CMU 15-896

Social choice 5: Ranking/Selection Systems

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Axiomatic approach

- Social choice theory often uses axioms to guide the design of voting rules
- Representation theorem: a set of axioms that uniquely characterize a popular rule
- This approach has been applied to ranking systems, collaborative filtering, recommendation systems, etc.
- Coming up: representation theorem for PageRank

The page ranking problem

- The internet is represented by a directed graph $G = (V, E)$
- Vertices V are webpages
- $(u, v) \in E$ represents a hyperlink from u to v
- Given G, a ranking system produces a ranking over V that represents the "power" or "relevance" of webpages
- From a social choice point of view, the sets of voters and alternatives coincide

PageRank

- Rank the vertices based on the stationary probability of a random walk on the graph
- Assume that the graph is strongly connected
- Define the matrix A_G

$$[A_G]_{ij} = \begin{cases} \frac{1}{|S(v_j)|} & (v_j, v_i) \in E \\ 0 & \text{Otherwise} \end{cases}$$

PageRank

- The PageRank of G is $\boldsymbol{r}$ such that

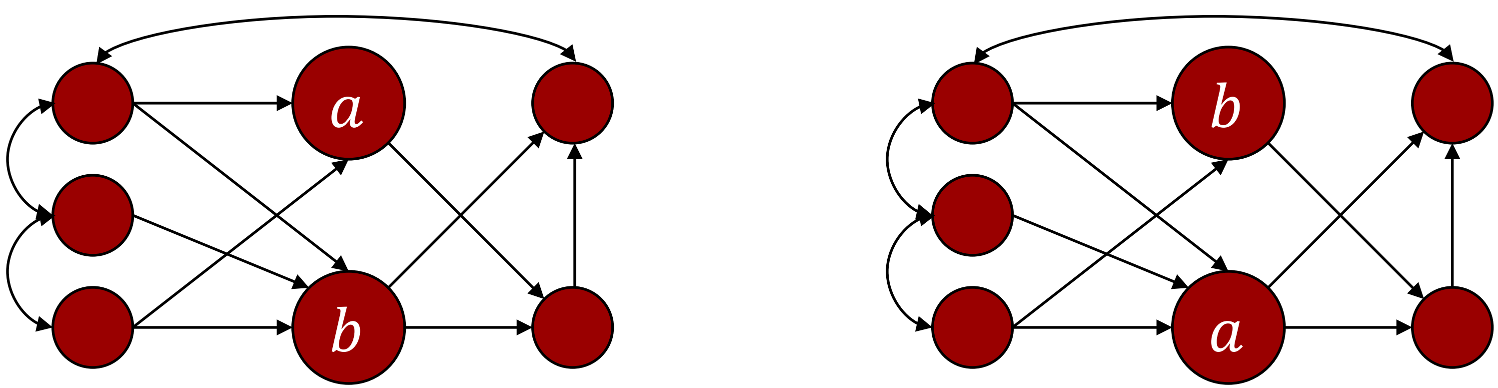
$$A_G \boldsymbol{r} = \boldsymbol{r}$$

- The PageRank ranking system ranks V according to $\boldsymbol{r}$:

$$v_i \succcurlyeq_{PR} v_j \Leftrightarrow r_i \geq r_j$$

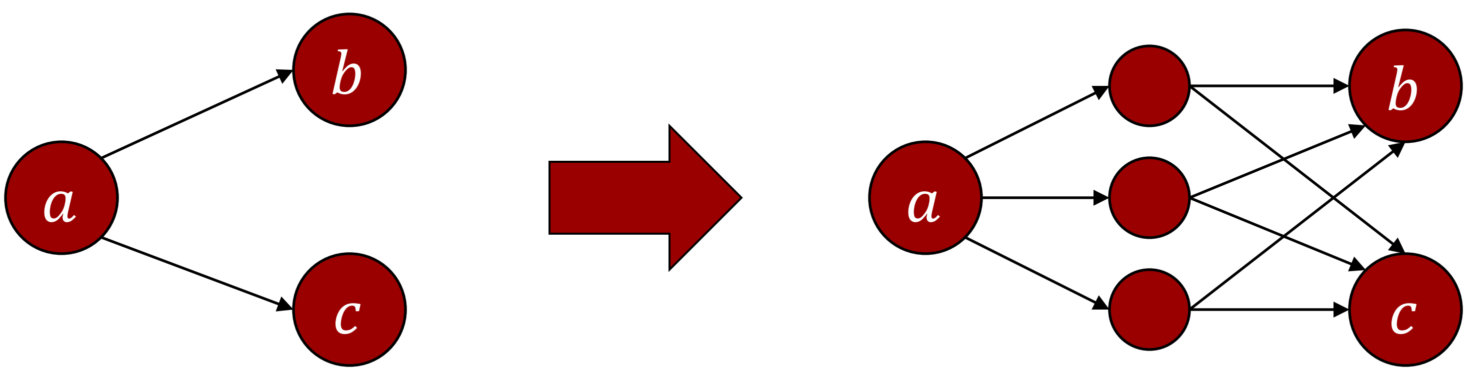
Axiom 1: isomorphism

- The ranking must not rely on the names of the vertices, only on the voting structure
- Clearly satisfied by PageRank



Axiom 2: Vote by committee

- A node may vote indirectly through intermediate nodes, each of which has the original votes



Vote by committee formalized

- Ranking system f satisfies vote by committee if for every $G = (V, E)$, for every $v, v', v'' \in V$, and for every $k \in \mathbb{N}$, if $G' = (V', E')$ where
$$V' = V \cup \{u_1, \dots, u_k\}$$
and
$$E' = E \setminus \{(v, x) | x \in S_G(v)\} \cup \{(v, u_i) | i = 1, \dots, k\}$$
$$\cup \{(u_i, x) | x \in S_G(v), i = 1, \dots, k\},$$
then $v' \succcurlyeq_G^f v'' \Leftrightarrow v' \succcurlyeq_{G'}^f v''$
- Lemma: PageRank satisfies vote by committee

Proof

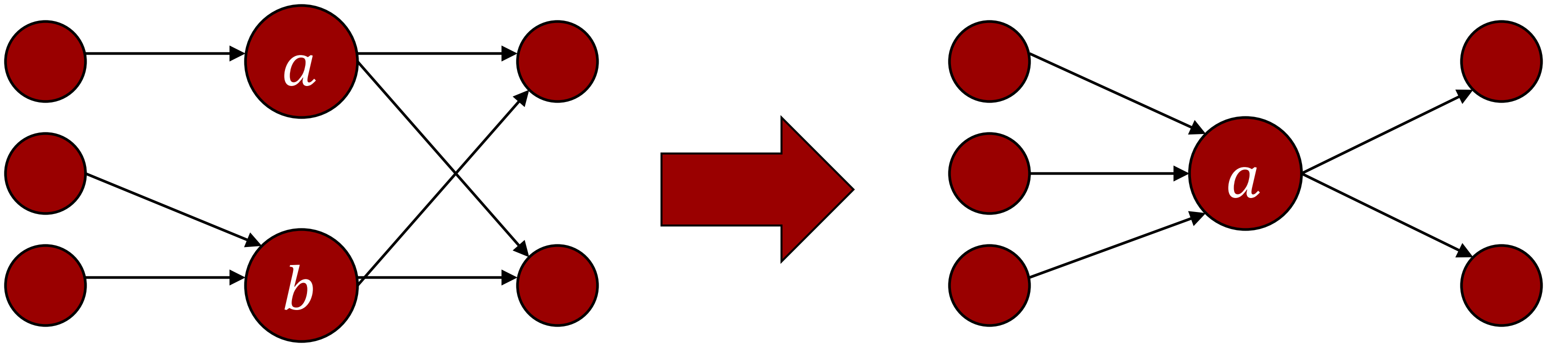
- Let $\boldsymbol{r}$ be a solution to $A_G\boldsymbol{r} = \boldsymbol{r}$
- $\boldsymbol{r}' = \left(r_1, \dots, r_n, \frac{r_1}{k}, \dots, \frac{r_1}{k}\right)^T$
- $A_{G'} = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} & a_{11} & \dots & a_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \dots & a_{nn} & a_{n1} & \dots & a_{n1} \\ \frac{1}{k} & & & & & & \\ \vdots & & & 0 & & & \\ \frac{1}{k} & & & & & & \end{pmatrix}$
- For i $= 1, \dots, n$: $[A_{G'}\boldsymbol{r}']_i = \sum_{j=2}^{n} a_{ij}r_j + ka_{i1} \cdot \frac{r_1}{k} = r_i$
- For $i = n+1, \dots, n+k$: $[A_{G'}\boldsymbol{r}']_i = \frac{1}{k} r_1$ ■

Axiom 3: self edge

- Adding a self edge to v strengthens v but does not change the ranking of other vertices

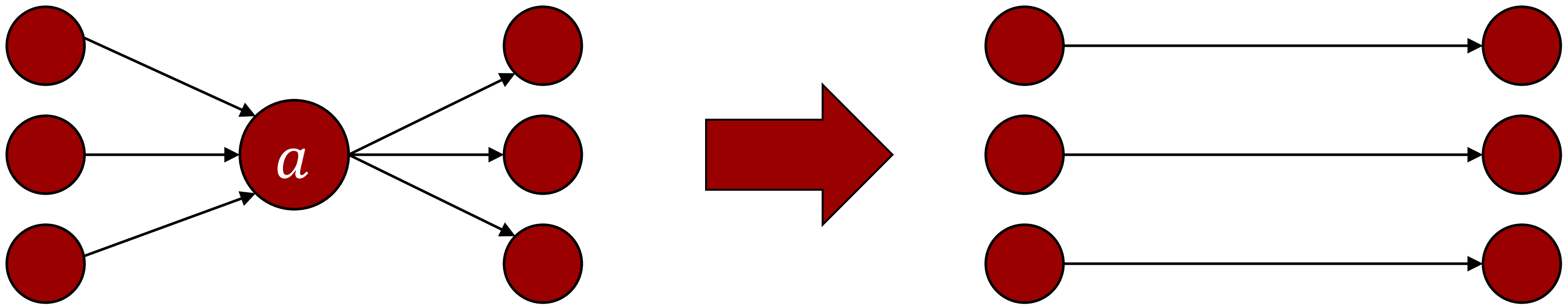
Axiom 4: collapsing

- Vertices that vote identically can be merged into a single vertex, with all the incoming edges of the original vertices
- The ranking of vertices that were not collapsed remains unchanged



Axiom 5: proxy

- k vertices of equal rank that voted for k alternatives via proxy can achieve the same result by voting for one alternative each



Representation theorem

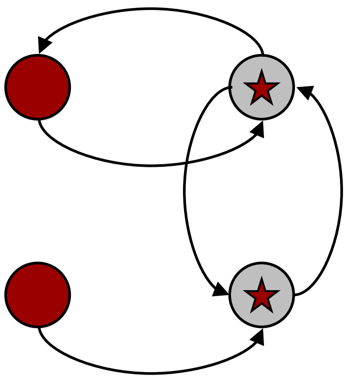
- Theorem [Altman and Tennenholtz 2005]: a ranking system satisfies axioms 1-5 if and only if it is the PageRank ranking system
- To show "only if": prove that the five axioms imply a unique ranking on each graph!

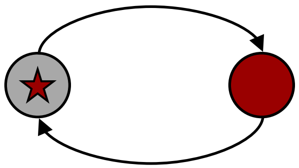
Selecting a subset

- A *k-selection system* receives a directed graph as input and outputs $V' \subseteq V$ such that $|V'| = k$
- Edges are interpreted as approval votes, trust, or support
- Think of graph as directed social network
- A k-selection system f is *impartial* if $i \in f(G)$ does not depend on the votes of i

Impartial approximations

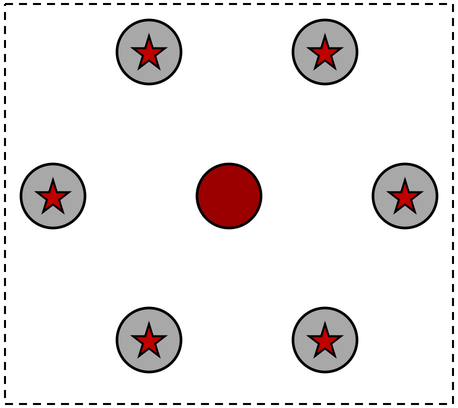
- Optimization target: sum of indegrees of selected agents
- Optimal solution: not impartial
- $k = n$: no problem
- $k = 1$: no finite impartial approx
- $k = n - 1$: no finite impartial approx!

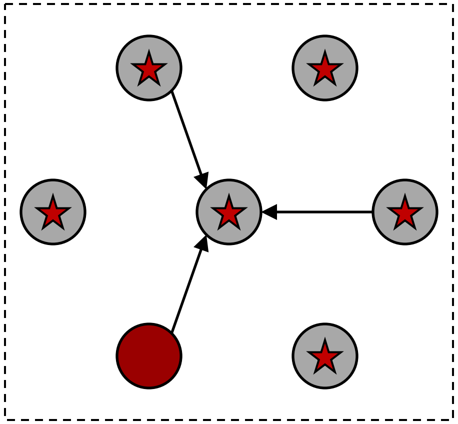




An impossibility result

- Theorem [Alon et al. 2011]: For all $k \in \{1, \dots, n-1\}$ there is no impartial k-selection system w. finite approx ratio
- Proof $(k = n - 1)$:
  - Assume for contradiction
  - Wlog n eliminated given empty graph
  - Consider stars with n as center, n cannot be eliminated
  - Function $f: \{0,1\}^{n-1} \setminus \{\vec{0}\} \to \{1, \dots, n-1\}$ satisfies $f(\vec{x}) = i \Leftrightarrow f(\vec{x} + e_i) = i$
  - $|f^{-1}(i)|$ even for all $i = 1, \dots, n-1 \Rightarrow |\mathrm{dom}(f)|$ is even; but $|\mathrm{dom}(f)| = 2^{n-1} - 1$ ■



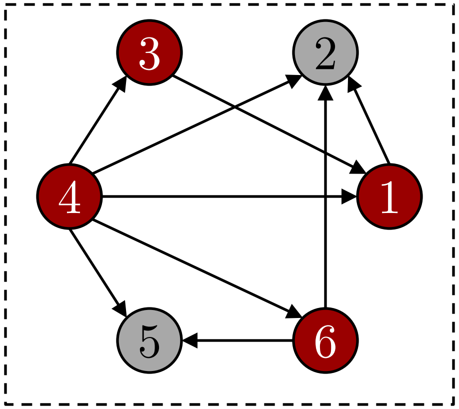


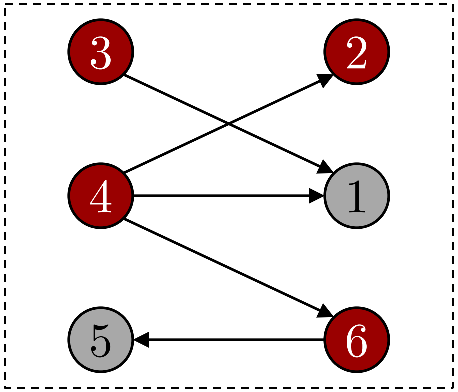
A mathematician's survivor

- Each tribe member votes for at most one member
- One member must be eliminated
- Impartial rule cannot have property: if unique member received votes he is not eliminated

Randomized systems

- The randomized m-partition system:
  - Assign vertices uniformly i.i.d. to m subsets
  - For each subset, select $\sim \frac{k}{m}$ agents with highest indegrees based on edges from other subsets
- The m-partition system is a distribution over impartial systems





Approximation

- Theorem [Alon et al. 2011]:
  1. The approx ratio is 4 with $m = 2$
  2. The approx ratio is $1 + O\left(\frac{1}{k^3}\right)$ for $m \sim k^{\frac{1}{3}}$
- Proof (only part 1):
  - Assume for ease of exposition: k is even
  - Let K be the optimal set
  - A partition $\pi = (\pi_1, \pi_2)$ divides K into two subsets $K_1^\pi = K \cap \pi_1$ and $K_2^\pi = K \cap \pi_2$
  - $d_1^\pi = \{(u, v) \in E | u \in \pi_2, v \in K_1^\pi\}$, d_2^π defined analogously
  - We get at least $\frac{d_1^\pi + d_2^\pi}{2}$
  - $\mathbb{E}[d_1^\pi + d_2^\pi] = \frac{OPT}{2}$ ■