Cooperative Games – The Shapley value and Weighted Voting

Yair Zick

The Shapley Value

Given a player i, and a set $S \subseteq N$, the marginal contribution of i to S is

$$m_i(S) = v(S \cup \{i\}) - v(S)$$

How much does i contribute by joining S?

Given a permutation $\sigma \in \Pi(N)$ of players, let the predecessors of i in σ be

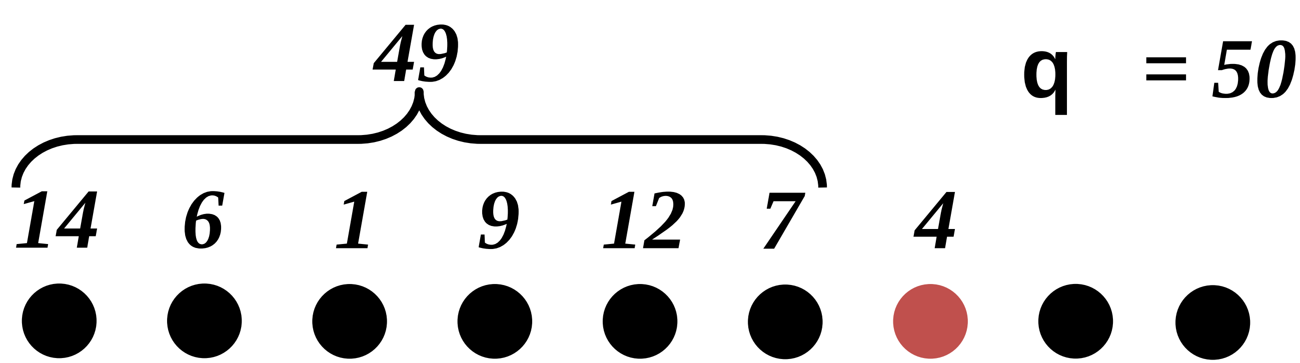
$$P_i(\sigma) = \{j \in N \mid \sigma(j) < \sigma(i)\}$$

We write $m_i(\sigma) = m_i(P_i(\sigma))$

The Shapley Value

Suppose that we choose an ordering of the players uniformly at random. The Shapley value of player i is

$$\phi_i = \mathbb{E}[m_i(\sigma)] = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i(\sigma)$$



The Shapley Value

Efficient: $\sum_{i \in N} \phi_i = v(N)$

Symmetric: players who contribute the same are paid the same.

Dummy: dummy players aren't paid.

Additive: $\phi_i(\mathcal{G}_1) + \phi_i(\mathcal{G}_2) = \phi_i(\mathcal{G}_1 + \mathcal{G}_2)$

The Shapley value is the only payoff division satisfying all of the above!

The Shapley Value

Theorem: if a value satisfies efficiency, additivity, dummy and symmetry, then it is the Shapley value.

Proof: let’s prove it on the board.

Computing Power Indices

The Shapley value has a brother – the Banzhaf value

$$\beta_i = \frac{1}{2^n} \sum_{S \subseteq N} m_i(S)$$

It uniquely satisfies a different set of axioms

Different distributional assumption – more biased towards sets of size $\frac{n}{2}$

Voting Power in the EU Council of Members

- The EU council of members is one of the governing members of the EU.
  - Each state has a number of representatives proportional to its population
  - Proportionality: “one person – one vote”
- In terms of voting power - $\phi_i \sim \frac{w_i}{w(N)}$

Image: Wikipedia

Image: Wikipedia

Voting Power in the EU Council of Members

- Changes to the voting system can achieve **better** proportional representation.

- **Changing the weights** – generally unpopular and politically delicate

- **Changing the quota** – easier to do, an “innocent” change.

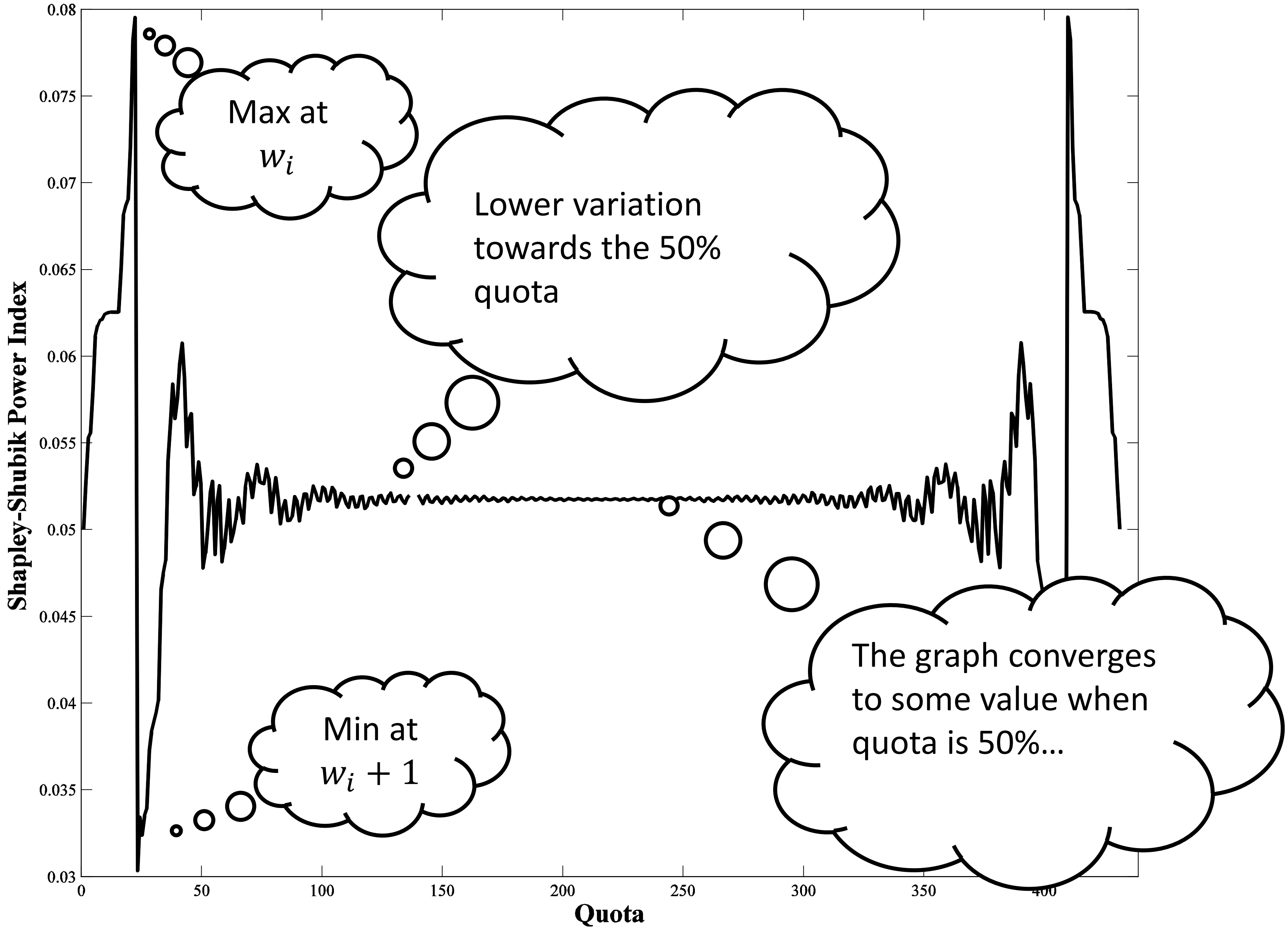
Selecting an appropriate quota (EU - about 62%), achieves proportional representation with a very small error!

Changing the Quota

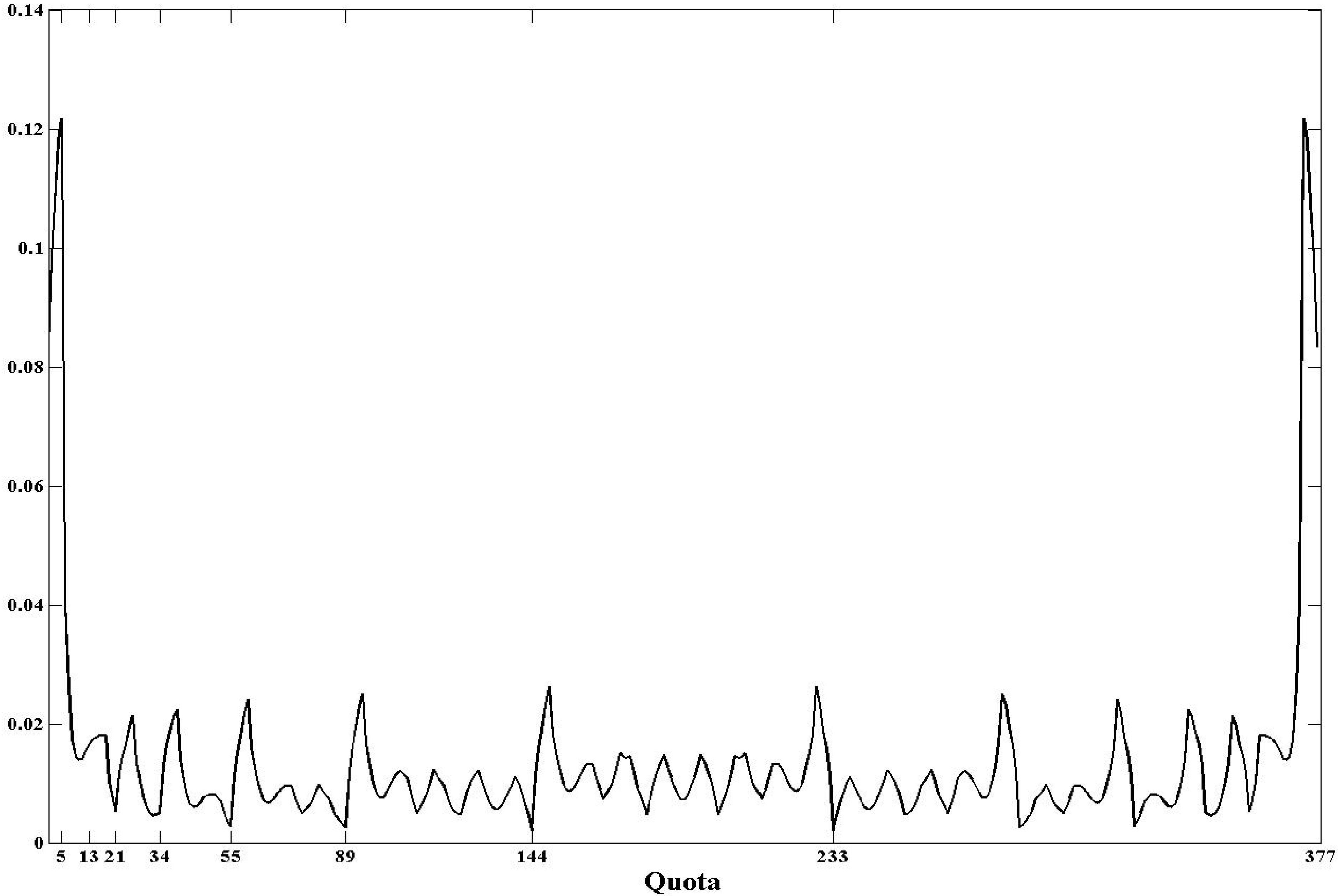
- Changes to the quota change players’ power.
- **What is the relation between quota selection and voting power?**

$$'_i \Rightarrow '_i(q)$$

A “typical” graph of '$_i$(q)



Weights are a Fibonacci Series



Maximizing $\phi_i(q)$

Theorem: $\boldsymbol{\phi_i(q)}$ is maximized at $\boldsymbol{q = w_i}$

Proof: two cases

$\boldsymbol{q \leq w_i}$**:** if i is pivotal for $\sigma \in \Pi(N)$ under q then $w(P_i(\sigma)) < q \leq w_i$, but $w(P_i(\sigma)) + w_i \geq q$. This implies that i is pivotal for σ when the threshold is w_i as well.

Maximizing $\phi_i(q)$

Lemma: let $\boldsymbol{T_i(x) = \{\sigma \in \Pi(\mathrm{N}) \mid w(P_i(\sigma)) < x\}}$
Then

$$\boldsymbol{|T_i(x)| + |T_i(y)| \geq |T_i(x+y)|}$$

for all $\boldsymbol{x, y \in \mathbb{N}}$

Proof: assume that $x \geq y$. We write

$$T_i(x, y) = \{\, \sigma \in \Pi(N) \mid x \leq w(P_i(\sigma)) < y \,\}$$

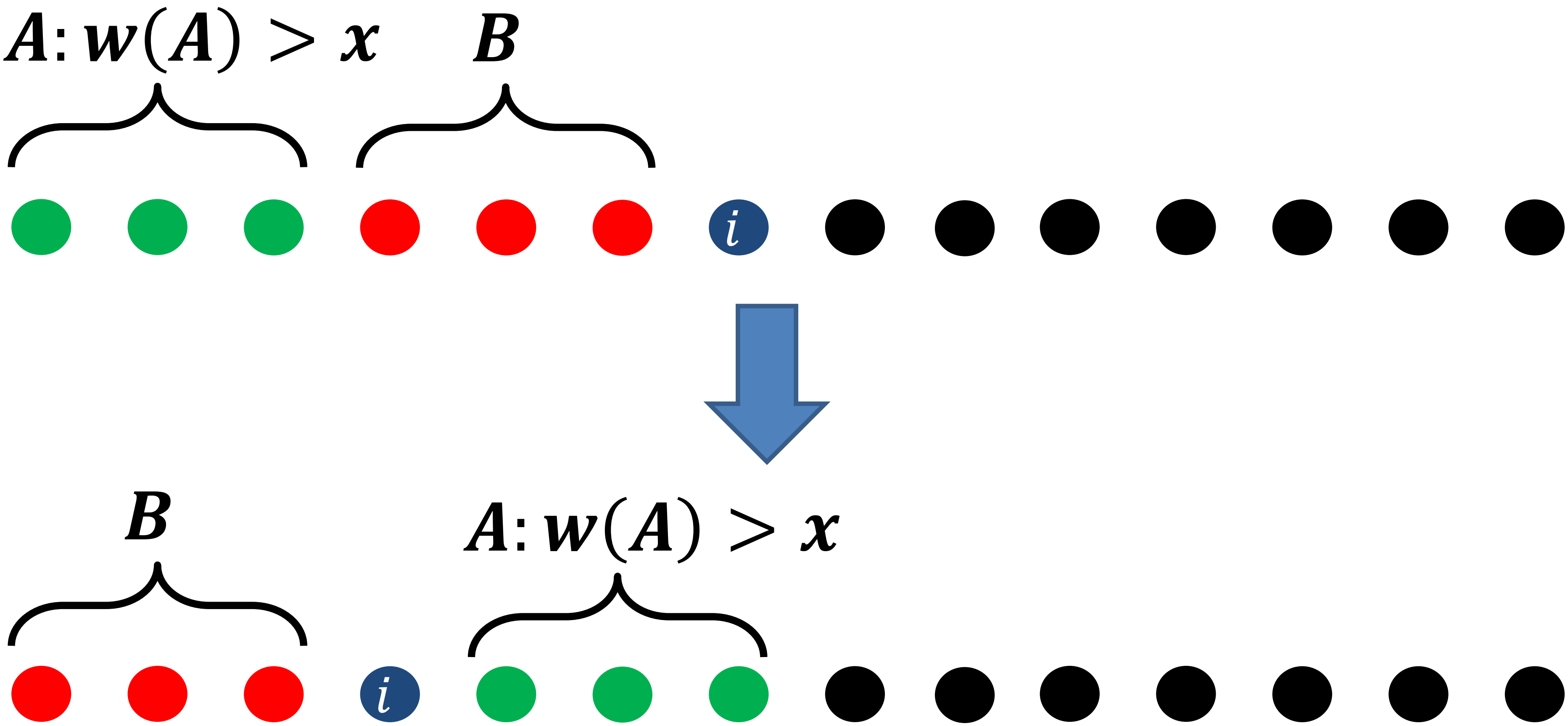
$T_i(x) \subseteq T_i(x+y)$, so $T_i(x+y) \setminus T_i(x) = T_i(x, x+y)$

Need to show that $|T_i(y)| \geq |T_i(x, x+y)|$

Maximizing $\phi_i(q)$

Need to show that $|T_i(y)| \geq |T_i(x, x+y)|$

Construct an injective mapping $\psi: T_i(x, x+y) \rightarrow T_i(y)$



Maximizing $\phi_i(q)$

Second case: $q > w_i$
Let $\Pi_i(q) = \{\sigma \in \Pi(N) \mid q - w_i \leq w(P_i(\sigma)) < q\}$, then
$\Pi_i(q) = T_i(q - w_i, q)$ and $\Pi_i(w_i) = T_i(w_i)$.
By Lemma

$$|\Pi_i(w_i)| = |T_i(w_i)| \geq |T_i(q)| - |T_i(q - w_i)| = |\Pi_i(q)|$$

which concludes the proof.

Minimizing $\phi_i(q)$

Not as easy, two strong candidate minimizers: $q = 1$ or $q = w_i + 1$.

Not always them, not clear which one to choose. For below-median players, setting $q = w_i + 1$ is worse.

Deciding whether a given quota is maximizing/minimizing is computationally intractable.

The expected behavior of $\phi_i(q)$

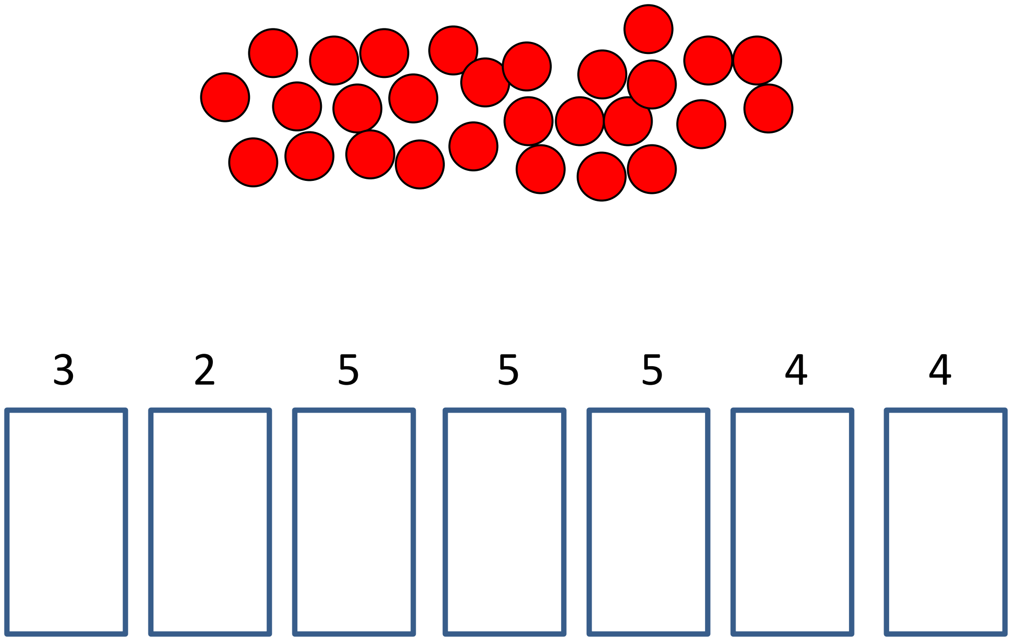
It seems that analyzing fixed weight vectors is not very effective…

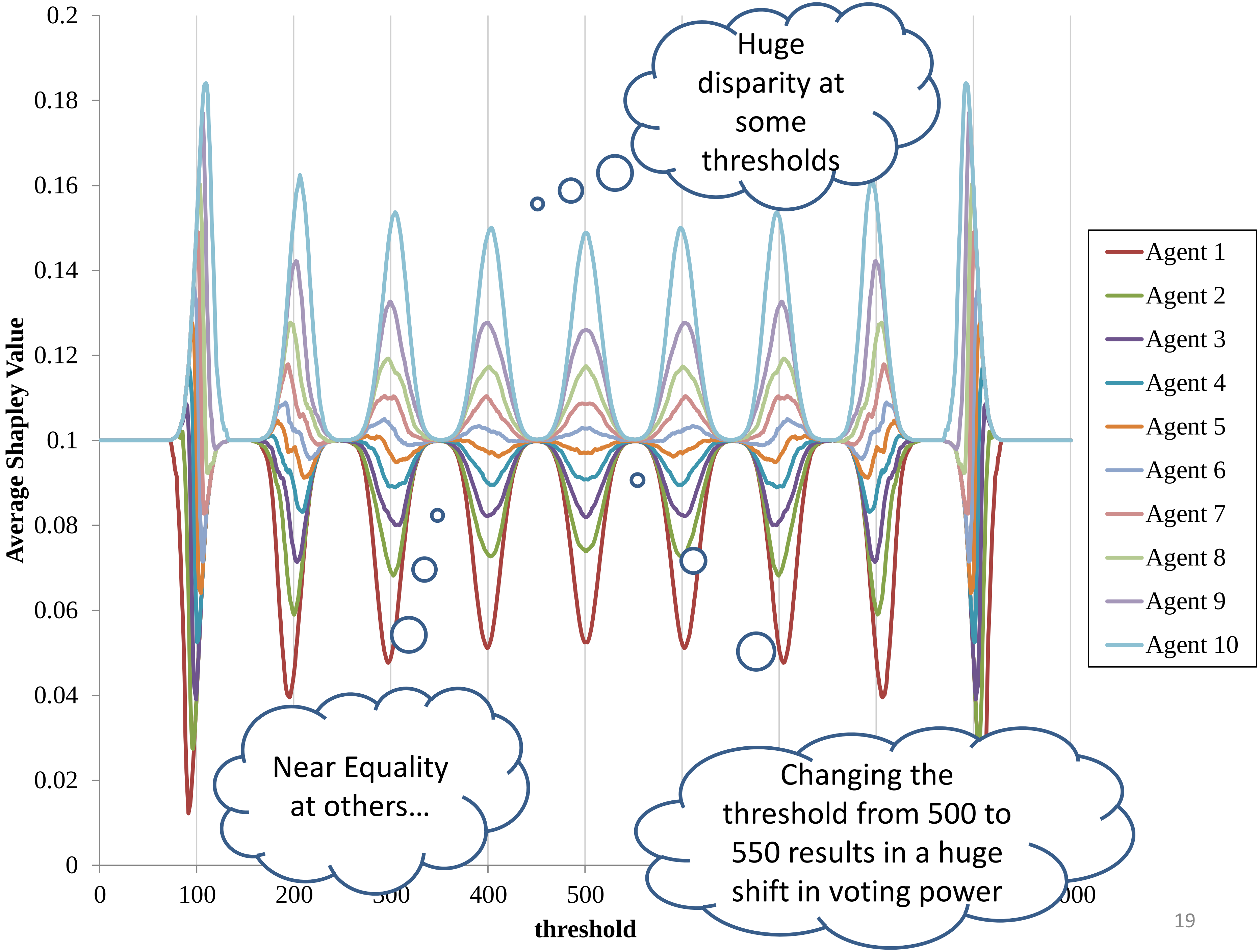
even small changes in quota can cause unpredictable behavior; worst-case guarantees are not great.

Can we say something about the **likely** Shapley value when weights are sampled from a distribution?

Balls and Bins Distributions

- We have m balls, n bins.
- A discrete probability distribution $(p_1, \ldots, p_n)$
- p_i is the probability that a ball will land in bin i





Huge disparity at some thresholds
Near Equality at others…
Changing the threshold from 500 to 550 results in a huge shift in voting power
Average Shapley Value
threshold
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
100
200
400
500
Agent 1
Agent 2
Agent 3
Agent 4
Agent 5
Agent 6
Agent 7
Agent 8
Agent 9
Agent 10

Balls and Bins: Uniform

- Suppose that the weights are generated from a uniform balls and bins process with m balls and n bins.
- **Theorem:** when the threshold is near integer multiples of m/n, there is a high disparity in voting power (w.h.p.)
- **Theorem:** when the threshold is well-away from integer multiples of m/n, all agents have nearly identical voting power (w.h.p.)

Balls and Bins: Exponential

- There are m voters. A voter votes for player i w.p. p^{i+1}
- The probability of high-index players getting votes is extremely low. Most votes go to a few candidates.
- **Theorem:** if weights are drawn from an exponential balls-and –bins distribution, then with high probability, the resulting weights are **super-increasing**
- A vector of weights $(w_1, \ldots, w_n)$ is called **super-increasing** if

$$\forall i \in N: \; w_i \geq \sum_{j<i} w_j$$

Balls and Bins: Exponential

- In order to study the Shapley value in the Balls and Bins exponential case, it suffices to understand super-increasing sequences of weights.
- Suppose that weights are $1,2,4,8,\ldots,2^{n-1}$ $(w_i = 2^{i-1})$
- Let us observe the (beautiful) graph that results.

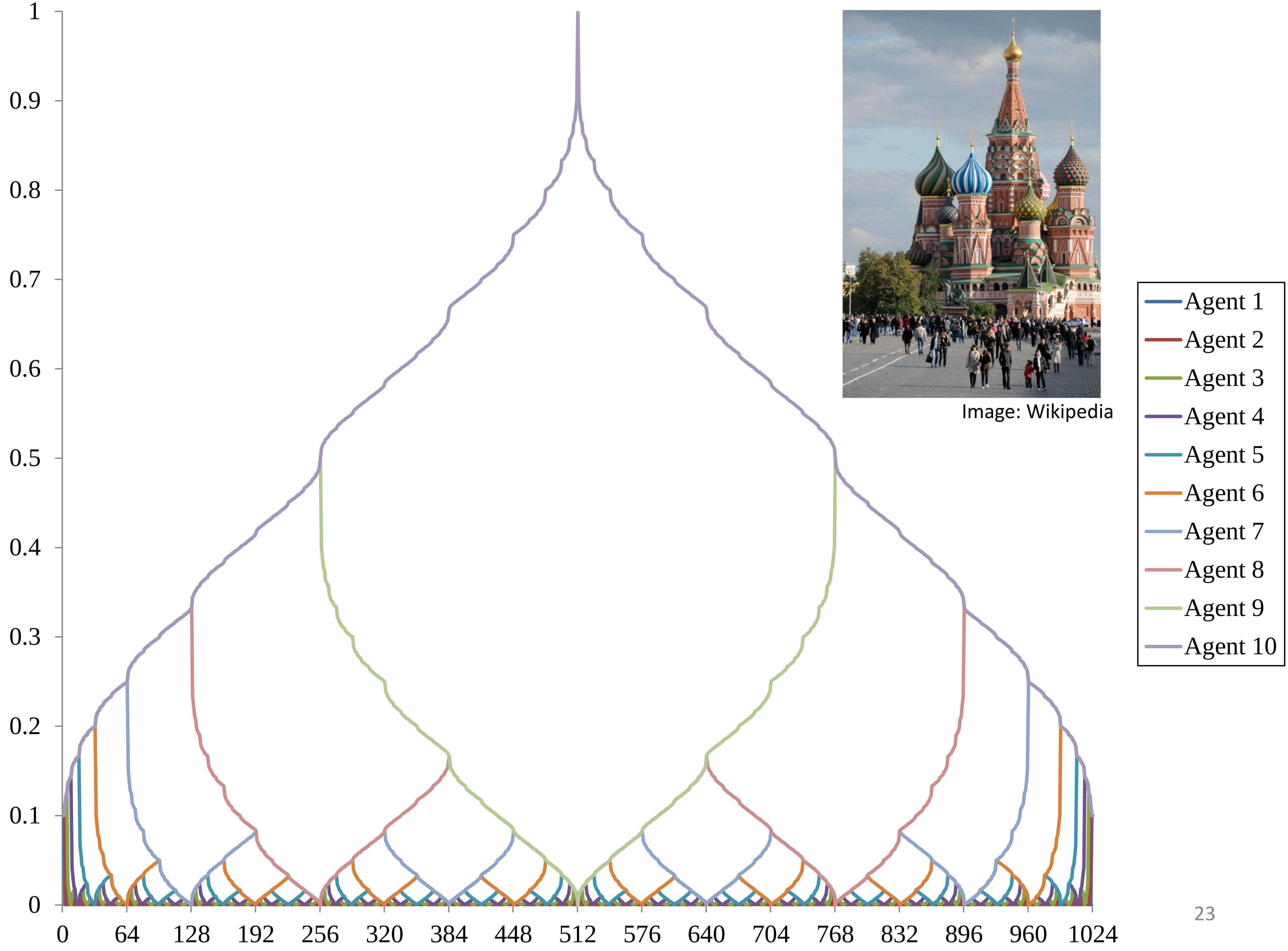


Image: Wikipedia

Super-Increasing Weights

- $\beta(S) = \sum_{i \in S} 2^{i-1}$: the binary representation of S
- $A(q)$: the minimal set $S \subseteq N$ such that $w(S) \geq q$
- **Claim:** if the weights are super-increasing, then
$$\varphi_i^{\boldsymbol{w}}(q) = \varphi_i^{\boldsymbol{\beta}}(\beta(A(q))$$
- the Shapley value when the threshold is q equals the Shapley value when the weights are powers of 2, and the threshold is $\beta\big(A(q)\big)$
- Computing the Shapley value for super-increasing weights boils down to computing it for powers of 2!
- Using this claim, we obtain a **closed-form formula** of the SV when the weights are super-increasing.

Conclusion

- Computation: generally, computing the Shapley value (and the Banzhaf value) is #P complete (counting complexity)
- It is easy when we know that the weights are not too large (pseudopolynomial time)
- It is easy to approximate them through random sampling in the case of simple games.

Further Reading

- Chalkiadakis et al. “Computational Aspects of Cooperative Game Theory”
- Zuckerman et al. “Manipulating the Quota in Weighted Voting Games” (JAIR’12)
- Zick et al. “The Shapley Value as a Function of the Quota in Weighted Voting Games” (IJCAI’11)
- Zick “On Random Quotas and Proportional Representation in Weighted Voting Games” (IJCAI’13)
- Oren et al. “On the Effects of Priors in Weighted Voting Games” (COMSOC’14)