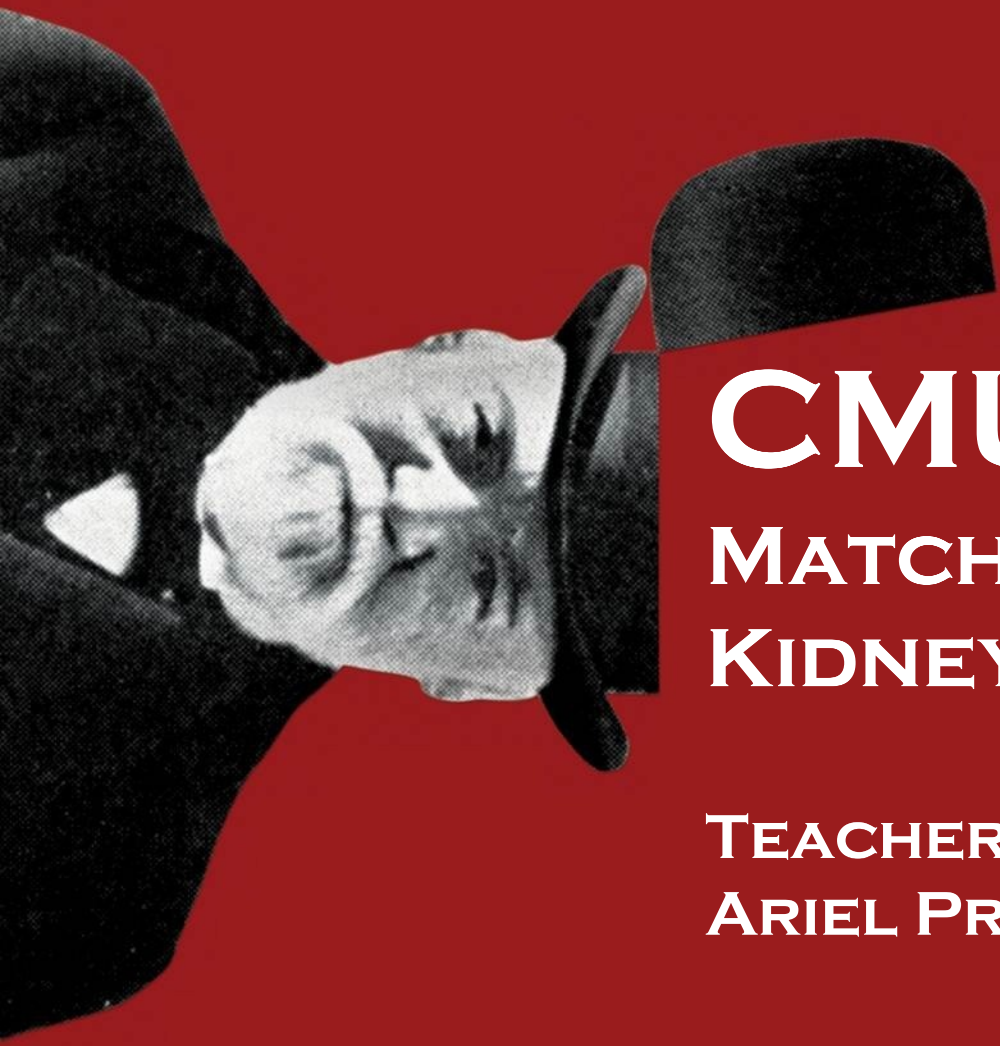

# CMU 15-896

## Matching 4: Kidney exchange

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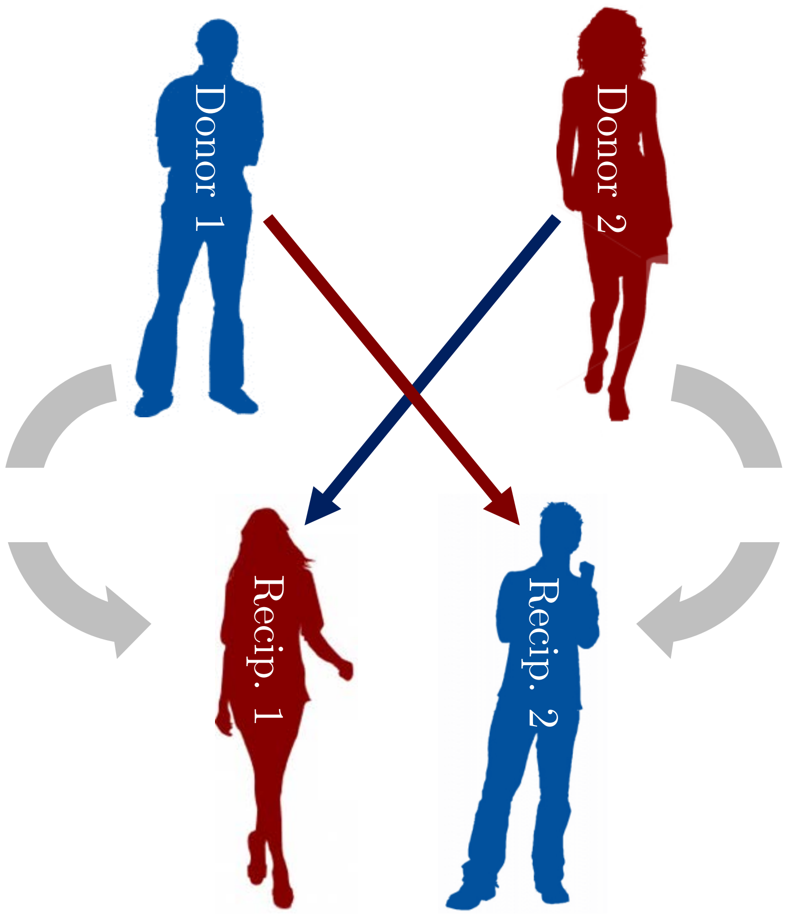
# Reminder: Kidney Exchange

- Kidney donations from live donors are common
- But some donors are incompatible with their patients
- Kidney exchange enables swaps



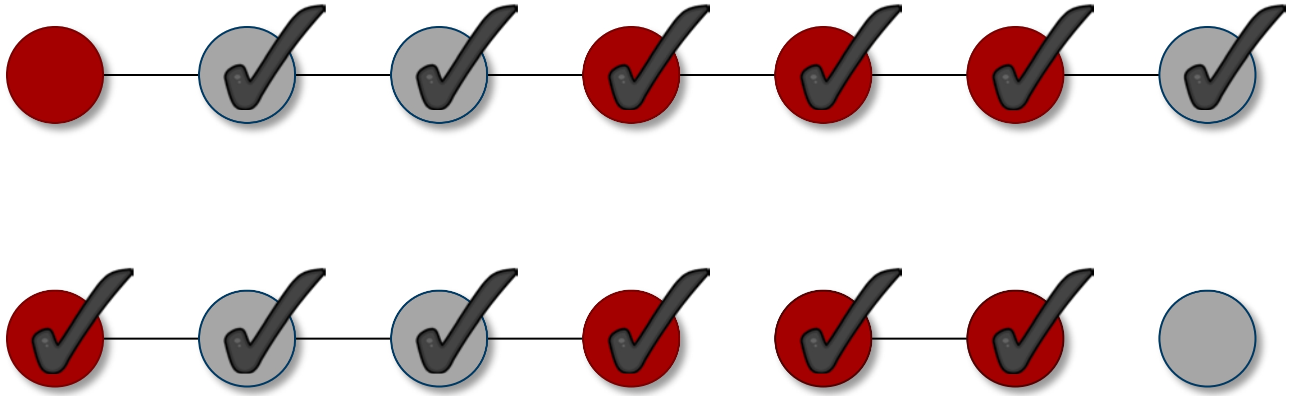
# Incentives

- A few years ago kidney exchanges were carried out by individual hospitals
- Today there are nationally organized exchanges; participating hospitals have little other interaction
- It was observed that hospitals match easy-to-match pairs internally, and enroll only hard-to-match pairs into larger exchanges
- Goal: incentivize hospitals to enroll all their pairs

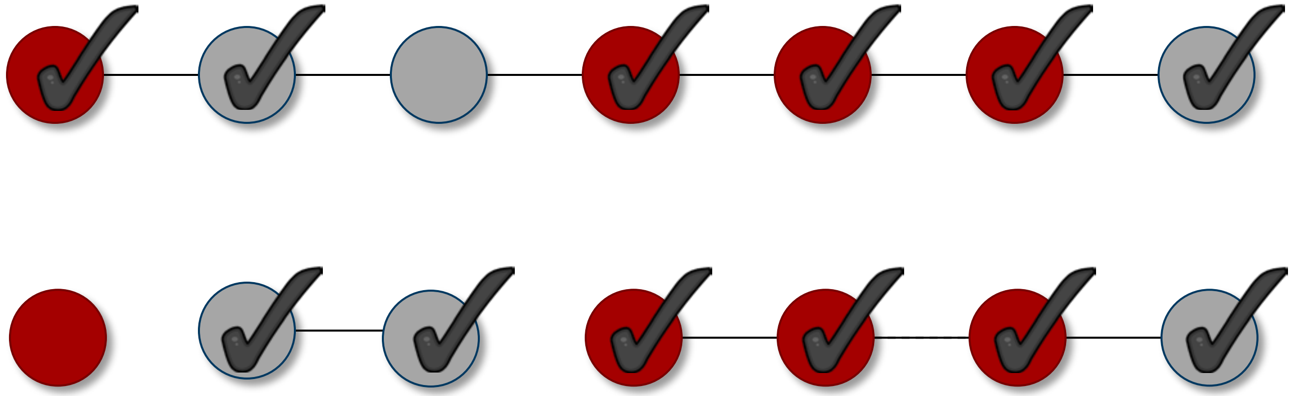
# The strategic model

- Undirected graph (only pairwise matches!)
  - Vertices = donor-patient pairs
  - Edges = compatibility
  - Each player controls subset of vertices
- Mechanism receives a graph and returns a matching
- Utility of player = # its matched vertices
- Target: # matched vertices
- Strategy: subset of revealed vertices
  - But edges are public knowledge
- Mechanism is strategyproof (SP) if it is a dominant strategy to reveal all vertices

# OPT IS MANIPULABLE



# OPT is manipulable



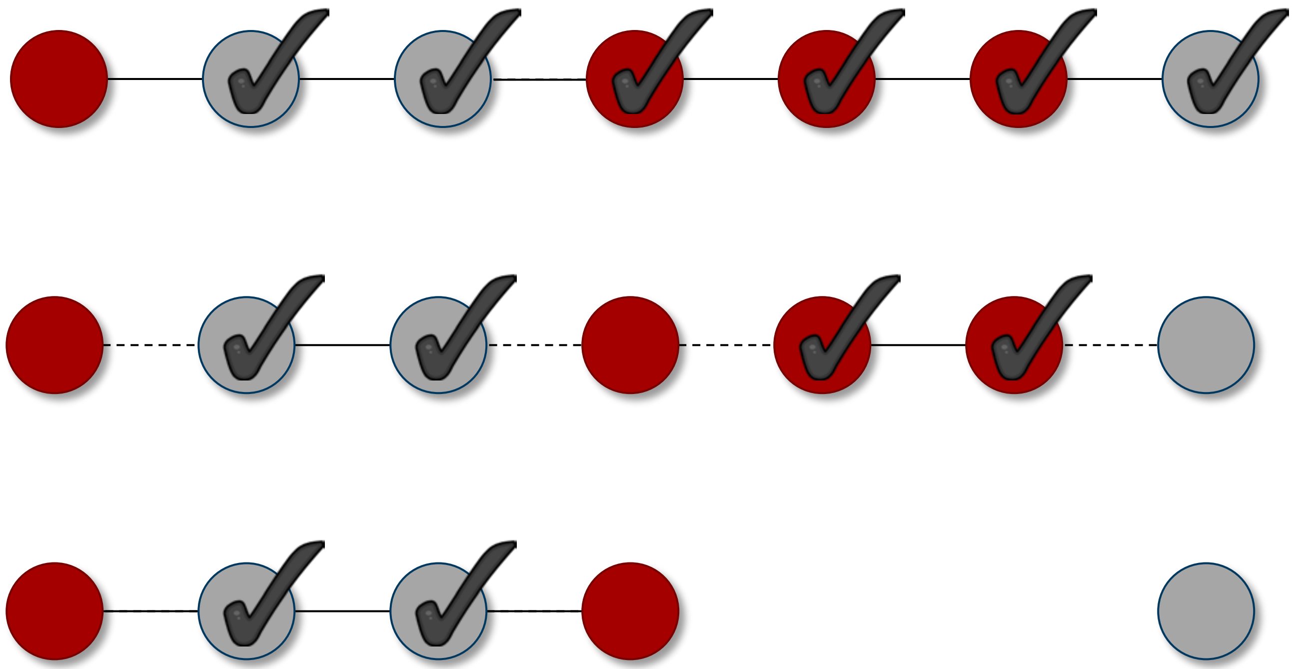
# Approximating SW

- Theorem [Ashlagi et al. 2010]: No deterministic SP mechanism can give a $2-\epsilon$ approximation
- Proof: We just proved it!
- Theorem [Kroer and Kurokawa 2013]: No randomized SP mechanism can give a $\frac{6}{5}-\epsilon$ approximation
- Proof: Homework 3

# SP mechanism: Take 1

- Assume two players
- The $\textsc{Match}_{\{\{1\},\{2\}\}}$ mechanism:
  - Consider matchings that maximize the number of "internal edges"
  - Among these return a matching with max cardinality

# Another example



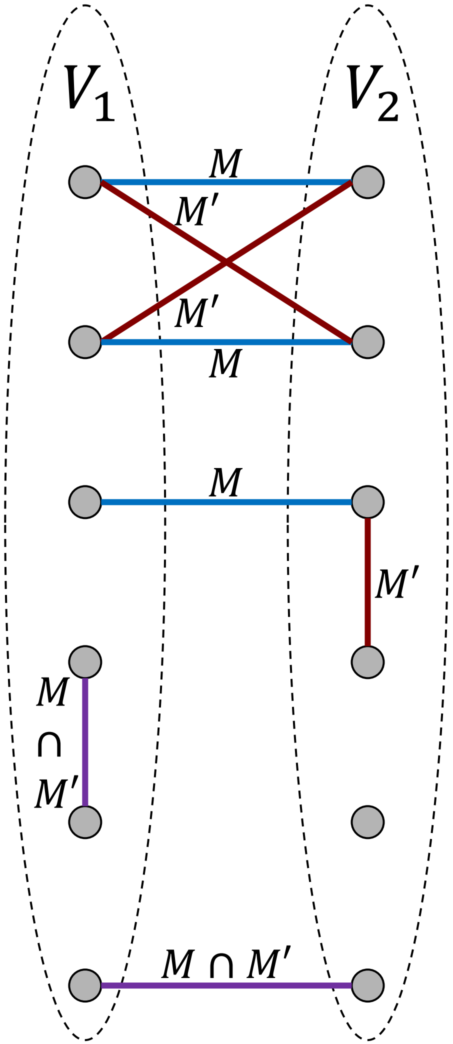
# Guarantees

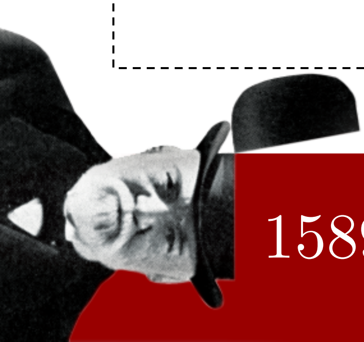
- $\textsc{Match}_{\{\{1\},\{2\}\}}$ gives a 2-approximation
  - Cannot add more edges to matching
  - For each edge in optimal matching, one of the two vertices is in mechanism's matching
- Theorem (special case): $\textsc{Match}_{\{\{1\},\{2\}\}}$ is strategyproof for two players

# Proof of theorem

- $M$ = matching when player 1 is honest, $M'$ = matching when player 1 hides vertices
- $M \Delta M'$ consists of paths and even-length cycles, each consisting of alternating $M, M'$ edges

What's wrong with the illustration on the right?



# Proof of theorem

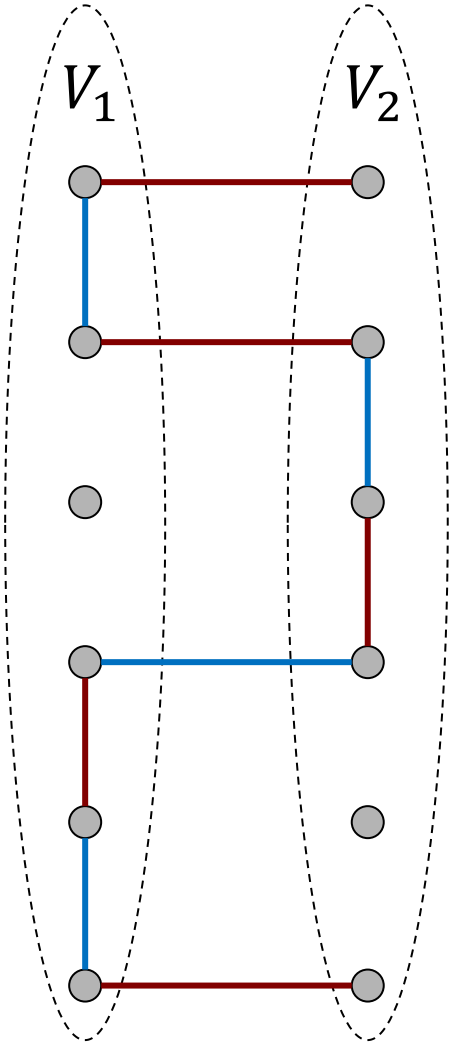
- Consider a path in $M \Delta M'$, denote its edges in $M$ by $P$ and its edges in $M'$ by $P'$
- For $i, j \in \{1,2\}$,

$$P_{ij} = \left\{(u,v) \in P : u \in V_i, v \in V_j\right\}$$
$$P'_{ij} = \left\{(u,v) \in P' : u \in V_i, v \in V_j\right\}$$

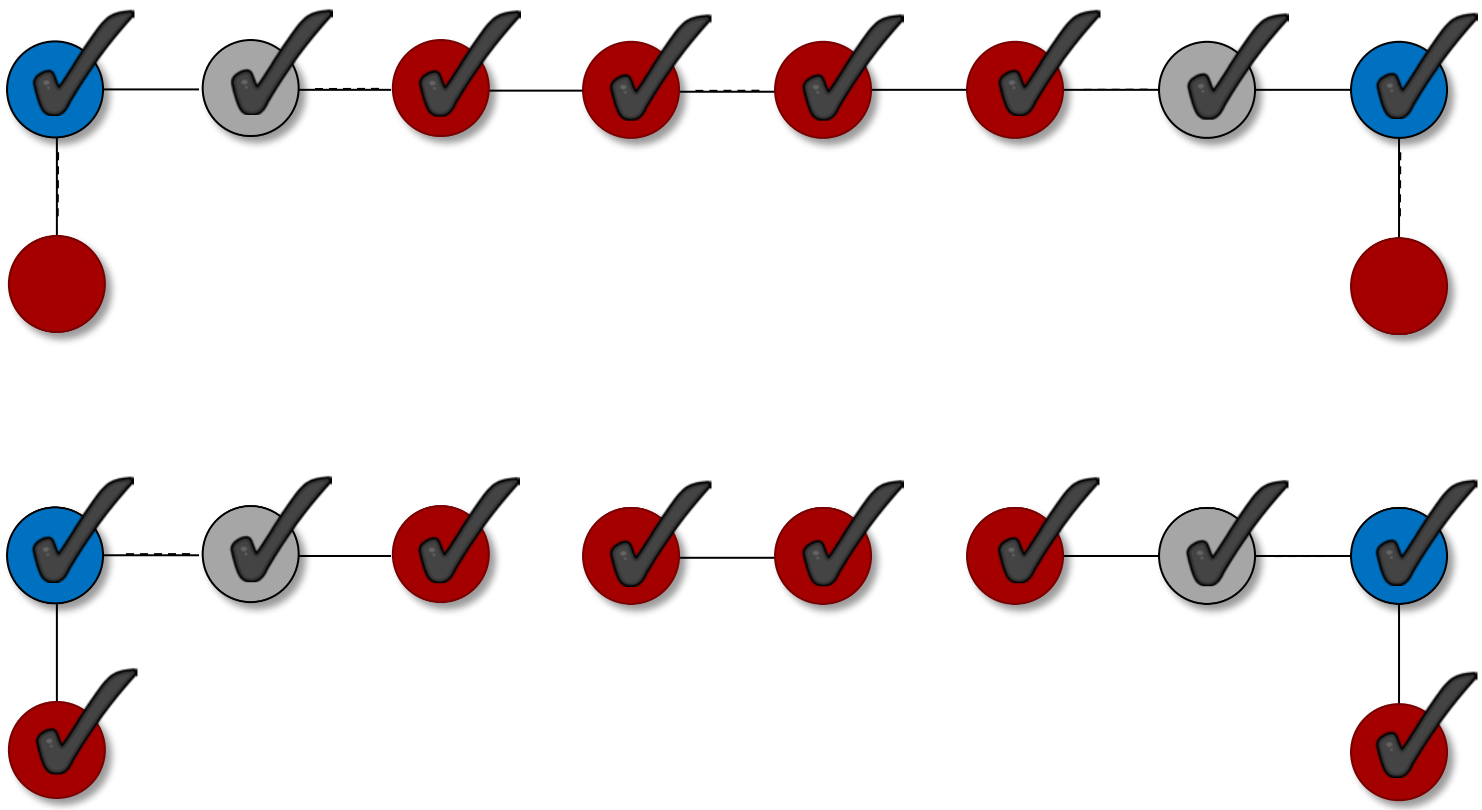
- $|P_{11}| \geq |P'_{11}|$, suppose $|P_{11}| = |P'_{11}|$
- It holds that $|P_{22}| = |P'_{22}|$
- $M$ is max cardinality $\Rightarrow |P_{12}| \geq |P'_{12}|$
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2|P'_{11}| + |P'_{12}| = U_1(P')$

# Proof of theorem

- Suppose $|P_{11}| > |P'_{11}|$
- $|P_{12}| \geq |P'_{12}| - 2$
  - Every subpath within $V_2$ is of even length
  - We can pair the edges of $P_{12}$ and $P'_{12}$, except maybe the first and the last
- $U_1(P) = 2|P_{11}| + |P_{12}| \geq 2(|P'_{11}| + 1) + |P'_{12}| - 2 = U_1(P')$




# The case of 3 players



# SP mechanism: Take 2

- Let $\Pi = (\Pi_1, \Pi_2)$ be a bipartition of the players
- The $\text{M}\textsc{atch}_\Pi$ mechanism:
  - Consider matchings that maximize the number of "internal edges" and do not have any edges between different players on the same side of the partition
  - Among these return a matching with max cardinality (need tie breaking)

# Eureka?

- Theorem [Ashlagi et al. 2010]: $\text{MATCH}_{\Pi}$ is strategyproof for any number of players and any partition $\Pi$
- Recall: for $n = 2$, $\text{MATCH}_{\{\{1\},\{2\}\}}$ guarantees a 2-approx

# Eureka?

Poll 1: approximation guarantees given by $\textsc{Match}_\Pi$ for $n = 3$ and $\Pi = \{\{1\},\{2,3\}\}$?

1. 2
2. 3
3. 4
4. More than 4

# The mechanism

- The Mix-and-Match mechanism:
  - Mix: choose a random partition $\Pi$
  - Match: Execute $\textsc{Match}_\Pi$
- Theorem [Ashlagi et al. 2010]: Mix-and-Match is strategyproof and guarantees a 2-approximation
- We only prove the approximation ratio

# Proof of theorem

- $M^* =$ optimal matching
- Create a matching $M'$ such that $M'$ is max cardinality on each $V_i$, and

$$\sum_i |M'_{ii}| + \frac{1}{2}\sum_{i\neq j} |M'_{ij}| \geq \sum_i |M^*_{ii}| + \frac{1}{2}\sum_{i\neq j} |M^*_{ij}|$$

  - $M^{**} =$ max cardinality on each $V_i$
  - For each path $P$ in $M^* \Delta M^{**}$, add $P \cap M^{**}$ to $M'$ if $M^{**}$ has more internal edges than $M^*$, otherwise add $P \cap M^*$ to $M'$
  - For every internal edge $M'$ gains relative to $M^*$, it loses at most two edges ■

# Proof of theorem

- Fix $\Pi$ and let $M^{\Pi}$ be the output of $\textsc{Match}_{\Pi}$
- The mechanism returns max cardinality across $\Pi$ subject to being max cardinality internally, therefore

$$\sum_i |M_{ii}^{\Pi}| + \sum_{i\in\Pi_1, j\in\Pi_2} |M_{ij}^{\Pi}| \geq \sum_i |M'_{ii}| + \sum_{i\in\Pi_1, j\in\Pi_2} |M'_{ij}|$$

# Proof of theorem

$$\mathbb{E}\left[\left|M^{\Pi}\right|\right] = \frac{1}{2^n}\sum_{\Pi}\left(\sum_{i}\left|M_{ii}^{\Pi}\right| + \sum_{i\in\Pi_1, j\in\Pi_2}\left|M_{ij}^{\Pi}\right|\right)$$

$$\geq \frac{1}{2^n}\sum_{\Pi}\left(\sum_{i}\left|M_{ii}'\right| + \sum_{i\in\Pi_1, j\in\Pi_2}\left|M_{ij}'\right|\right)$$

$$= \sum_{i}\left|M_{ii}'\right| + \frac{1}{2^n}\sum_{\Pi}\sum_{i\in\Pi_1, j\in\Pi_2}\left|M_{ij}'\right|$$

$$= \sum_{i}\left|M_{ii}'\right| + \frac{1}{2}\sum_{i\neq j}\left|M_{ij}'\right| \geq \sum_{i}\left|M_{ii}^{*}\right| + \frac{1}{2}\sum_{i\neq j}\left|M_{ij}^{*}\right|$$

$$\geq \frac{1}{2}\sum_{i}\left|M_{ii}^{*}\right| + \frac{1}{2}\sum_{i\neq j}\left|M_{ij}^{*}\right| = \frac{1}{2}\left|M^{*}\right| \quad \blacksquare$$