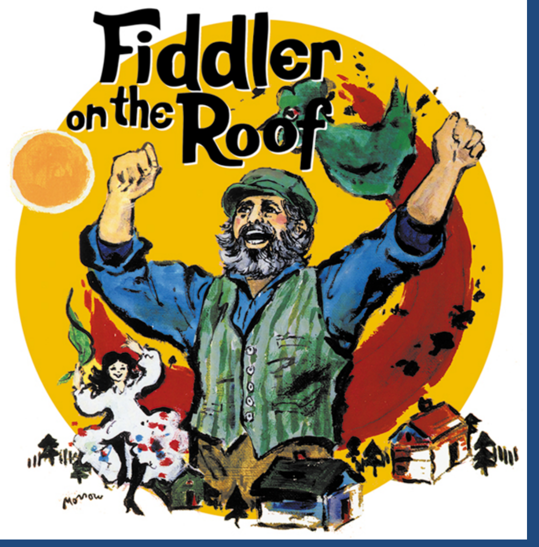
Next lecture!



Stable marriage problem

- Complete bipartite graph with equal sides:
  - n men and n women (old school terminology ☹)
- Each man has a strict, complete preference ordering over women, and vice versa
- **Want**: a **stable** matching

Stable matching: No unmatched man and woman both prefer each other to their current spouses



Example preference profiles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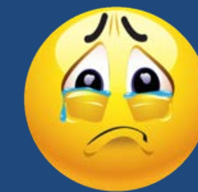


Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Example matching #1

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Is this a **stable matching?**

Example matching #1

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

No.

Albert and Emily form a **blocking pair.**

Example matching #2

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

What about this matching?

Example matching #2

Albert	Diane	Emily	Fergie
Bradley	Emily	Diane	Fergie
Charles	Diane	Emily	Fergie

Diane	Bradley	Albert	Charles
Emily	Albert	Bradley	Charles
Fergie	Albert	Bradley	Charles

Yes!

(Fergie and Charles are unhappy, but helpless.)

Some questions

- Does a stable solution to the marriage problem always exist?
- Can we compute such a solution efficiently?
- Can we compute **the best** stable solution efficiently?

Lloyd Shapley

David Gale

Gale-Shapley [1962]

1. Everyone is unmatched
2. While some man *m* is unmatched:
   - *w* := *m*'s most-preferred woman to whom he has not proposed yet
   - If *w* is also unmatched:
     - *w* and *m* are engaged
   - Else if *w* prefers *m* to her current match *m'*
     - *w* and m are engaged, *m'* is unmatched
   - Else: *w* rejects *m*
3. Return matched pairs

Claim

GS terminates in polynomial time (at most n^2 iterations of the outer loop)

Proof:

- Each iteration, one man proposes to someone to whom he has never proposed before
- n men, n women → $n \times n$ possible events

(Can tighten a bit to $n(n - 1) + 1$ iterations.)

Claim

GS results in a perfect matching

Proof by contradiction:

- Suppose BWOC that *m* is unmatched at termination
- *n* men, *n* women → *w* is unmatched, too
- Once a woman is matched, she is never unmatched; she only swaps partners. Thus, nobody proposed to *w*
- *m* proposed to everyone (by def. of GS): ><

Claim

GS results in a stable matching (i.e., there are no blocking pairs)

Proof by contradiction (1):

- Assume *m* and *w* form a blocking pair

Case #1: *m* never proposed to *w*

- GS: men propose in order of preferences
- *m* prefers current partner *w'* > *w*
- → *m* and *w* are not blocking

Claim

GS results in a stable matching (i.e., there are no blocking pairs)

Proof by contradiction (2):

Case #2: *m* proposed to *w*

- *w* rejected *m* at some point
- *GS:* women only reject for better partners
- *w* prefers current partner *m'* > *m*
- → *m* and *w* are not blocking

Case #1 and #2 exhaust space. ><

Recap: Some questions

- Does a stable solution to the marriage problem always exist?
- Can we compute such a solution efficiently?
- Can we compute **the best** stable solution efficiently?

We'll look at a specific notion of "the best" – optimality with respect to one side of the market

Man optimality/pessimality

- Let $\mathcal{S}$ be the set of stable matchings
- *m* is a **valid partner** of *w* if there exists some stable matching *S* in $\mathcal{S}$ where they are paired
- A matching is **man optimal** if each man receives his *best* valid partner
  - Is this a perfect matching? Stable?
- A matching is **man pessimal** if each man receives his *worst* valid partner

Claim

GS – with the man proposing – results in a man-optimal matching

Proof by contradiction (1):

- Men propose in order → at least one man was rejected by a valid partner
- Let *m* and *w* be the first such reject in *S*
- This happens because *w* chose some *m'* > *m*
- Let *S'* be a stable matching with *m*, *w* paired
  (*S'* exists by def. of valid)

Claim

GS – with the man proposing – results in a man-optimal matching

Proof by contradiction (2):

- Let *w'* be partner of *m'* in *S'*
- *m'* was not rejected by valid woman in *S* before *m* was rejected by *w* (by assump.)
  → *m' prefers w to w'*
- Know *w* prefers *m'* over *m*, her partner in *S'*
  → *m'* and *w* form a blocking pair in *S'* ><

Recap: Some questions

- Does a stable solution to the marriage problem always exist? ✓
- Can we compute such a solution efficiently? ✓
- Can we compute **the best** stable solution efficiently? ✓*

For one side of the market. What about the other side?

Claim

GS – with the man proposing – results in a woman-pessimal matching

Proof by contradiction:

- *m* and *w* matched in *S*, *m* is not worst valid
- → exists stable *S′* with *w* paired to *m′* < *m*
- Let *w′* be partner of *m* in *S′*
- *m* prefers to *w* to *w′* (by man-optimality)
- → *m* and *w* form blocking pair in *S′* ><

Incentive issues

- Can either side benefit by misreporting?
  - (Slight extension for rest of talk: participants can mark possible matches as unacceptable – a form of preference list truncation)

Any algorithm that yields woman- (man-)optimal matching
→ truthful revelation by women (men) is dominant strategy [Roth 1982]

In GS with men proposing, women can benefit by misreporting preferences

Truthful reporting

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	Albert
Emily	Albert	Bradley

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	Albert
Emily	Albert	Bradley

Strategic reporting

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	⃠
Emily	Albert	Bradley

Albert	Diane	Emily
Bradley	Emily	Diane

Diane	Bradley	⃠
Emily	Albert	Bradley

Claim

There is **no** matching mechanism that:

1. is strategy proof (for both sides); and
2. always results in a stable outcome (given revealed preferences)

Extensions to stable marriage

One-to-many matching

- The **hospitals/residents problem** (aka college/students problem aka admissions problem):
  - Strict preference rankings from each side
  - One side (hospitals) can accept $q > 1$ residents
- Also introduced in [Gale and Shapley 1962]

Deferred acceptance: Redux

1. Residents unmatched, empty waiting lists
2. All residents apply to first choice
3. Each hospital places top *q* residents on waiting list
4. Rejected residents apply to second choice
5. Hospitals update waiting lists with new top *q*

   …

   …
6. Repeat until all residents are on a list or have applied to all hospitals

Hospitals/Residents != Marriage

- For ~20 years, most people thought these problems had very similar properties
- Roth [1985] shows:
  - No stable matching algorithm exists s.t. truth-telling is dominant strategy for hospitals

NRMP: Matching in practice

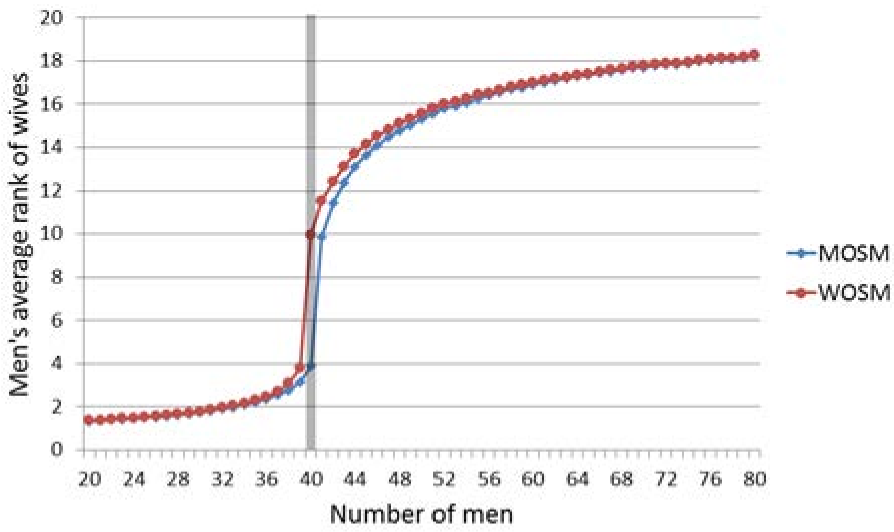
- 1940s: decentralized resident-hospital matching
  - Market "unraveled", offers came earlier and earlier, quality of matches decreased
- 1950s: NRMP introduces hospital-proposing deferred acceptance algorithm
- 1970s: couples increasingly don't use NRMP
- 1998: matching with couple constraints
  - (Stable matching may not exist anymore …)

Take-home message

Looks like: M.D.s aren't the only type of doctor who help people!

Imbalance [Ashlagi et al. 2013]

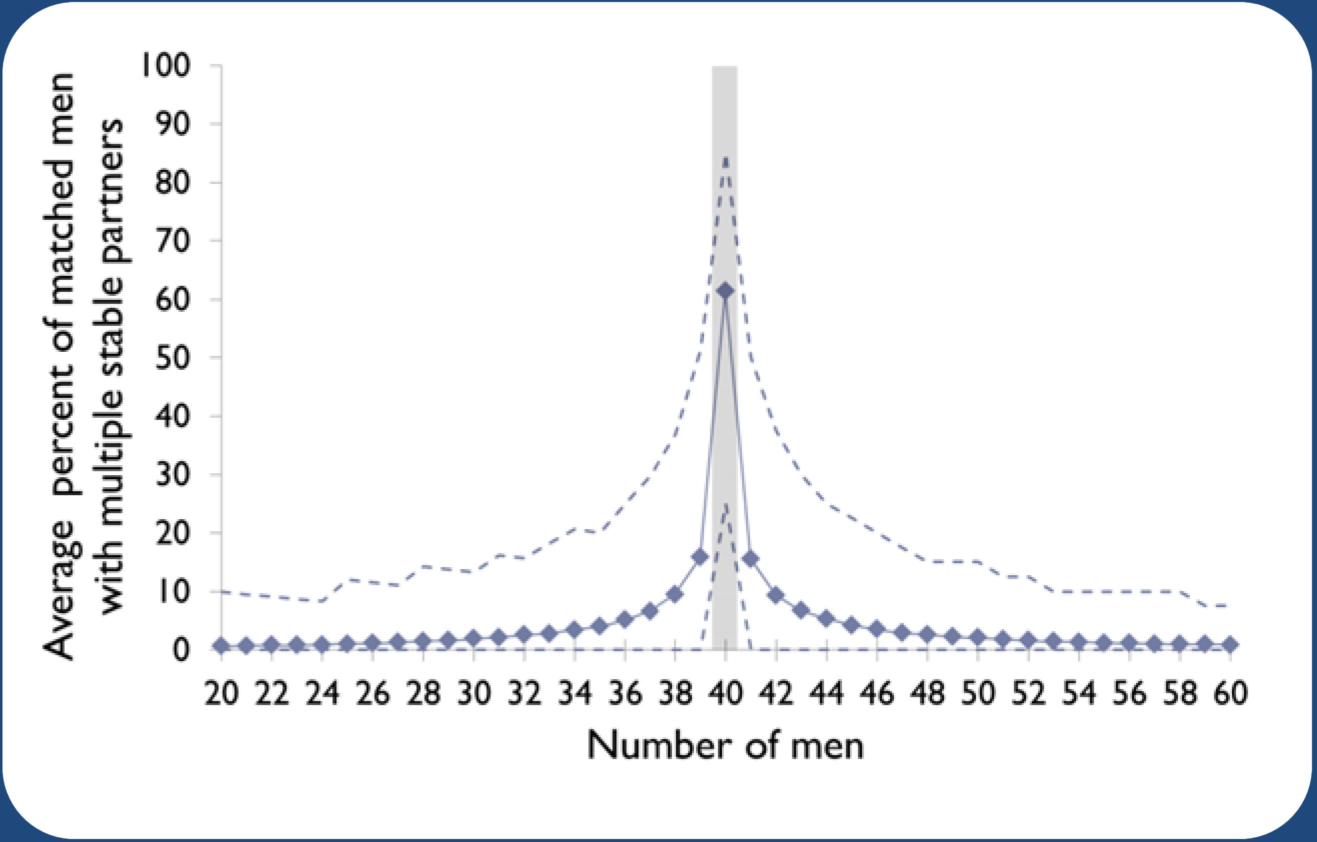
- What if we have n men and $n' \neq n$ women?
- How does this affect participants? Core size?



- Being on short side of market: good!
- W.h.p., short side get rank ~$log(n)$
- … long side gets rank ~random

Imbalance [Ashlagi et al. 2013]

- Not many stable matchings with even small imbalances in the market



Imbalance [Ashlagi et al. 2013]

- "Rural hospital theorem" [Roth 1986]:
  - The set of residents and hospitals that are unmatched is **the same** for all stable matchings
- Assume *n* men, *n+1* women
  - One woman *w* unmatched in all stable matchings
  - → Drop *w*, same stable matchings
- Take stable matchings with *n* women
  - Stay stable if we add in *w* **if** no men prefer *w* to their current match
  - → average rank of men's matches is low

Online arrival [Khuller et al. 1993]

- Random preferences, men arrive over time, once matched nobody can switch
- **Algorithm**: match *m* to highest-ranked free *w*
  - On average, $O(n\log(n))$ unstable pairs
- No deterministic **or randomized** algorithm can do better than $\Omega(n^2)$ unstable pairs!
  - Not better with randomization ☹

Incomplete prefs [Manlove et al. 2002]

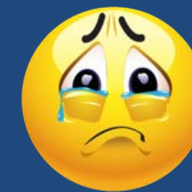
- Before: complete + strict preferences
  - Easy to compute, lots of nice properties
- Incomplete preferences
  - May exist: stable matchings of **different sizes**
- Everything becomes hard!
  - Finding max or min cardinality stable matching
  - Determining if <*m*,*w*> are stable
  - Finding/approx. finding "egalitarian" matching

Moving along to 2015 …

Non-bipartite graph …?

- Matching is defined on **general graphs:**
  - “Set of edges, each vertex included at most once”
  - (Finally, no more “men” or “women” …)
- The **stable roommates problem** is stable marriage generalized to any graph
- Each vertex ranks all *n-1* other vertices
  - (Variations with/without truncation)
- Same notion of stability

Is this different than stable marriage?

 >  > 

Alana	Brian	Cynthia	Dracula
Brian	Cynthia	Alana	Dracula
Cynthia	Alana	Brian	Dracula
Dracula 	(Anyone)	(Anyone)	(Anyone)

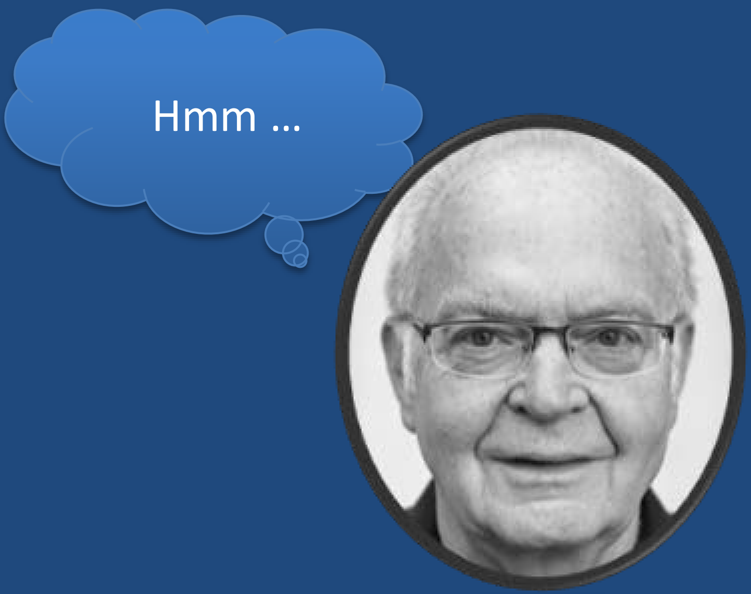
No stable matching exists!

Anyone paired with Dracula (i) prefers some other *v* and (ii) is preferred by that *v*

Hopeless?

- Can we build an algorithm that:
  - Finds a stable matching; or
  - Reports nonexistence

… In polynomial time?



- Yes! [Irving 1985]
  - Builds on Gale-Shapley ideas and work by McVitie and Wilson [1971]

Irving's algorithm: Phase 1

- Run a deferred acceptance-type algorithm
- If at least one person is unmatched: nonexistence
- Else: create a reduced set of preferences
  - *a* holds proposal from *b* → *a* truncates all *x* after *b*
  - Remove *a* from *x*'s preferences
  - Note: *a* is at the top of *b*'s list
- If any truncated list is empty: nonexistence
- Else: this is a "stable table" – continue to Phase 2

Stable tables

1. *a* is first on *b*'s list iff *b* is last on *a*'s
2. *a* is not on *b*'s list iff
   – *b* is not on *a*'s list
   – *a* prefers last element on list to *b*
3. No reduced list is empty

- Note 1: stable table with all lists length 1 is a stable matching
- Note 2: any stable **subtable** of a stable table can be obtained via **rotation eliminations**

Irving's algorithm: Phase 2

- Stable table has length 1 lists: return matching
- Identify a **rotation**:

$(a_0,b_0),(a_1,b_1),\ldots,(a_{k-1},b_{k-1})$ such that:

- b_i is first on ai's reduced list
- b_{i+1} is second on ai's reduced list ($i+1$ is mod k)

- Eliminate it:
  - a_0 rejects b_0, proposes to b_1 (who accepts), etc.
- If any list becomes empty: nonexistence
- If the subtable hits length 1 lists: return matching

Claim

Irving's algorithm for the stable roommates problem terminates in polynomial time – specifically O(n^2).

- This requires some data structure considerations
  - Naïve implementation of rotations is ~O(n^3)

Acknowledgments

- Algorithm Design (Tardos and Kleinberg)
- Princeton CS 403 lecture notes (Wayne)
- The Stable Marriage Problem: Structure and Algorithms (Gusfield and Irving)
- Wikipedia / Creative Commons (images)
- Combinatorics and more (Kalai)
- https://nrmp.org (images)
- Matching and Market Design (Kojima)