Cooperative Games

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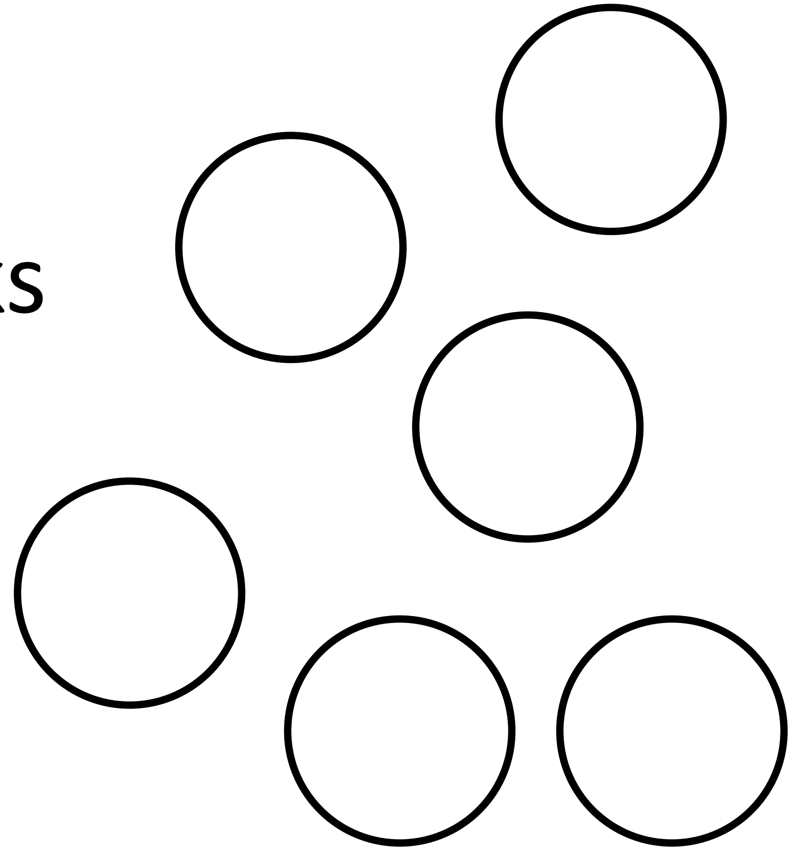
Cooperative Games

Players divide into coalitions to perform tasks

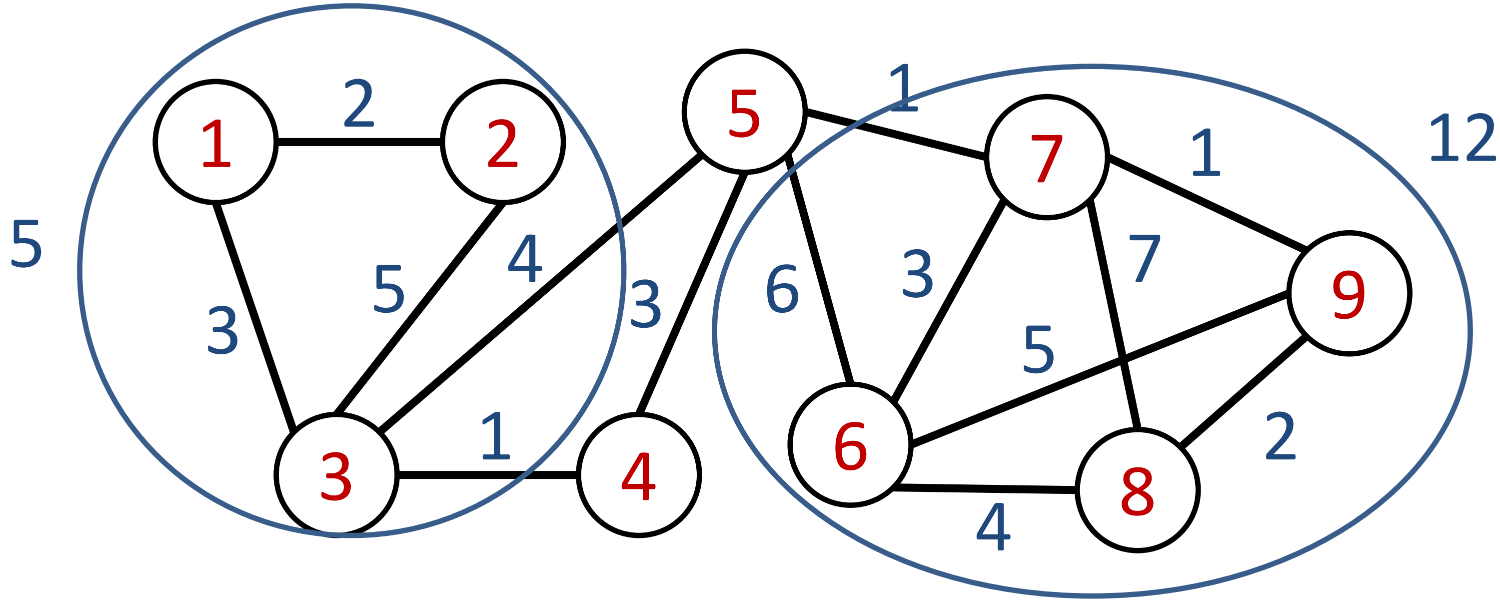


Coalition members can freely divide profits.

How should profits be divided?

Matching Games

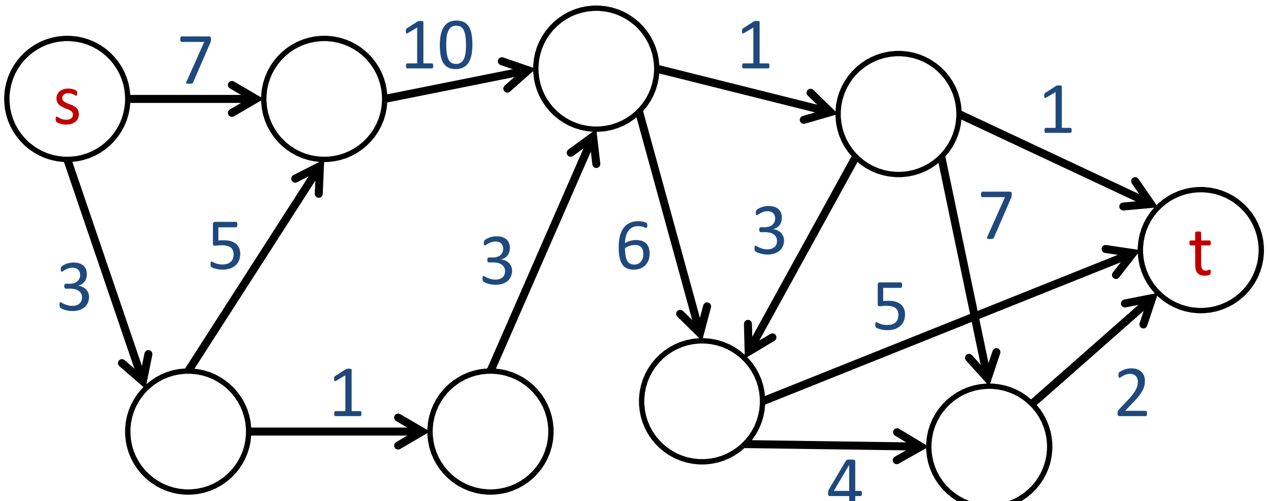
- We are given a weighted graph



- Players are nodes; value of a coalition is the value of the max. weighted matching on the subgraph.
- Applications: markets, collaboration networks.

Network Flow Games

- We are given a weighted, directed graph



- Players are edges; value of a coalition is the value of the max. flow it can pass from s to t.
- Applications: computer networks, traffic flow, transport networks.

Weighted Voting Games

- We are given a list of weights and a threshold.
- $(w_1, \ldots, w_n; q)$
- Each player i has a weight w_i; value of a coalition is 1 if its total weight is more than q (winning), and 0 otherwise (losing).
- Applications: models parliaments, UN security council, EU council of members.

Bankruptcy Problem

- In the Talmud:
- A business goes bankrupt, leaving several debts behind.
- Creditors want to collect the debt.
- The business has a net value of L to divide.
- Each creditor has a claim c_i
- Problem: claims total is more than the net value: $c_1 + \cdots + c_n > L$
- How should L be divided?
- Applications: legal matters (divorce law, bankruptcy)

Cost Sharing

- A group of friends shares a cab on the way back from a club; how should taxi fare be divided?
- How to split a bill?
- A number of users need to connect to a central electricity supplier; how should the cost of setting up the electricity network be divided? (should a central location be charged as much as a far-off location?)

Cooperative Games

- A set of players - $N = \{1, \ldots, n\}$
- Characteristic function - $v: 2^N \rightarrow \mathbb{R}$
- $v(S)$ – value of a coalition S.
- CS – a partition of N; a coalition structure.
- $OPT(\mathcal{G}) = \max_{CS} \sum_{S \in CS} v(S)$
- Imputation: a vector $\mathbf{x} \in \mathbb{R}^n$ satisfying efficiency: $\sum_{i \in S} x_i = v(S)$ for all S in CS

Cooperative Games

A game $\mathcal{G} = \langle N, v \rangle$ is called **simple** if

$$v(S) \in \{0,1\}$$

$\mathcal{G}$ is **monotone** if for any $S \subseteq T \subseteq N$:

$$v(S) \leq v(T)$$

$\mathcal{G}$ is **superadditive** if for disjoint $S, T \subseteq N$:

$$v(S) + v(T) \leq v(S \cup T)$$

$\mathcal{G}$ is **convex** if for $S \subseteq T \subseteq N$ & $i \in N \setminus T$:

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$$

Dividing Payoffs in Cooperative Games

The core, the Shapley value and the Nucleolus

The Core

An imputation $\mathbf{x}$ is in the core if

$$\sum_{i \in S} x_i = x(S) \geq v(S), \forall S \subseteq N$$

- Each subset of players is getting at least what it can make on its own.
- A notion of stability; no one can deviate.

The Core

The core is a polyhedron: a set of vectors in $\mathbb{R}^n$ that satisfies linear constraints

$$\sum_{i \in N} x_i = v(N)$$

$$\sum_{i \in S} x_i \geq v(S), \forall S \subseteq N$$

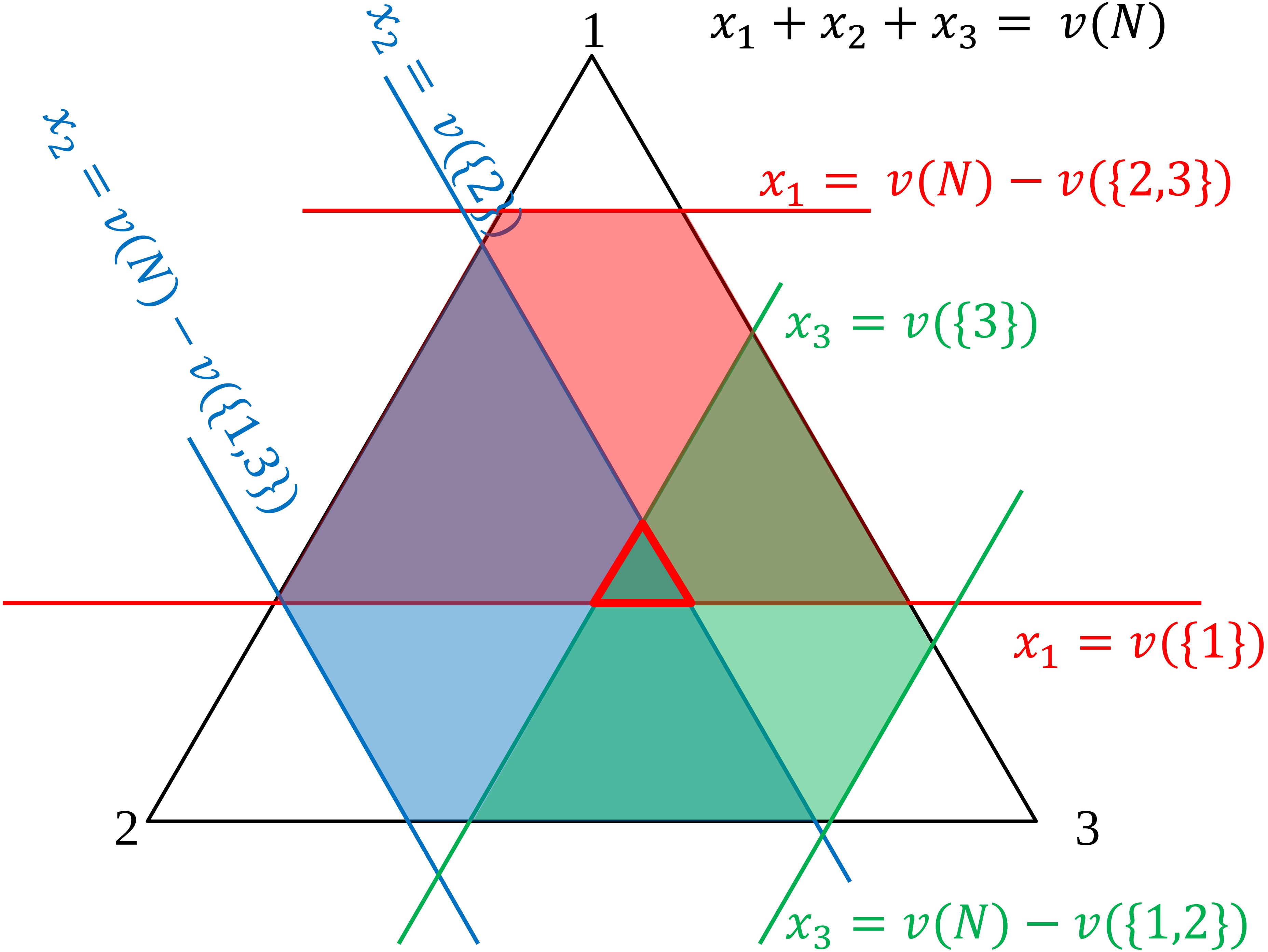
The Core

For three players, $N = \{1,2,3\}$

$$x_1 + x_2 + x_3 = v(N)$$

$$x_i \geq v(\{i\}), \forall i \in N$$

$$x_2 + x_3 \geq v(\{2,3\}) \Rightarrow x_1 \leq v(N) - v(\{2,3\})$$
$$x_1 + x_3 \geq v(\{1,3\}) \Rightarrow x_2 \leq v(N) - v(\{1,3\})$$
$$x_1 + x_2 \geq v(\{1,2\}) \Rightarrow x_3 \leq v(N) - v(\{1,2\})$$



1
2
3
$x_1 + x_2 + x_3 = v(N)$
$x_1 = v(N) - v(\{2,3\})$
$x_1 = v(\{1\})$
$x_2 = v(\{2\})$
$x_2 = v(N) - v(\{1,3\})$
$x_3 = v(\{3\})$
$x_3 = v(N) - v(\{1,2\})$

Is the Core Empty?

The core can be empty…

Core-Empty: given a game $\mathcal{G} = \langle \boldsymbol{N}, \boldsymbol{v} \rangle$, is the core of $\mathcal{G}$ empty?

Note that we are "cheating" here: a naïve representation of $\mathcal{G}$ is a list of 2^n vectors

We are generally dealing with

a. Games with a compact representation

b. Oracle access to $\mathcal{G}$

… and obtaining algorithms that are $poly(n)$

Is the Core Empty?

Simple Games: a game is called simple if $v(S) \in \{0,1\}$ for all $S \subseteq N$.

Coalitions with value 1 are winning;

those with value 0 are losing.

A player is called a veto player if she is a member of every winning coalition (can't win without her).

Core Nonemptiness: Simple Games

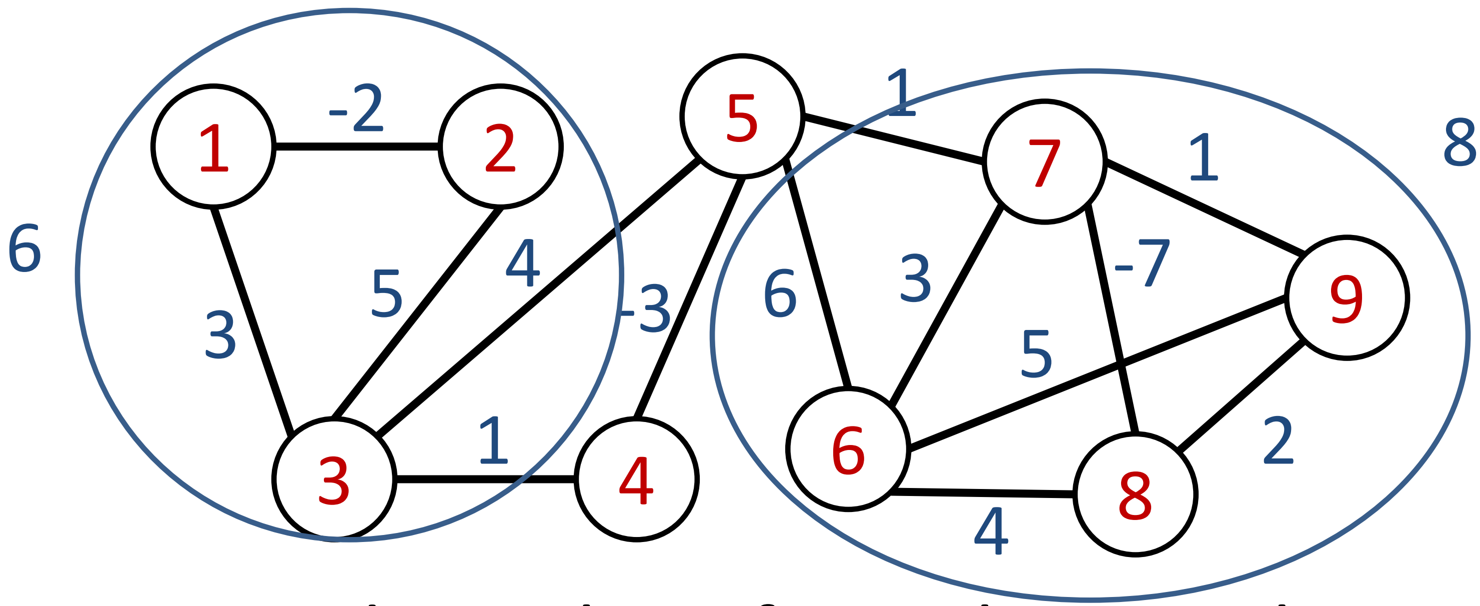
Theorem: let $\mathcal{G} = \langle N, v \rangle$ be a simple game; then $Core(\mathcal{G}) \neq \emptyset$ iff $\mathcal{G}$ has veto players.

Corollary: Core-Empty is easy when restricted to weighted voting games.

The General Case

Theorem: Core-Empty is NP-hard

Proof: we will show this claim for a class of games called induced-subgraph games



Players are nodes, value of a coalition is the weight of its induced subgraph.

The General Case

Lemma: the core of an induced subgraph game is not empty iff the graph has no negative cut.

Proof: we will show first that if there is no negative cut, then the core is not empty.

Consider the payoff division that assigns each node half the value of the edges connected to it

$$\phi_i = \frac{1}{2} \sum_{j \in N} w(i,j)$$

Need to show that $\phi(S) \geq v(S)$ for all $S \subseteq N$.

The General Case

$$\phi(S) = \sum_{i \in S} \phi_i = \sum_{i \in S} \sum_{j \in N} \frac{1}{2} w(i,j)$$

$$= \sum_{i \in S} \sum_{j \in S} \frac{1}{2} w(i,j) + \sum_{i \in S} \sum_{j \in N \setminus S} \frac{1}{2} w(i,j)$$

$$= v(S) + \frac{1}{2} Cut(S, N \setminus S)$$

Since there are no negative cuts, the last expression is at least $v(S)$

Note: we haven't shown efficiency, i.e. $\phi(N) = v(N)$

The General Case

Now, suppose that there is some negative cut; i.e. there is some $S \subseteq N$ such that

$$\sum_{i \in S} \sum_{j \in N \setminus S} w(i, j) < 0$$

Take any imputation x; then

$$\sum_{i \in N} x_i = x(S) + x(N \setminus S) = v(N)$$

$$= \phi(S) + \phi(N \setminus S)$$

The General Case

Therefore:

$$x(S) - v(S) + x(N \setminus S) - v(N \setminus S) =$$

$$\phi(S) - v(S) + \phi(N \setminus S) - v(N \setminus S) \quad =$$

$$\sum_{i \in S} \sum_{j \in N \setminus S} \frac{1}{2} w(i,j) + \sum_{i \in N \setminus S} \sum_{j \in S} \frac{1}{2} w(i,j) \; =$$

$$Cut(S, N \setminus S) < 0$$

So, it is either the case that $x(S) < v(S)$ or $x(N \setminus S) < v(N \setminus S)$; hence x cannot be in the core.

The General Case

Lemma: deciding whether a graph has a negative cut is NP-complete.

Proof: we reduce from the **Max-Cut problem**. Given a weighted, undirected graph $\Gamma = \langle V, E \rangle$, where $w(i,j) \geq 0$ for all $(i,j) \in E$, and an integer K, is there a cut $(S, V \setminus S)$ of Γ whose weight is more than K?

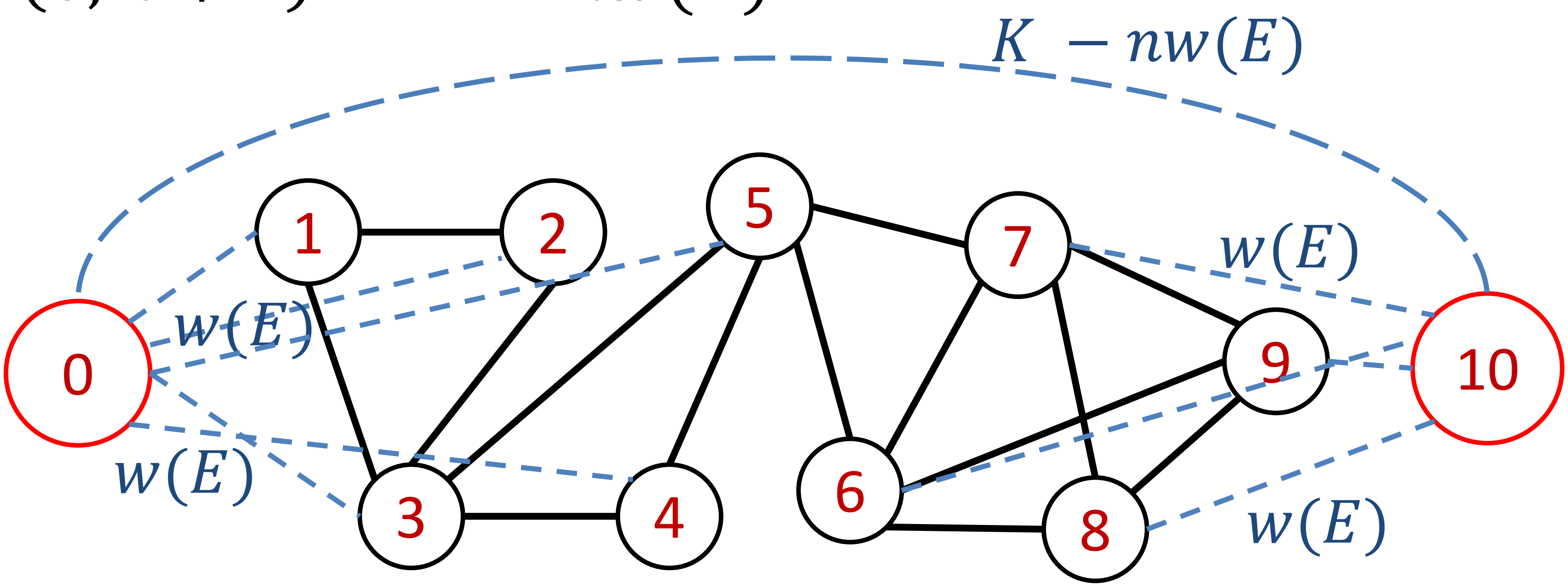
The General Case

We write $V = \{1, \ldots, n\}$. We define a graph $\Gamma' = \langle V', E' \rangle$ with capacities as follows

$c(i, j) = -w(i, j)$ for all $(i, j) \in E$

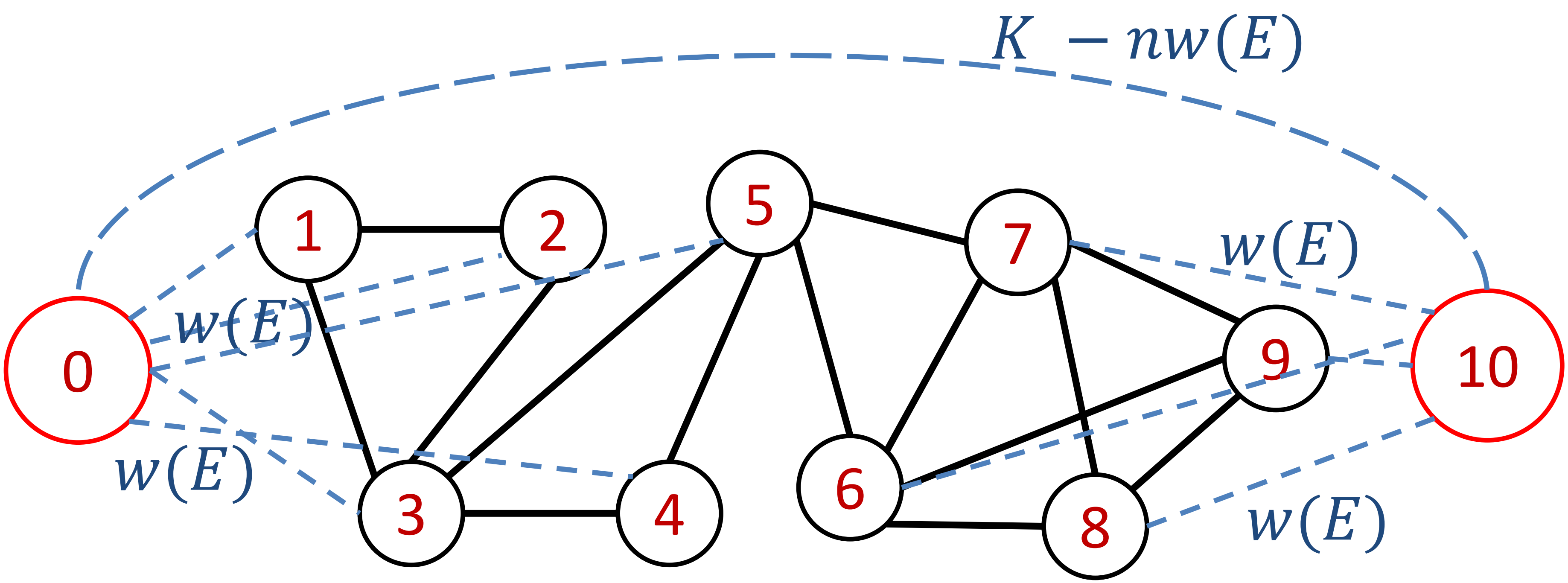
$c(0, j) = c(n + 1, j) = w(E)$ for all $j \in V$

$c(0, n + 1) = K - nw(E)$



The General Case

Any negative cut in this graph must separate 0 and $n + 1$. It must also have exactly n edges with capacity $w(E)$. Therefore, it is negative iff the original graph has a cut with weight at least K.



The General Case

Notes:

this proof is one of the first complexity results on the stability of cooperative games, and appears in Deng & Papadimitriou's seminal paper "On the Complexity of Cooperative Solution Concepts" (1994).

The payoff division ϕ_i is special: it is in fact the **Shapley value** for induced graph games.

Core Extensions

- What if the core is empty?
- The players cannot generate enough value to satisfy everyone.
- We can increase the total value with a subsidy

$\alpha \cdot OPT(\mathcal{G})$

Core Extensions

As an LP:

$$\min \alpha$$

$$\text{subject to:}$$

$$x(N) = \alpha \cdot OPT(\mathcal{G})$$

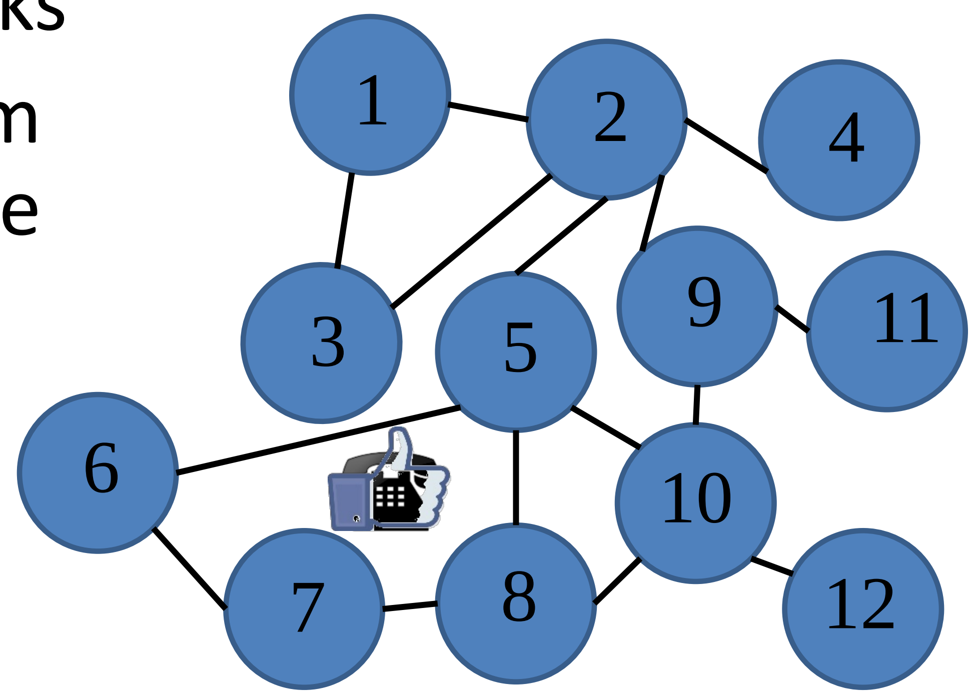
$$x(S) \geq v(S), \forall S \subseteq N$$

If $\alpha = 1$ then the core is not empty.

The value of α in an optimal solution of the above LP is called **the cost of stability of** $\mathcal{G}$,
and referred to as $CoS(\mathcal{G})$

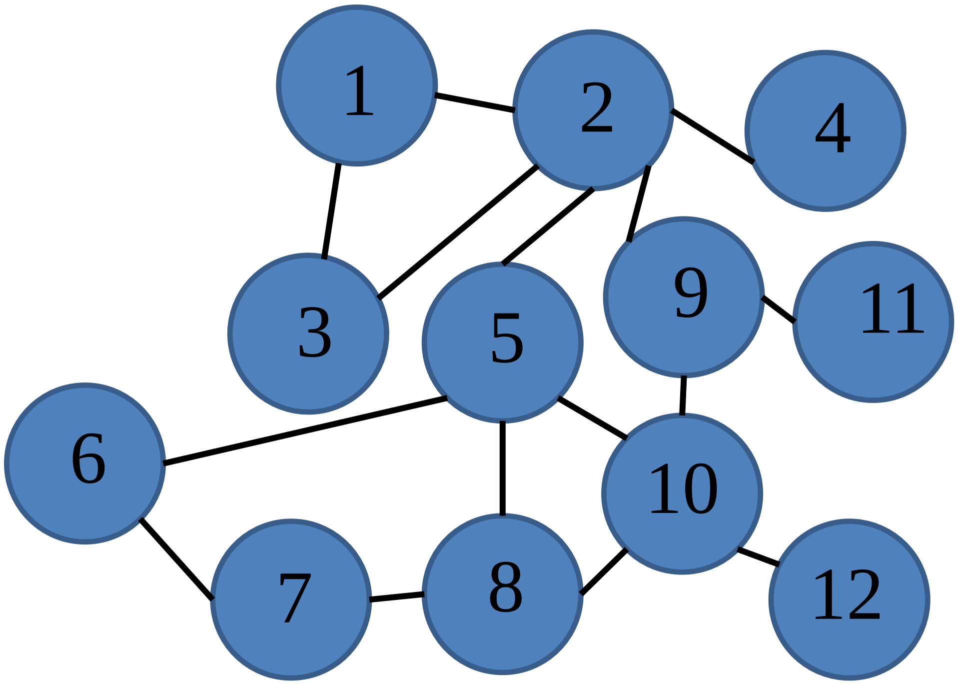
Restricted Cooperation

- Some coalitions may be impossible or unlikely due to practical reasons
- Interaction networks [Myerson '77]:
  - Nodes are agents
  - Edges are social links
  - A coalition can form only if its agents are connected



Restricted cooperation - example

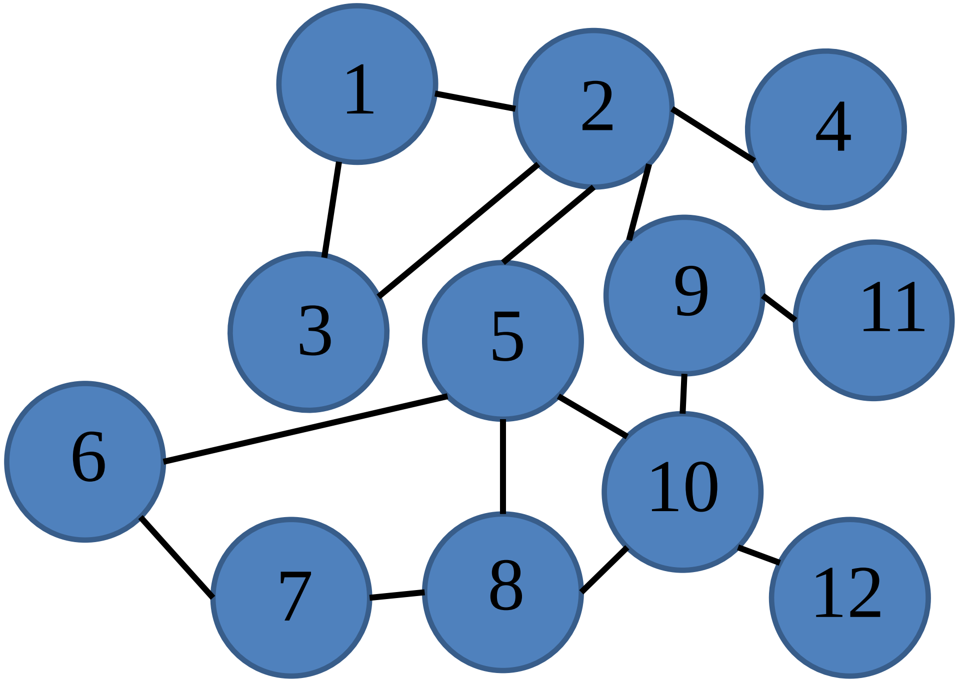
- The coalition {2,9,10,12} is allowed
- The coalition {3,6,7,8} is not allowed



Restricted cooperation increases stability

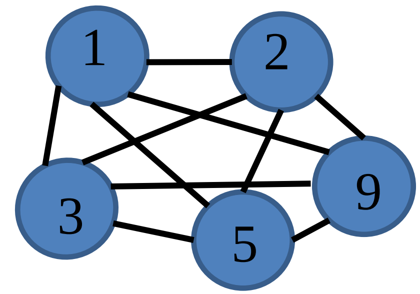
Theorem [Demange'04]: If the underlying network H is a *tree*, then the core of $\mathcal{G}|_H$ is non-empty

Moreover, a core outcome can be computed efficiently

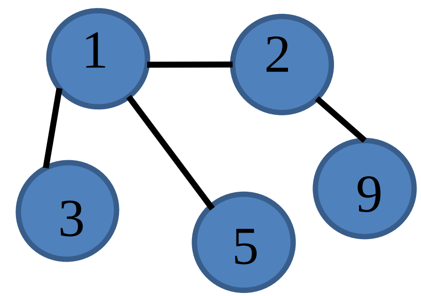


CoS with restricted cooperation

- Generally, $CoS(\mathcal{G})$ can be as high as $\sqrt{n}$
  - See example in [Bachrach et al.'09]



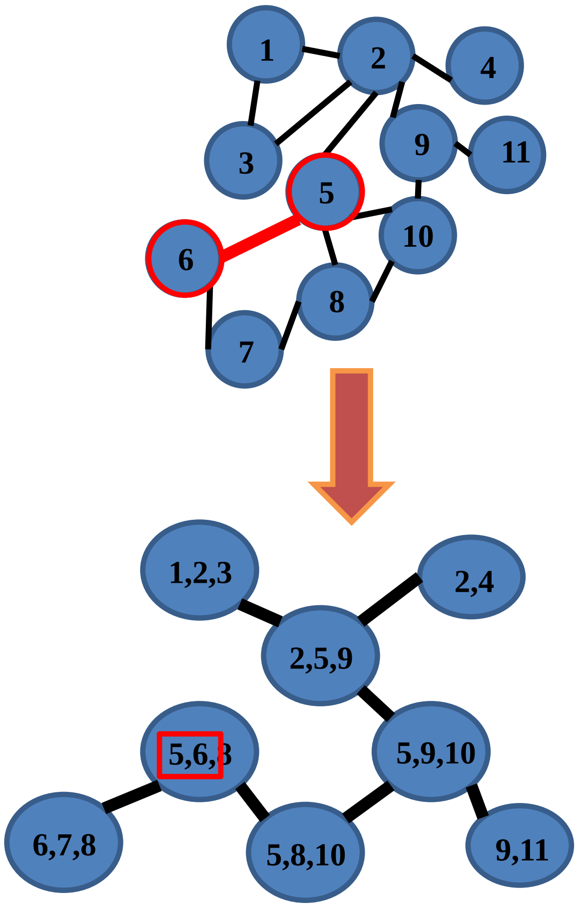
- By [Demange'04]: if H is a *tree*, the core is non-empty (i.e. $CoS(\mathcal{G}|_H) = 1$)



What is the connection between network complexity and the cost of stability?

Graphs and tree-width

- Combinatorial measures to the "complexity" of a graph. E.g.:
  - Average/max degree
  - Expansion
  - Connectivity
  - **Tree-width**



Graphs and tree-width

Given a graph $\Gamma = \langle V, E \rangle$, a tree decomposition of Γ is a tree $\mathcal{T} = \langle \mathcal{B}, \mathcal{E} \rangle$ where

- The nodes of $\mathcal{T}$ are subsets of V
- If $(i, j) \in E$, then there exists some $S \in \mathcal{B}$ such that $i, j \in S$
- If $S, T \in \mathcal{B}$ contain $i \in V$, then S, T are connected in $\mathcal{T}$.

Graphs and tree-width

Given a tree decomposition $\mathcal{T} = \langle \mathcal{B}, \mathcal{E} \rangle$ of Γ, define

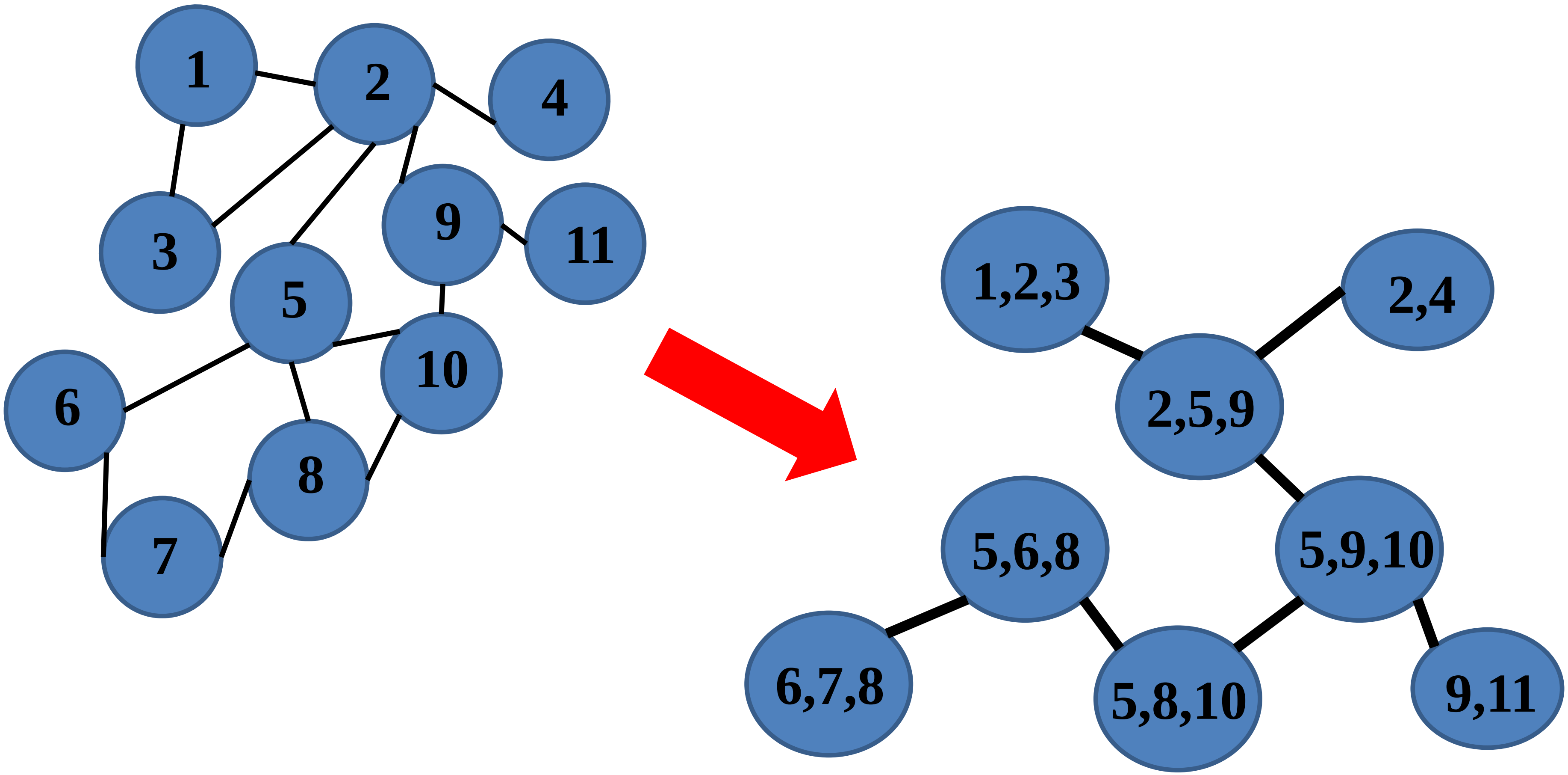
$$width(\mathcal{T}) = \max_{S \in \mathcal{B}} |S| - 1$$

The treewidth of Γ is

$$tw(\Gamma) = \min width(\mathcal{T})$$

Where the minimum is taken over all possible tree decompositions of Γ.

Graphs and tree-width



Tree-Width bounds Complexity

Many NP-hard graph combinatorial problems are FPT in $tw(\Gamma)$:

- Coloring
- Hamiltonian cycle
- Constraint solving
- Bayesian inference
- Computing equilibrium
- more…

Tree-Width bounds the CoS

Theorem **[Meir,Z.,Elkind,Rosenschein, AAAI’13]**:
For any $\mathcal{G}$ with an interaction graph H

$$CoS(\mathcal{G}|_H) \leq \; tw(H) + 1$$

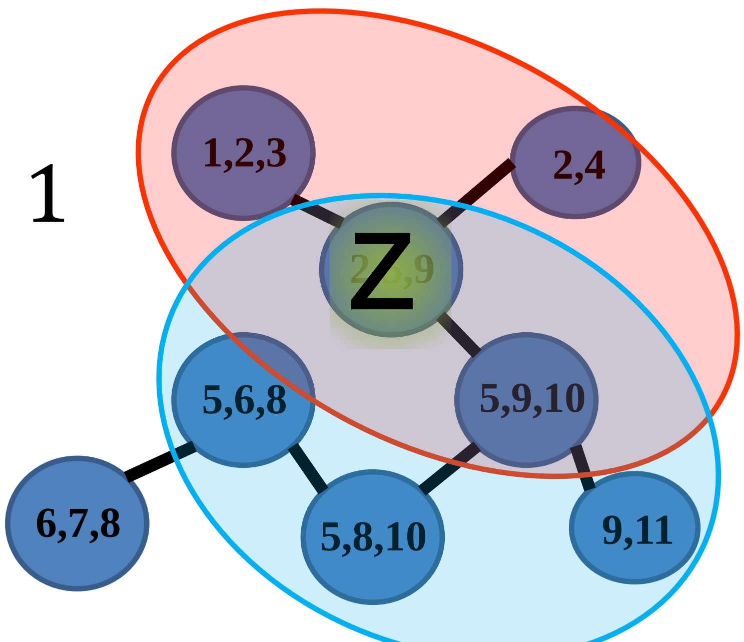
and this bound is tight for all non-trees.

A Simple Case

- Consider a **simple** and **superadditive** game
- Every two winning coalitions intersect
- Every coalition induces a subtree
- Thus all "winning subtrees" intersect at some node Z

For example: $v(\{1,2,3,\mathbf{5}\}) = 1$

and $v(\{\mathbf{5},11\}) = 1$



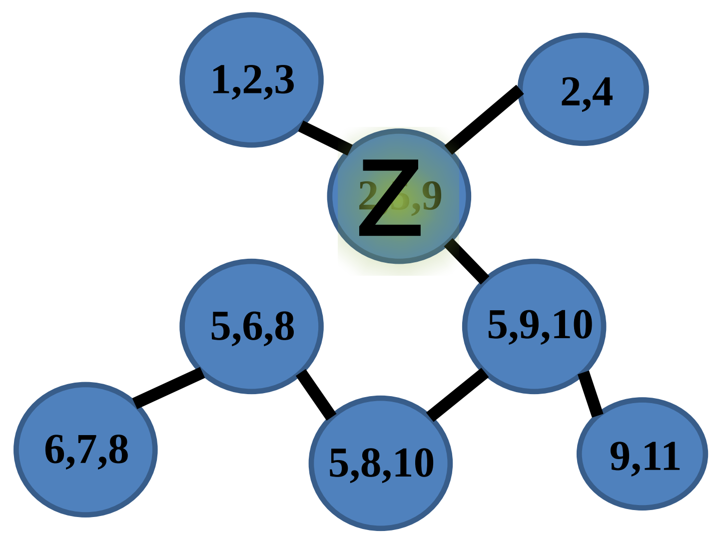
A Simple Case

- All winning coalitions intersect some node Z
- Pay 1 to every agent in Z
- Every winning coalition gets at least 1
- Total payoff is at most $|Z| \leq tw(H) + 1$

$v(S) = 1 \rightarrow \exists i \in S \cap Z$

If $i \in Z$ then $x_i = 1$ so…

$x(S) \geq x_i \geq 1 = v(S)$

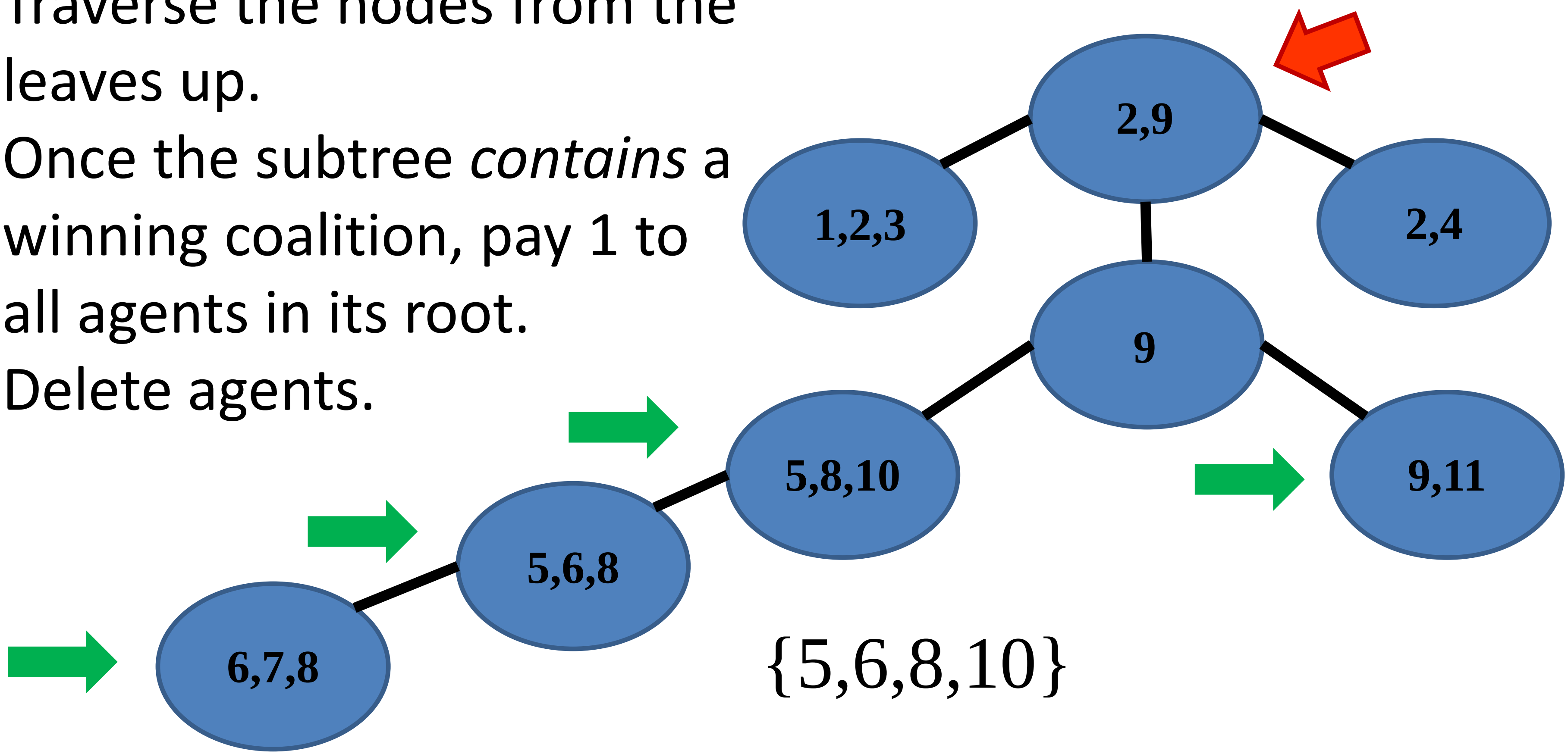


Step 1 – Simple Games

(w/o superadditivity or monotonicity)

$$(x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10})$$
$$(\;0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0 \quad 1\;)$$

1. Traverse the nodes from the leaves up.
2. Once the subtree *contains* a winning coalition, pay 1 to all agents in its root.
3. Delete agents.



$\{5,6,8,10\}$

Lemma: For any simple $\mathcal{G}$ with an interaction graph H, the algorithm produces a stable imputation x such that

$$x(N) \leq (tw(H) + 1)OPT(G|_H)$$

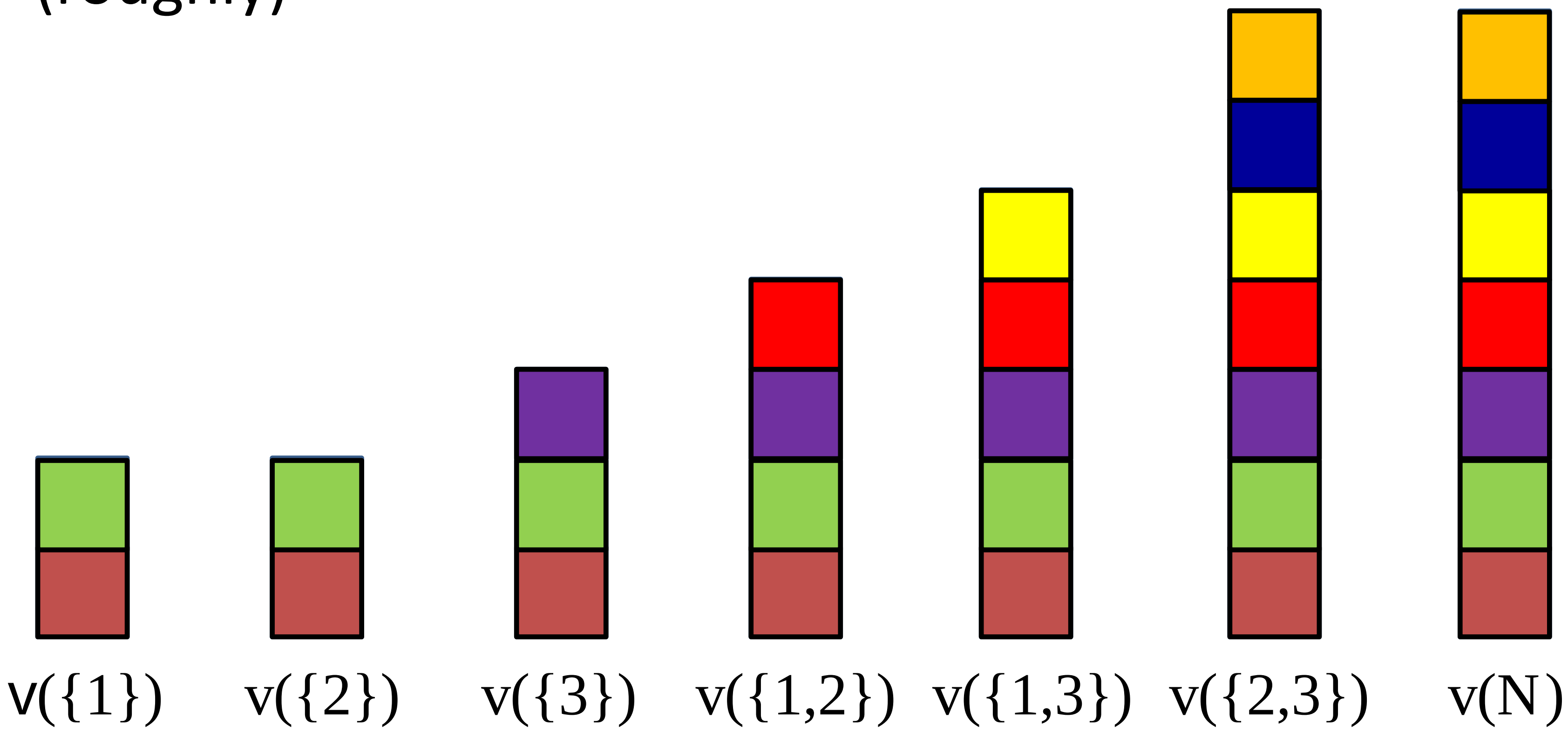
- **Stability:** every winning coalition intersects a node in the tree decomposition that was paid by the algorithm; thus gets at least 1.

- **Bounded payoff:** let S_t be the set of agents that were removed at time t.
  - S_t contains a winning coalition W_t
  - We can partition the agents into a coalition structure $CS = \{\{W_t\}_{t \in T^*}, L\}$.
  - T^* is the set of all times where sets were pruned by the algorithm.
  - The value of CS is at most $|T^*|$.

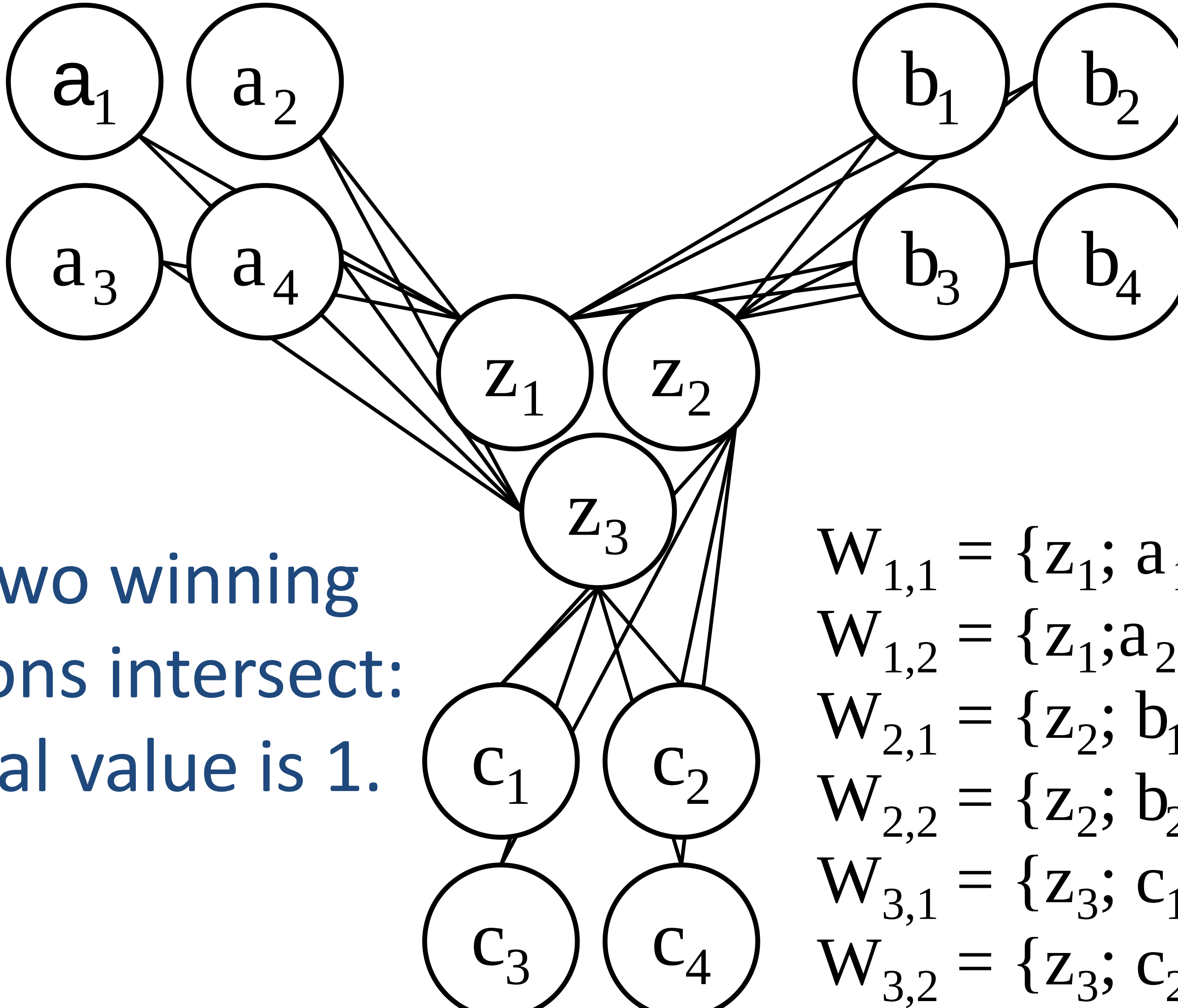
$$x(N) \leq (tw(H) + 1)\,|T^*|$$
$$\leq (tw(H) + 1) OPT(G|_H)$$

Step 2 – The General Case

1. Given a general (integer) game, split it into simple games and stabilize each individually.
2. Sum the resulting stable imputations.
(roughly)



Tightness



Any two winning coalitions intersect: optimal value is 1.

$W_{1,1} = \{z_1; a_1; a_4; b_3; b_1\}$

$W_{1,2} = \{z_1; a_2; a_3; b_2; b_4\}$

$W_{2,1} = \{z_2; b_1; b_4; c_3; c_1\}$

$W_{2,2} = \{z_2; b_2; b_3; c_2; c_4\}$

$W_{3,1} = \{z_3; c_1; c_4; a_3; a_1\}$

$W_{3,2} = \{z_3; c_2; c_3; a_2; a_4\}$

Tightness

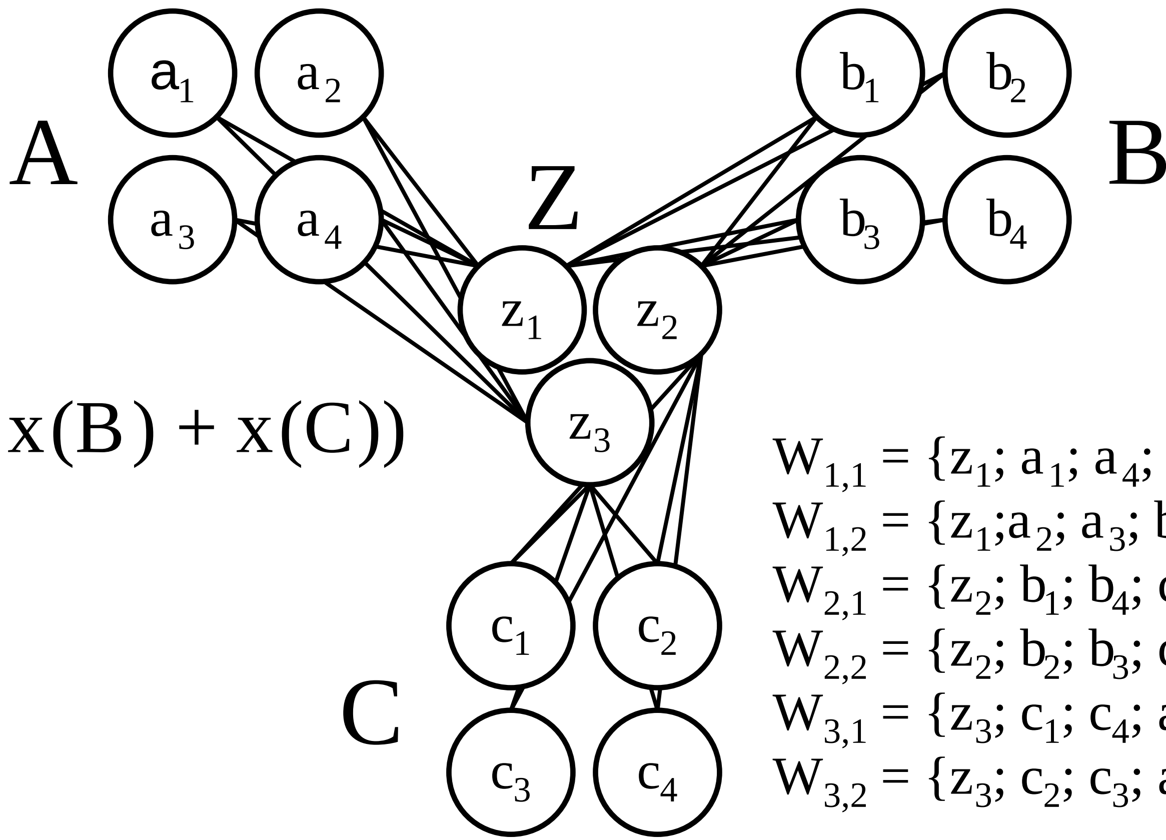
$$\left.\begin{array}{l} x(W_{1,1}) \geq 1 \\ x(W_{1,2}) \geq 1 \end{array}\right\} \; x_{z_1} \geq 1 - \tfrac{1}{2}(x(A) + x(B))$$

$$\Downarrow$$

$$x(Z) \geq 3 - (x(A) + x(B) + x(C))$$

$$\Downarrow$$

$$x(N) \geq 3$$



$W_{1,1} = \{z_1; a_1; a_4; b_3; b_1\}$
$W_{1,2} = \{z_1; a_2; a_3; b_2; b_4\}$
$W_{2,1} = \{z_2; b_1; b_4; c_3; c_1\}$
$W_{2,2} = \{z_2; b_2; b_3; c_2; c_4\}$
$W_{3,1} = \{z_3; c_1; c_4; a_3; a_1\}$
$W_{3,2} = \{z_3; c_2; c_3; a_2; a_4\}$

Implications

- The structure of the underlying social network determines stability of cooperation
- Results can be applied on many games that are based on graphs/hypergraphs:
  - Induced subgraph games [Deng & Papadimitriou ’94];
  - Matching, Covering, and Coloring games [Deng et al. ’99];
  - Social distance games [Branzei & Larson ’11];
  - Synergy coalition groups [Conitzer & Sandholm ’06];
  - Marginal contribution nets [Ieong & Shoham ’05].