CMU 15-896

Social choice 2: Manipulation

Teacher:
Ariel Procaccia

Reminder: Voting

- Set of voters $N = \{1, \dots, n\}$
- Set of alternatives $A, |A| = m$
- Each voter has a ranking over the alternatives
- $x \succ_i y$ means that voter i prefers x to y
- Preference profile $\vec{\succ}$ = collection of all voters' rankings
- Voting rule f = function from preference profiles to alternatives
- Important: so far voters were honest!

Manipulation

- Using Borda count
- Top profile: b wins
- Bottom profile: a wins
- By changing his vote, voter 3 achieves a better outcome!

1	2	3
b	b	a
a	a	b
c	c	c
d	d	d

1	2	3
b	b	a
a	a	c
c	c	d
d	d	b

Borda responds to critics

My scheme is intended only for honest men!

Random 18th Century French Dude

Strategyproofness

- A voting rule is strategyproof (SP) if a voter can never benefit from lying about his preferences:

$$\forall \vec{\succ}, \forall i \in N, \forall \succ_i', f(\vec{\succ}) \succcurlyeq_i f(\succ_i', \vec{\succ}_{-i})$$

Maximum value of m for which plurality is SP?

Strategyproofness

- A voting rule is dictatorial if there is a voter who always gets his most preferred alternative
- A voting rule is constant if the same alternative is always chosen
- Constant functions and dictatorships are SP

Dictatorship

Constant function

Gibbard-Satterthwaite

- A voting rule is onto if any alternative can win
- Theorem (Gibbard-Satterthwaite): If $m \geq 3$ then any voting rule that is SP and onto is dictatorial
- In other words, any voting rule that is onto and nondictatorial is manipulable

Gibbard

Satterthwaite

Proof sketch of G-S

- Lemmas (prove in HW1):
  - Strong monotonicity: f is SP rule, $\vec{\prec}$ profile, $f(\vec{\prec}) = a$. Then $f(\vec{\prec}') = a$ for all profiles $\vec{\prec}'$ s.t. $\forall x \in A, i \in N$: $[a \succ_i x \Rightarrow a \succ_i' x]$
  - Pareto optimality: f is SP+onto rule, $\vec{\prec}$ profile. If $a \succ_i b$ for all $i \in N$ then $f(\vec{\prec}) \neq b$
- Let us assume that $m \geq n$, and neutrality:

$$f\left(\pi(\vec{\prec})\right) = \pi\left(f(\vec{\prec})\right) \text{ for all } \pi: A \to A$$

Proof sketch of G-S

- Say $n = 4$ and $A = \{a, b, c, d, e\}$
- Consider the following profile

$\overrightarrow{\succsim} =$

1	2	3	4
a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c
e	e	e	e

- Pareto optimality $\Rightarrow e$ is not the winner
- Suppose $f(\overrightarrow{\succsim}) = a$

Proof sketch of G-S

1	2	3	4
a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c
e	e	e	e

$\overrightarrow{\precsim}$

1	2	3	4
a	d	d	d
d	a	a	a
b	b	b	b
c	c	c	c
e	e	e	e

$\overrightarrow{\precsim}^1$

- Strong monotonicity $\Rightarrow f\left(\overrightarrow{\precsim}^1\right) = a$

1	2	3	4
a	d	d	d
d	a	a	a
b	b	b	b
c	c	c	c
e	e	e	e

$\overrightarrow{\precsim}^1$

1	2	3	4
a	d	d	d
d	b	a	a
b	c	b	b
c	e	c	c
e	a	e	e

$\overrightarrow{\precsim}^2$

Poll 1: How many options are there for $f\left(\overrightarrow{\precsim}^2\right)$?

1. 1
2. 2
3. 3
4. 4

1	2	3	4
a	d	d	d
b	b	a	a
c	c	b	b
d	e	c	c
e	a	e	e

$\overrightarrow{\precsim}^2$

1	2	3	4
a	d	d	d
b	b	b	a
c	c	c	b
d	e	e	c
e	a	a	e

$\overrightarrow{\precsim}^3$

1	2	3	4
a	d	d	d
b	b	b	b
c	c	c	c
d	e	e	e
e	a	a	a

$\overrightarrow{\precsim}^4$

- Pareto optimality $\Rightarrow f\left(\overrightarrow{\precsim}^j\right) \notin \{b, c, e\}$
- $[\text{SP} \Rightarrow f\left(\overrightarrow{\precsim}^j\right) \neq d] \Rightarrow f\left(\overrightarrow{\precsim}^j\right) = a$
- Strong monotonicity $\Rightarrow f(\overrightarrow{\precsim}) = a$ for every $\overrightarrow{\precsim}$ where 1 ranks a first
- Neutrality $\Rightarrow$ 1 is a dictator

Circumventing G-S

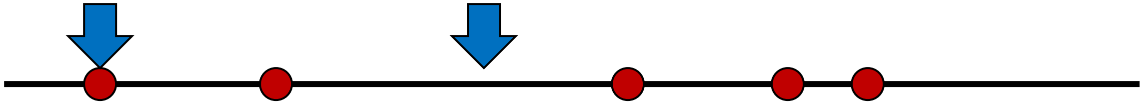
- Restricted preferences (this lecture)
- Money ⇒ mechanism design (not here)
- Computational complexity (this lecture)

Single peaked preferences

- We want to choose a location for a public good (e.g., library) on a street
- Alternatives = possible locations
- Each voter has an ideal location (peak)
- The closer the library is to a voter's peak, the happier he is

Single peaked preferences

- Leftmost point mechanism: return the leftmost point
- Midpoint mechanism: return the average of leftmost and rightmost points

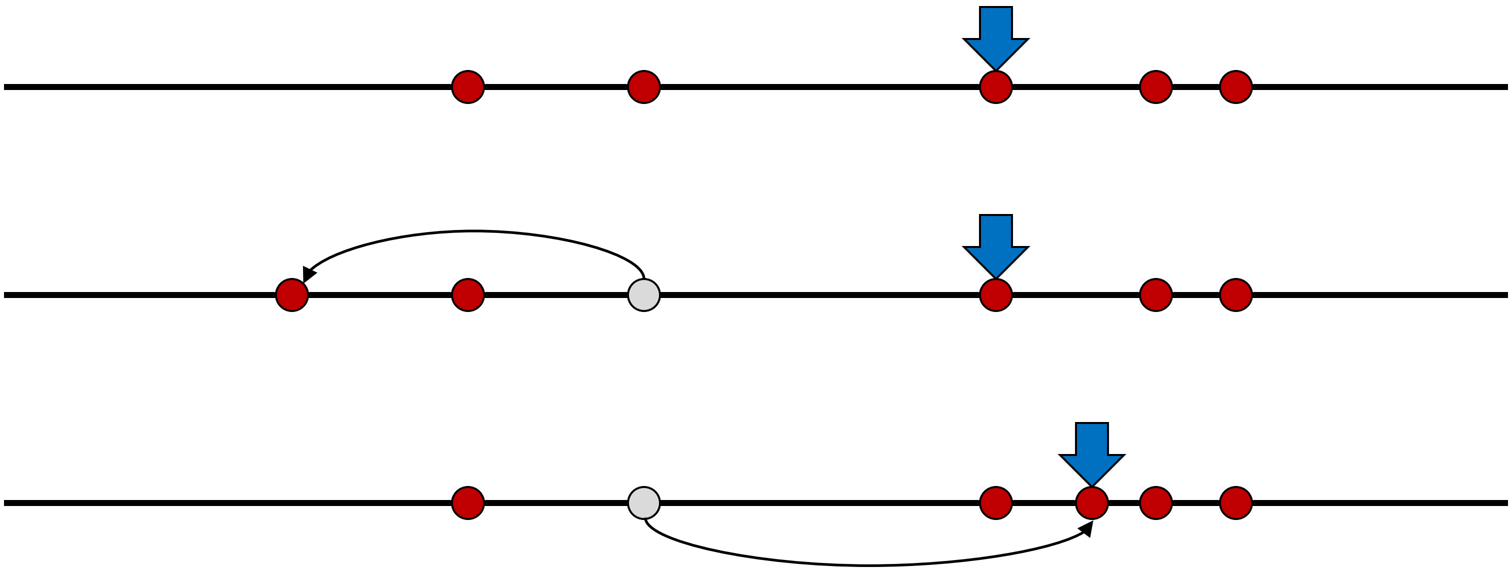


Which of the two mechanisms is SP?

The median

- Select the median peak
- The median is a Condorcet winner!
- The median is onto
- The median is nondictatorial

The median is SP



Complexity of manipulation

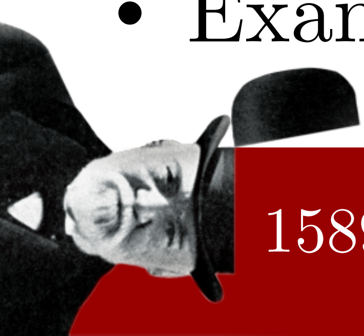
- Manipulation is always possible in theory
- But can we design voting rules where it is difficult in practice?
- Are there “reasonable” voting rules where manipulation is a hard computational problem? [Bartholdi et al., SC&W 1989]

The computational problem

- f-Manipulation problem:
  - Given votes of nonmanipulators and a preferred candidate p
  - Can manipulator cast vote that makes p (uniquely) win under f?
- Example: Borda, $p = a$

1	2	3
b	b	
a	a	
c	c	
d	d	

1	2	3
b	b	a
a	a	c
c	c	d
d	d	b



A greedy algorithm

- Rank p in first place
- While there are unranked alternatives:
    - If there is an alternative that can be placed in next spot without preventing p from winning, place this alternative
    - Otherwise return false

Example: Borda

1	2	3
b	b	a
a	a	
c	c	
d	d	

1	2	3
b	b	a
a	a	b
c	c	
d	d	

1	2	3
b	b	a
a	a	c
c	c	
d	d	

1	2	3
b	b	a
a	a	c
c	c	b
d	d	

1	2	3
b	b	a
a	a	c
c	c	d
d	d	

1	2	3
b	b	a
a	a	c
c	c	d
d	d	b

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	
c	d	b	b	
d	e	a	a	
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	2	-	3	1
d	0	0	1	-	2
e	2	2	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	
d	e	a	a	
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	0	1	-	2
e	2	2	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	d
d	e	a	a	
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	1	1	-	3
e	2	2	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	d
d	e	a	a	e
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	1	1	-	3
e	2	3	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	d
d	e	a	a	e
e	c	d	d	b

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	1	1	-	3
e	2	3	3	2	-

Pairwise elections

When does the alg work?

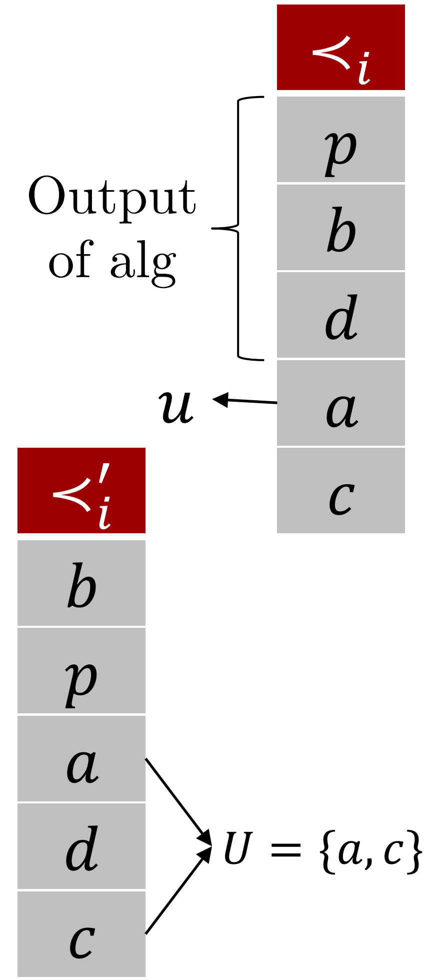
- Theorem [Bartholdi et al., SCW 89]: Fix $i \in N$ and the votes of other voters. Let f be a rule s.t. ∃function $s(\prec_i, x)$ such that:
  1. For every $\prec_i$ chooses a candidate that uniquely maximizes $s(\prec_i, x)$
  2. $\{y\colon y \prec_i x\} \subseteq \{y\colon y \prec_i' x\} \Rightarrow s(\prec_i, x) \leq s(\prec_i', x)$

  Then the algorithm always decides f-Manipulation correctly

What is s for plurality?

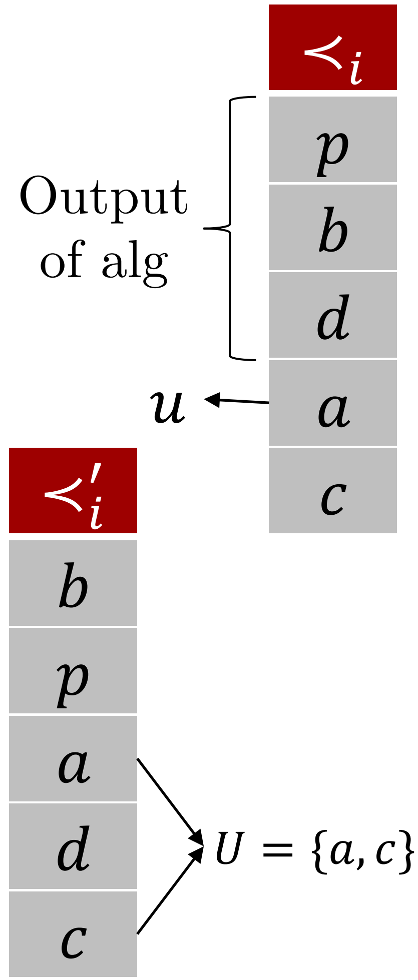
Proof of Theorem

- Suppose the algorithm failed, producing a partial ranking $\prec_i$
- Assume for contradiction $\prec_i'$ makes p win
- $U \leftarrow$ alternatives not ranked in $\prec_i$
- $u \leftarrow$ highest ranked alternative in U according to $\prec_i'$
- Complete $\prec_i$ by adding u first, then others arbitrarily



Proof of Theorem

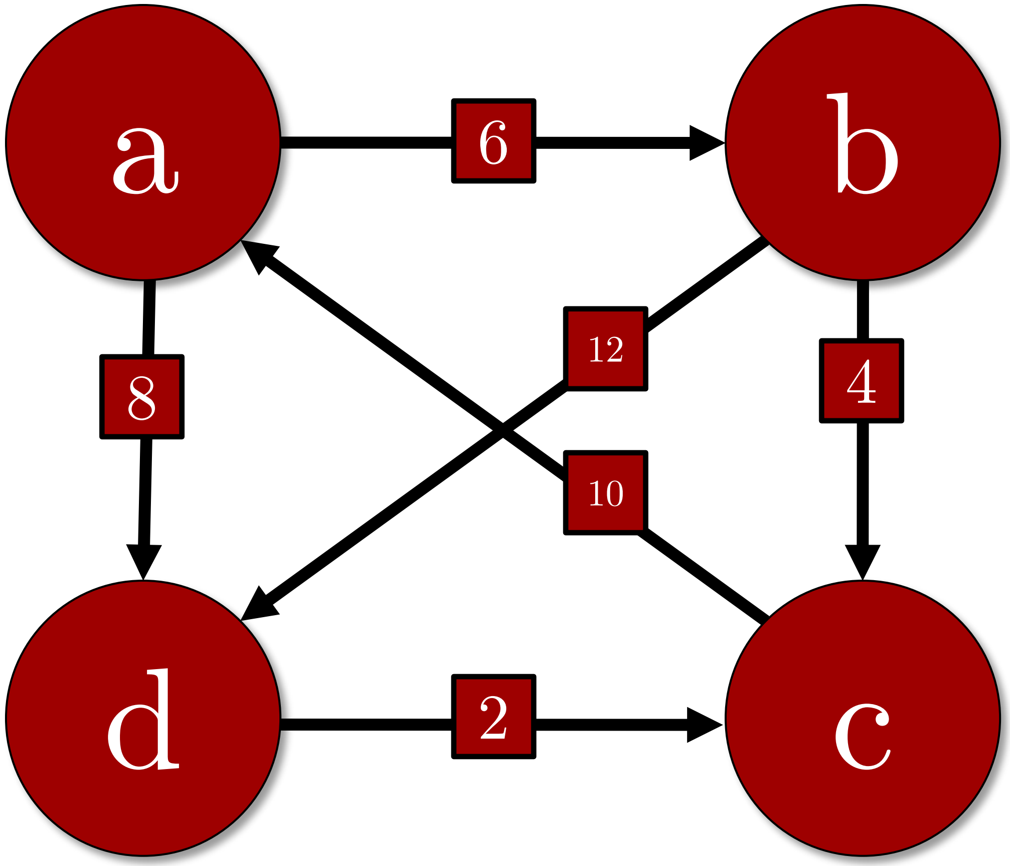
- Property 2 $\Rightarrow s(\prec_i, p) \geq s(\prec_i', p)$
- Property 1 and $\prec'$ makes p the winner $\Rightarrow s(\prec_i', p) > s(\prec_i', u)$
- Property 2 $\Rightarrow s(\prec_i', u) \geq s(\prec_i, u)$
- Conclusion: $s(\prec_i, p) > s(\prec_i, u)$, so the alg could have inserted u next ∎



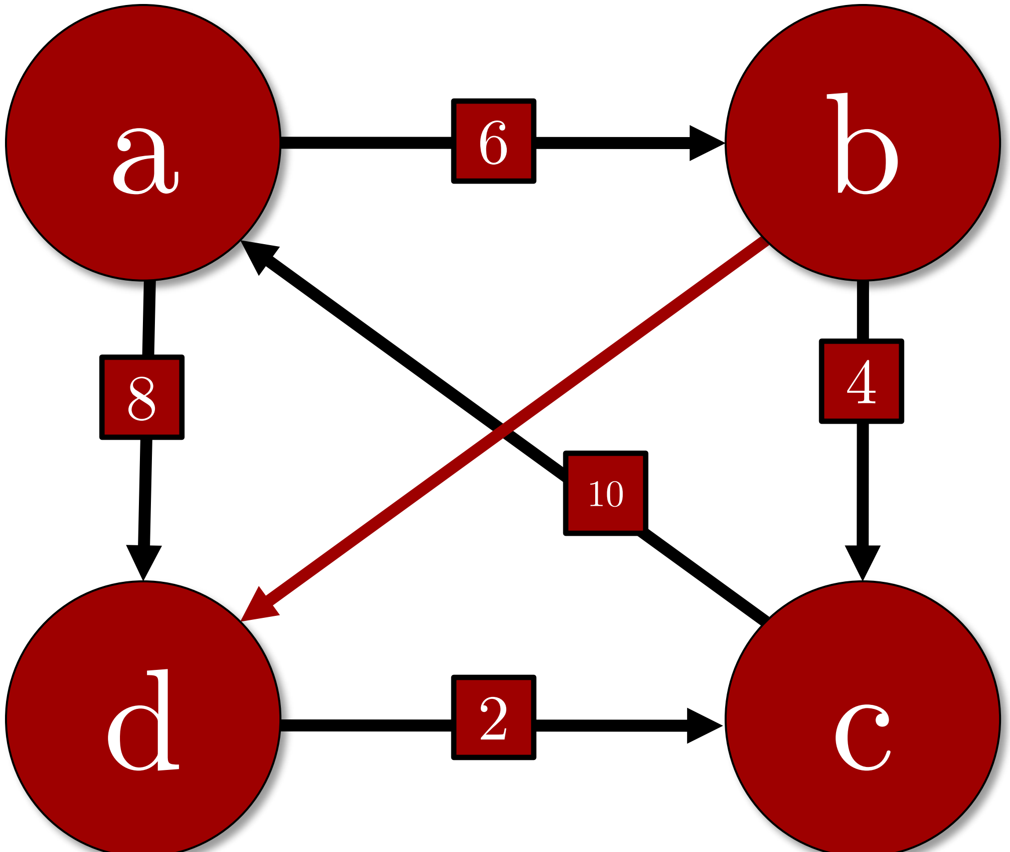
Voting rules that are hard to manipulate

- Natural rules
  - Copeland with second order tie breaking [Bartholdi et al., SCW 89]
  - STV [Bartholdi&Orlin, SCW 91]
  - Ranked Pairs [Xia et al., IJCAI 09]
    - Order pairwise elections by decreasing strength of victory
    - Successively lock in results of pairwise elections unless it leads to cycle
    - Winner is the top ranked candidate in final order
- Can also "tweak" easy to manipulate voting rules [Conitzer&Sandholm, IJCAI 03]

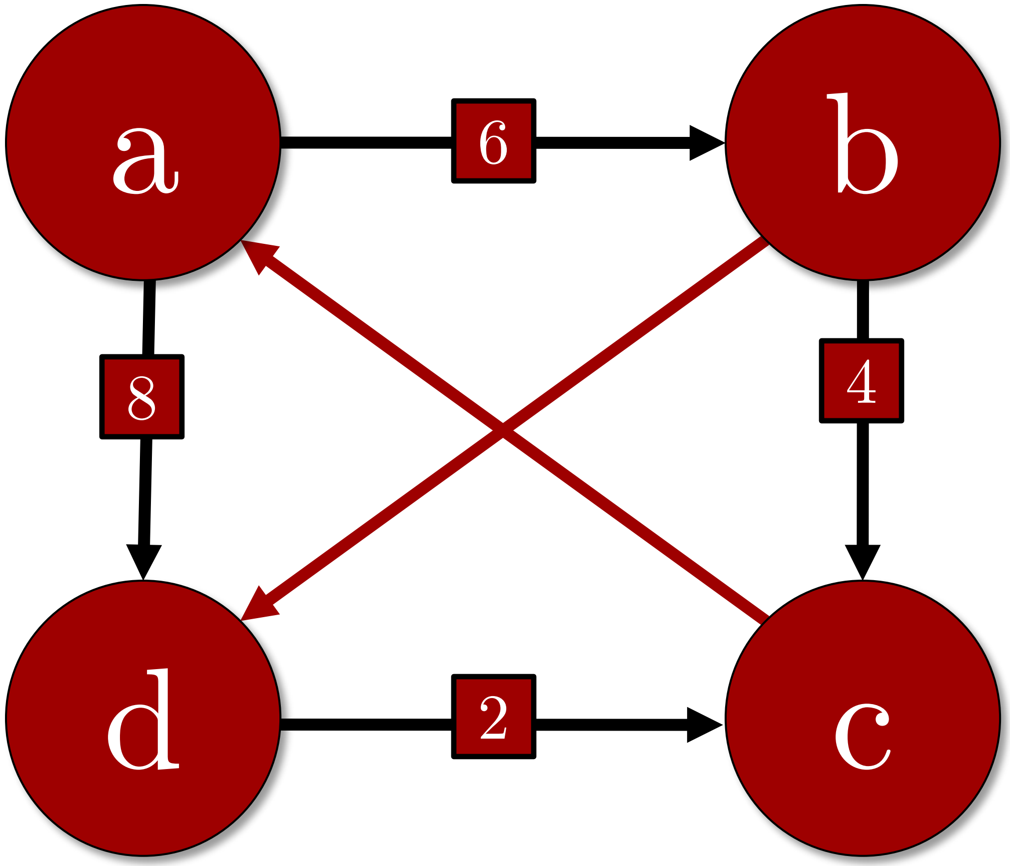
Example: ranked pairs



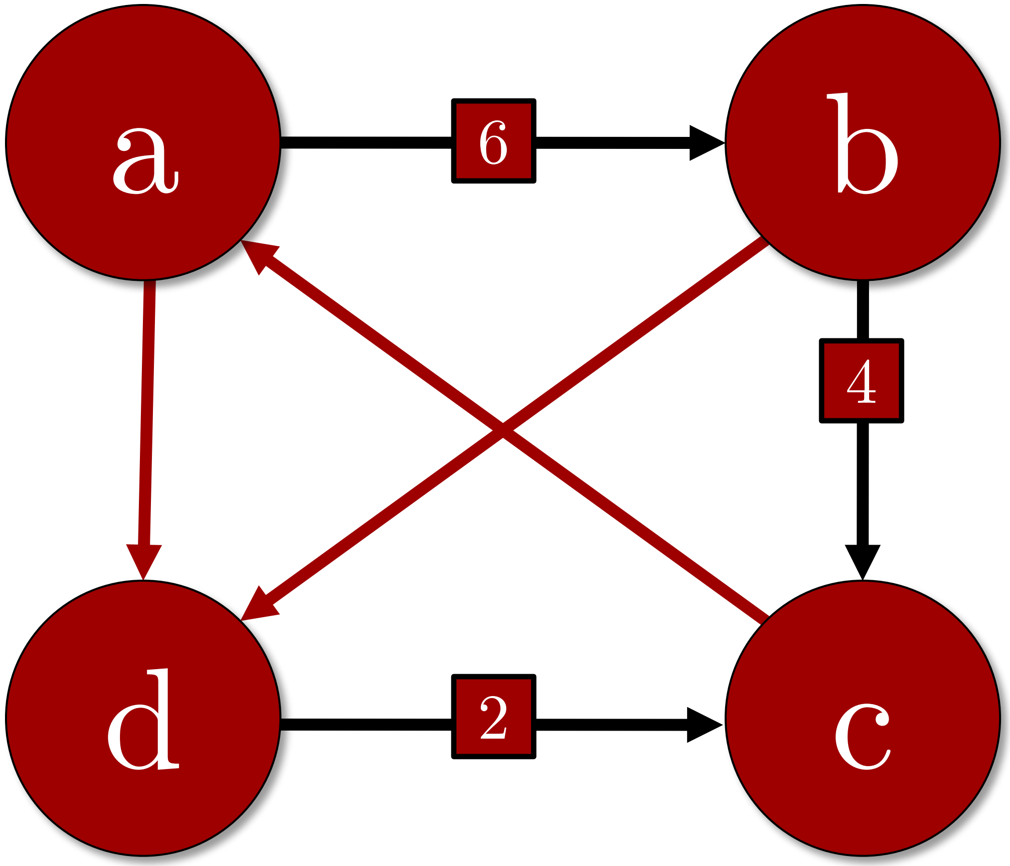
Example: ranked pairs



Example: ranked pairs



Example: ranked pairs



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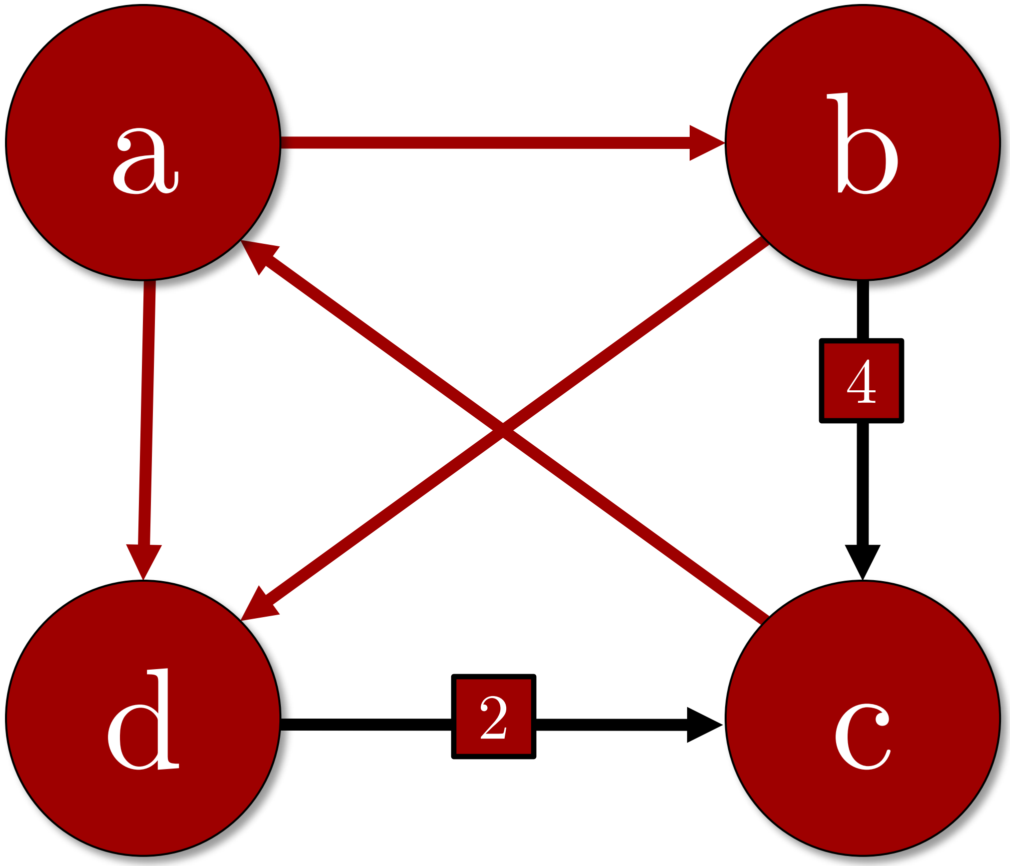
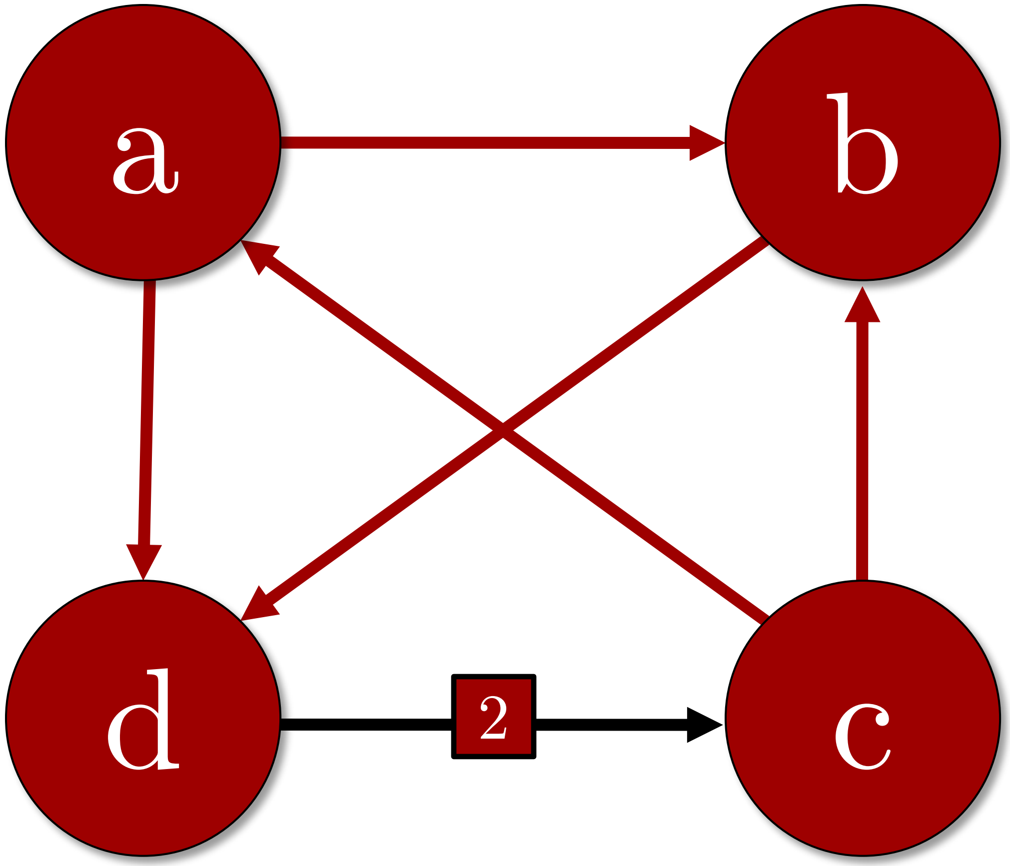


Example: ranked pairs



Example: ranked pairs