15-859(B) Machine Learning Theory

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Lecture 4: The Perceptron Algorithm (+ continuing on Winnow)

Recap from last time

- Winnow algorithm for learning a disjunction of r out of n variables. eg $f(x)= x_3 \vee x_9 \vee x_{12}$
- h(x): predict pos iff $w_1x_1 + \ldots + w_nx_n \geq n$.
- Initialize $w_i = 1$ for all i.
  - Mistake on pos: $w_i \leftarrow 2w_i$ for all $x_i=1$.
  - Mistake on neg: $w_i \leftarrow 0$ for all $x_i=1$.
- Thm: Winnow makes at most O(r log n) mistakes.

Recap from last time

- Winnow algorithm for learning a k-of-r function: e.g., $x_3 + x_9 + x_{10} + x_{12} \geq 2$.
- h(x): predict pos iff $w_1x_1 + \ldots + w_nx_n \geq n$.
- Initialize $w_i = 1$ for all i.
  - Mistake on pos: $w_i \leftarrow w_i(1+\epsilon)$ for all $x_i=1$.
  - Mistake on neg: $w_i \leftarrow w_i/(1+\epsilon)$ for all $x_i=1$.
  - Use $\epsilon = 1/2k$.
- Thm: Winnow makes at most O(rk log n) mistakes.

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Analysis:

- Each m.o.p. adds at least k relevant chips, and each m.o.n removes at most k-1 relevant chips. At most $r(1/\epsilon)\log n$ relevant chips total.

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Analysis:

- Each m.o.p. adds at least k relevant chips, and each m.o.n removes at most k-1 relevant chips. At most $r(1/\epsilon)\log n$ relevant chips total.
- Each m.o.n. removes almost as much total weight as each m.o.p. adds. Can make (1+1/(2k)) m.o.n. for every m.o.p. $\Rightarrow$ Mistake bound $O((r/\epsilon)\log n)$.

How about learning general LTFs?

E.g., $4x_3 - 2x_9 + 5x_{10} + x_{12} \geq 3$.

Will look at two algorithms today, each with different types of guarantees:

- Winnow (same as before)
- Perceptron

Winnow for general LTFs

E.g., $4x_3 - 2x_9 + 5x_{10} + x_{12} \geq 3$.

- First, add variable $y_i = 1 - x_i$ so can assume all weights positive.

E.g., $4x_3 + 2y_9 + 5x_{10} + x_{12} \geq 5$.

- Also conceptually scale so that all weights w_i^* of target are integers (not needed but easier to think about)

Winnow for general LTFs

- Idea: suppose we made W copies of each variable, where $W = w_1^* + \ldots + w_n^*$.
- Then this is just a "w_0^* out of W" function!

E.g., $4x_3 + 2y_9 + 5x_{10} + x_{12} \geq 5$.

- So, Winnow makes $O(W^2 \log(Wn))$ mistakes.
- And here is a cool thing: this is equivalent to just initializing each w_i to W and using threshold of nW. But that is same as original Winnow!

Winnow for general LTFs

More generally, can show the following (will do the analysis on hwk2):

Suppose $\exists$ w^* s.t.:

- $w^* \cdot x \geq c$ on positive x,
- $w^* \cdot x \leq c - \gamma$ on negative x.

Then mistake bound is

- $O((L_1(w^*)/\gamma)^2 \log n)$

Multiply by $L_\infty(X)$ if examples not in {0,1}

Perceptron algorithm

An even older and simpler algorithm, with a bound of a different form.

Suppose $\exists$ w^* s.t.:

- $w^* \cdot x \geq \gamma$ on positive x,
- $w^* \cdot x \leq -\gamma$ on negative x.

Then mistake bound is

- $O((L_2(w^*)L_2(x)/\gamma)^2)$

L_2 margin of examples

Perceptron algorithm

Thm: Suppose data is consistent with some LTF $w^* \cdot x > 0$, where $||w^*||=1$ and

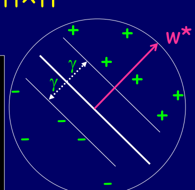
$$\gamma = \min_x |w^* \cdot x|/||x||$$

Then # mistakes $\leq 1/\gamma^2$.

Algorithm:

Initialize $w=0$. Use $w \cdot x > 0$.

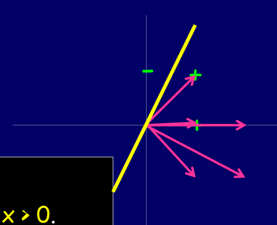
- Mistake on pos: $w \leftarrow w+x$.
- Mistake on neg: $w \leftarrow w-x$.



(Pre-scale examples to be in unit ball)

Perceptron algorithm

Example: (0,1) –
(1,1) +
(1,0) +



Algorithm:

Initialize $w=0$. Use $w \cdot x > 0$.

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Analysis

Thm: Suppose data is consistent with some LTF $w^* \cdot x > 0$, where $||w^*||=1$ and

$\gamma = \min_x |w^* \cdot x|$ (after scaling so all $||x||=1$)

Then # mistakes $\leq 1/\gamma^2$.

Proof: consider $|w \cdot w^*|$ and $||w||$

- Each mistake increases $|w \cdot w^*|$ by at least γ.
  $(w + x) \cdot w^* = w \cdot w^* + x \cdot w^* \geq w \cdot w^* + \gamma$.
- Each mistake increases $w \cdot w$ by at most 1.
  $(w + x) \cdot (w + x) = w \cdot w + 2(w \cdot x) + x \cdot x \leq w \cdot w + 1$.
- So, in M mistakes, $\gamma M \leq |w \cdot w^*| \leq ||w|| \leq M^{1/2}$.
- So, $M \leq 1/\gamma^2$.

What if no perfect separator?

In this case, a mistake could cause $|w \cdot w^*|$ to drop.

The γ-hinge-loss of $w^* = \sum_x \max[0, 1 - l(x)(x \cdot w^*)/\gamma]$

(by how much, in units of γ, would you have to move the points to all be correct by γ)

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Mistakes(perceptron) $\leq 1/\gamma^2 + 2(\gamma\text{-hinge-loss}(w^*))$

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Kernel functions

See board…