An $\tilde{O}(n^{3/14})$-Coloring Algorithm for 3-Colorable Graphs

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Abstract

We show how the results of Karger, Motwani, and Sudan [6] and Blum [3] can be combined in a natural manner to yield a polynomial-time algorithm for $\tilde{O}(n^{3/14})$-coloring any n-node 3-colorable graph. This improves on the previous best bound of $\tilde{O}(n^{1/4})$ colors [6].

Keywords: Graph coloring, approximation algorithms, analysis of algorithms.

1 Introduction

A k-coloring of an n-node graph is an assignment of one of k colors to each of the vertices in the graph so that no two adjacent vertices receive the same color. It is well known that the question: "given a graph G, can it be colored with k or fewer colors?" is NP-hard for any $k \geq 3$. In addition, recent results show that coloring 3-colorable graphs with 4 or fewer colors is NP-hard [5] and much stronger approximation hardness results are known for the general chromatic number problem [4]. Wigderson [9] describes a very simple method to color any n-node 3-colorable graph with $O(\sqrt{n})$ colors. This approximation guarantee was improved by Blum [2, 3] to $\tilde{O}(n^{3/8})$ colors and most recently by Karger, Motwani, and Sudan (KMS) [6] to $\tilde{O}(n^{1/4})$ colors, where the $\tilde{O}$ notation hides logarithmic factors. Related work has been done by Schiermeyer [8]. In this paper we show how ideas used in the results of Blum [3] and Karger Motwani and Sudan [6] can be combined to produce a polynomial-time $\tilde{O}(n^{3/14})$-coloring of any n-node 3-colorable graph.

Our basic approach is similar to Wigderson's coloring algorithm. Wigderson's algorithm considers two cases. If the there is a vertex with large degree, then its neighborhood is two-colored and set aside. The algorithm then recurses on the remaining graph using new colors.

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This process uses up few colors since each iteration colors many vertices with only two new colors. Once the maximum degree is some small Δ, the algorithm then $(\Delta + 1)$-colors the graph remaining.

The basic idea of our algorithm is simply to use stronger "handlers" for the two cases described above. When the graph has large degree, we use the techniques of [3] to color large "neighborhoods" of a vertex set. When the degree is reduced to some small Δ, we use the techniques of [6] to $\tilde{O}(\Delta^{1/3})$-color the remaining graph.

2 Preliminaries and Definitions

We will use $N(v)$ to denote the neighborhood of a vertex v, and $N(S)$ to denote the neighborhood of a set of vertices S: that is, $N(S)$ is the union of the neighborhoods of all vertices in S.

It will be convenient to talk about "making progress" towards a desired approximate coloring. Specifically, suppose that for some set of objects we can prove the following implication: if on any 3-colorable graph we can always find one of these objects in polynomial time, then there is a polynomial time algorithm that $C(n)$-colors any 3-colorable n-node graph. In that case we will say that finding one of these objects "makes progress towards a $C(n)$-coloring". For example, it is not hard to see that finding an independent set of size $n^{1-\alpha}$ makes progress towards an $O(n^{\alpha})$-coloring, because if we can find such a set of vertices, we can give those vertices one color, remove them from the graph, and recurse on the remaining graph. Similarly, if our 3-colorable graph has a vertex with degree exceeding $2n^{1-\alpha}$, then we can two-color its neighborhood to find an independent set of size $n^{1-\alpha}$ and make progress. More generally, the following useful fact is given in [3].

Lemma 1 (From Lemma 1 and Theorem 3 in [3]) *Finding any of the following objects in an n-vertex 3-colorable graph G makes progress towards an $O(n^{\alpha})$-coloring of G.*

1. *An independent or 2-colorable set of size $\Omega(n^{1-\alpha})$.*
2. *Two vertices that must be the same color under any legal 3-coloring of G.*
3. *An independent set S having a neighborhood of size $O(|S|n^{\alpha})$.*
4. *Two vertices that share $\Omega(n^{1-2\alpha})$ neighbors in common.*

Proof outline: We have already discussed why finding an independent (or 2-colorable) set of size $\Omega(n^{1-\alpha})$ is useful. It is obvious why Item (2) is useful: we can simply merge the two vertices together and continue on the graph remaining with one less vertex. The idea behind Item (3) is that the set S found can be colored with one color, and then that color can be reused in the graph in which S and its neighborhood have been removed. For instance, if we find a set S satisfying this condition, remove S and its neighborhood from the graph, and then repeat this process until $n/2$ nodes have been removed from G, then the union of the sets S found is an independent set of size $\Omega(n^{1-\alpha})$. The most interesting part of the above lemma, Item (4), follows from noticing that if two vertices u and v belong to different

color classes in a legal 3-coloring of the graph, then their shared neighborhood S belongs to a single color class, and therefore *its* neighborhood T is 2-colorable. Thus, if we find that T is not 2-colorable or S is not independent then we can apply Item (2). If T is 2-colorable and large, we can apply Item (1). Otherwise, T is small (and 2-colorable) so we can apply Item (3). A similar argument is used in Section 3. ■

This "progress lemma" will be useful when the graph has high degree. We now introduce a second part of our algorithm, which replaces Δ-coloring for low-degree graphs.

Theorem 2 ([6]) *There is a (randomized) polynomial-time algorithm that will color any 3-colorable graph having maximum degree Δ with $\tilde{O}(\Delta^{1/3})$ colors.*

Corollary 3 *Suppose we have an algorithm A that makes progress towards an $O(n^\alpha)$-coloring on any n-node 3-colorable graph having average degree at least cn^ρ for some constant $c > 0$. Then we can color any n-node 3-colorable graph with $\tilde{O}(\max(n^{\rho/3}, n^\alpha))$ colors.*

Proof: If our given graph has average degree at least cn^ρ then we simply apply algorithm A. Otherwise, the graph has at least $n/2$ vertices of degree less than $2cn^\rho$. The subgraph induced by those vertices clearly has maximum degree at most $2cn^\rho$, and therefore by Theorem 2 we can color that subgraph with $\tilde{O}(n^{\rho/3})$ colors. This coloring must contain an independent set of size $\tilde{\Omega}(n^{1-\rho/3})$ and therefore makes progress towards an $\tilde{O}(n^{\rho/3})$-coloring of the original graph. ■

The remainder of this paper will use the above corollary. We will give algorithms that make progress on graphs with large average degree, and then apply Corollary 3 to handle all graphs.

3 A simpler result

Before describing the $\tilde{O}(n^{3/14})$-coloring algorithm, we give a simpler argument that yields an $\tilde{O}(n^{2/9})$-coloring.

Lemma 4 *There is a polynomial time algorithm to make progress towards an $O(n^{\frac{2}{3}(1-\rho)})$ coloring of any n-node 3-colorable graph G with average degree exceeding $2n^\rho$, for $\rho \geq 4/7$.*

Proof: Let G be the given graph, which by definition has more than $n^{1+\rho}$ edges. Let G' be the graph remaining after removing all vertices whose degree (in the current graph) is at most n^ρ; we know that G' is nonempty because each time a vertex is removed, the number of edges decreases by at most n^ρ.

Let v be some vertex in G'. Vertex v has at least $1 + n^\rho$ neighbors in G', each with at least n^ρ neighbors in G' in addition to v. Thus, within G' there are at least $n^{2\rho}$ length-two paths from v that terminate in vertices besides v. Therefore at least $n^{2\rho-1}$ of them must terminate

at some specific other vertex u. It follows that u and v have at least $n^{2\rho-1}$ neighbors in G' in common.

We now proceed as follows, using an argument similar to that for Item (4) of Lemma 1. Let $S = N(u) \cap N(v) \cap V(G')$ be the shared neighborhood of u and v within G' which we have argued has size at least $n^{2\rho-1}$, and let T be the neighborhood of S. Notice that if u and v have different colors in a legal 3-coloring of G, then S is monochromatic and therefore T may be 2-colored. Thus, if we find that T is *not* 2-colorable, we know u and v must have the same color in any legal 3-coloring and so we can make progress by Item (2) of Lemma 1. Also, if we find that T *is* 2-colorable and $|T| \geq \frac{1}{2}n^{\frac{1}{3}(1+2\rho)}$, then we can make progress towards an $O(n^{\frac{2}{3}(1-\rho)})$-coloring by Item (1) of Lemma 1. Therefore, we may assume that $|T| < \frac{1}{2}n^{\frac{1}{3}(1+2\rho)}$.

We now show that there must be two vertices in S that share at least $\frac{1}{2}n^{\frac{1}{3}(4\rho-1)}$ neighbors. Let S' be an arbitrary subset of S of size $n^{\frac{1}{3}(1-\rho)}$ (since $\rho \geq 4/7$, it follows that $n^{\frac{1}{3}(1-\rho)} \leq n^{2\rho-1}$) and suppose for contradiction that no two vertices of S' share $\frac{1}{2}n^{\frac{1}{3}(4\rho-1)}$ neighbors. Since each vertex of S' has degree at least n^{ρ} it follows that each vertex of S' has at least: $n^{\rho} - n^{\frac{1}{3}(1-\rho)}(\frac{1}{2}n^{\frac{1}{3}(4\rho-1)}) = \frac{1}{2}n^{\rho}$ neighbors *not* in common with *any* other vertex of S'. Therefore, $|N(S')| \geq n^{\frac{1}{3}(1-\rho)} \cdot \frac{1}{2}n^{\rho} = \frac{1}{2}n^{\frac{1}{3}(1+2\rho)}$. But this contradicts our assumption from the previous paragraph that $|T| < \frac{1}{2}n^{\frac{1}{3}(1+2\rho)}$. Therefore, there must be two vertices in S' that share at least $\frac{1}{2}n^{\frac{1}{3}(4\rho-1)}$ neighbors, so we can make progress towards an $\tilde{O}(n^{\frac{2}{3}(1-\rho)})$-coloring by Item (4) of Lemma 1. ■

Corollary 5 *We can $\tilde{O}(n^{2/9})$-color any 3-colorable graph in polynomial time.*

Proof: Set $\rho = 2/3$ in the above lemma and apply Corollary 3. ■

4 The $\tilde{O}(n^{3/14})$-coloring

In this section, we improve the previous section's result to the following:

Lemma 6 *In any n-vertex 3-colorable graph with average degree exceeding $2n^{\rho}$, we can make progress towards an $\tilde{O}(n^{\alpha})$-coloring where $\alpha = \frac{3}{5}(1-\rho)$.*

Our main theorem is now a simple corollary:

Theorem 7 *There is a polynomial time algorithm to color any n-vertex 3-colorable graph with $\tilde{O}(n^{3/14})$ colors.*

Proof: Apply the above lemma to Corollary 3 with $\rho = \frac{9}{14}$. ■

The remainder of this section proves the lemma. We will need the following definitions from [3]. Let $d(v)$ denote the degree of vertex v. If S and T are sets of vertices, let $d_T(v) = |N(v) \cap T|$, and let $D_T(S) = \sum_{v \in S} d_T(v)$. Write $D_V(S)$, which is just the sum of degrees of vertices in S, as $D(S)$. Define (as before) $N(S) = \bigcup_{v \in S} N(v)$.

Let $\delta = \delta(n) = \frac{1}{5 \log n}$. We define $I_j = \{v \in V \mid d(v) \in [(1+\delta)^j, (1+\delta)^{j+1})\}$ for $j = 0, 1, 2, \ldots$. That is, we divide the set of vertices of degree at least 1 into bins I_j so that in each bin, the ratio of the degrees of any two vertices is less than $(1+\delta)$. For a set of vertices S, let $N_i(S) = \{v \in N(S) \mid d_S(v) \in [(1+\delta)^i, (1+\delta)^{i+1})\}$ for $i = 0, 1, 2, \ldots$. In other words, $N_i(S)$ is the subset of vertices in $N(S)$ that are hit by at least $(1+\delta)^i$ and less that $(1+\delta)^{i+1}$ edges from S.

We now present slightly simplified versions of three theorems from [3]. Let R (red) be a color class in a correct 3-coloring of G such that $D(R) \geq \frac{1}{3} D(V)$, or equivalently, $D(R) \geq \frac{1}{2} D(V - R)$. Notice that $D_R(V - R) = D(R)$ so $D_R(V - R) \geq \frac{1}{2} D(V - R)$; or in words: our definition of "red" is such that on average, a non-red vertex has at least half its neighbors colored red.

Theorem 8 (Theorem 7 of [3]) *Given a 3-colorable n-vertex graph $G = (V, E)$, there is some $v \in R$ and some $j \in \{0, 1, \ldots, \log_{1+\delta} n\}$ such that the set $S = N(v) \cap I_j$ satisfies the following:*

1. $|S| \geq \delta^2 D(R) / (|R| \log_{1+\delta} n)$,
2. $D_R(S) \geq \frac{1}{2}(1 - 3\delta) D(S)$.

In other words, for some $v \in R$, the set $S = N(v) \cap I_j$ is reasonably large and almost half of the edges incident to it have their other endpoint in R.

The next theorem considers the neighbors of S and states that for some i, the set $T = N_i(S)$ is nearly half red in the correct 3-coloring. This is useful because given a set of vertices T which contains an independent set of size at least $\frac{1}{2}(1 - \frac{1}{\log n})|T|$, one can *find* an independent set of size at least $|T| / \log |T|$ using an algorithm developed independently by Bar-Yehuda and Even [1] and by Monien and Speckenmeyer [7]. Thus, if T is sufficiently large we can make progress towards our desired coloring.

Theorem 9 (Theorem 8 of [3]) *Given an n-vertex 3-colorable graph $G = (V, E)$, and $\lambda \in [0, 1]$:*

For any set S such that $D_R(S) \geq \lambda D(S)$, there must exist some $i < \log_{1+\delta} n$ such that the set $T = N_i(S)$ satisfies the following:

1. $D_T(S) \geq \delta D_R(S) / \log_{1+\delta} n$,
2. $|T \cap R| / |T| \geq (1 - 2\delta)\lambda$.

The final theorem from [3] that we will use implies that if the set T in the above theorem is much smaller than what one would expect given the number of edges between S and T, then one can make progress towards the desired coloring.

Theorem 10 (Theorem 13 of [3]) *Given sets of vertices S and T in an n-vertex 3-colorable graph G, such that*

1. *S is 2-colored under some legal 3-coloring of G,*
2. *$D_T(S) = \Omega(|S| n^{1-2\alpha} \log^2 n)$, and*
3. *$[D_T(S)]^3 = \Omega\left(\left[|S| + \max_{v \in S} d_T(v)\right] \times \left[|S||T|^2 n^{1-2\alpha} \log n + |T||S|^2 n^{2-4\alpha}\right]\right)$,*

we can make progress towards an $O(n^\alpha)$-coloring of G.

We now combine these theorems to make progress towards $\tilde{O}(n^{\frac{3}{5}(1-\rho)})$-coloring any n-vertex 3-colorable graph with average degree exceeding $2n^\rho$, thus proving our lemma. Subsequently, we fix $\alpha = \frac{3}{5}(1-\rho)$.

Proof of Lemma 6: Let G be the given graph. As in the proof of Lemma 4, let G' be the graph remaining after removing all vertices whose degree (in the current graph) is at most n^ρ. For the remainder of this proof, unless explicitly mentioned otherwise, we will restrict our attention to the subgraph G'.

Theorems 8 and 9 imply that for some vertex v and some indices $i, j \in \{0, \ldots, \log_{1+\delta} n\}$, the sets $S = N(v) \cap I_j$ and $T = N_i(S)$ satisfy the following properties.

$$\begin{aligned} |T \cap R|/|T| &\geq (1-2\delta)(1-3\delta)\frac{1}{2} \\ &\geq \frac{1}{2}(1 - \frac{1}{\log n}) \end{aligned}$$

and

$$\begin{aligned} D_T(S) &= \Omega(\delta D(S)/\log_{1+\delta} n) \\ &= \Omega(\delta n^\rho |S|/\log_{1+\delta} n). \text{ (because of the minimum degree)} \end{aligned} \quad (1)$$

(In the algorithm, we simply try all choices of v, i, j.)

Since T has an independent set of size at least $\frac{1}{2}|T|(1 - \frac{1}{\log n})$, the vertex-cover approximation algorithm developed independently by Bar-Yehuda and Even [1] and Monien and Speckenmeyer [7] (or see Lemma 11 in [3]) will find in T an independent set of size $|T|/\log|T|$. Therefore, we have made progress towards an $\tilde{O}(n^\alpha)$-coloring of G if $|T| = \Omega(n^{1-\alpha})$.

If on the other hand $|T| = o(n^{1-\alpha})$, we will make progress using Theorem 10, with $\alpha = \frac{3}{5}(1-\rho)$, as follows. First, S satisfies property (1) of the theorem since it is part of the neighborhood of some node. Second, property (2) is satisfied easily since it is a weaker statement than Equation 1 above (since $\rho < 1$, we know that $1 - 2\alpha = 1 - 2\frac{3}{5}(1-\rho) < \rho$).

The key constraint is property (3), which is what results in the specific exponent of $\alpha = \frac{3}{5}(1-\rho)$. Define $\hat{d}$ to be the average degree of vertices in S. In fact, by definition of S, all vertices in S have nearly the same degree, so $\max_{v \in S} d_T(v) \leq \max_{v \in S} d(v) = O(\hat{d})$. By Equation 1 we have that $D_T(S) = \tilde{\Omega}(D(S)) = \tilde{\Omega}(\hat{d}|S|)$. Thus, we can reduce property (3) to showing that:

$$[\hat{d}|S|]^3 = \tilde{\Omega}\left((|S| + \hat{d})(|S||T|^2 n^{1-2\alpha} + |S|^2|T|n^{2-4\alpha}\right).$$

Clearly the worst case for this is when T is largest, but we are assuming that $|T| = o(n^{1-\alpha})$. Thus it suffices to show that

$$[\hat{d}|S|]^3 = \tilde{\Omega}\left((|S| + \hat{d})(|S|n^{3-4\alpha} + |S|^2 n^{3-5\alpha}\right).$$

This in turn is clearly worst-case when S and $\hat{d}$ are minimal. But we have argued that both have size $\tilde{\Omega}(n^\rho)$ (For $|S|$, this fact follows from item 1 of Theorem 8). Thus we need only verify that

$$n^{6\rho} = \tilde{\Omega}(n^\rho(n^{\rho+3-4\alpha} + n^{2\rho+3-5\alpha})),$$

or in other words,

$$n^{4\rho} = \tilde{\Omega}(n^{3-4\alpha} + n^{\rho+3-5\alpha}).$$

From our choice of $\alpha = \frac{3}{5}(1-\rho)$, we find that $4\rho > 3 - 4\alpha$ while $4\rho = \rho + 3 - 5\alpha$, so this condition is satisfied. One final point: our goal is to make progress towards an $\tilde{O}(n^\alpha)$ coloring of G, not of G'. However, sets S and T satisfy all conditions of Theorem 10 with respect to G as well as G' (the only one to check is that S is 2-colored under a legal 3-coloring, which is true since S is contained within the neighborhood of some node). Therefore we have proven the lemma. ■

References

[1] R. Bar-Yehuda and S. Even. A local-ratio theorem for approximating the weighted vertex cover problem. *Annals of Discrete Mathematics*, 25:27–46, 1985.

[2] A. Blum. An $\tilde{O}(n^{0.4})$-approximation algorithm for 3-coloring (and improved approximation algorithms for k-coloring). In *Proceedings of the Twenty-First Annual ACM Symposium on Theory of Computing*, pages 535–542, Seattle, May 1989.

[3] A. Blum. New approximation algorithms for graph coloring. *JACM*, 31(3):470–516, 1994.

[4] M. Bellare and M. Sudan. Improved Non-approximability Results. In *Proceedings of the Twenty-Sixth ACM Symposium on Theory of Computing*, pp. 184–193, 1994.

[5] S. Khanna, N. Linial, and S. Safra. On the Hardness of Approximating the Chromatic Number. In *Proceedings 2nd Israeli Symposium on Theory and Computing Systems*, pp. 250–260, 1992.

[6] D. Karger, R. Motwani, and M. Sudan. Approximate graph coloring by semidefinite programming. In *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, November 1994.

[7] B. Monien and E. Speckenmeyer. Ramsey numbers and an approximation algorithm for the vertex cover problem. *Acta Informatica*, 22:115–123, 1985.

[8] I. Schiermeyer. An approximation algorithm for 3-colorability. In *Proceedings of the 21st International Workshop on Graph Theoretic Concepts in Computer Science*, Springer-Verlag Lecture Notes # 1017, pp. 146–151, June 1995.

[9] A. Wigderson. Improving the performance guarantee for approximate graph coloring. *JACM*, 30(4):729–735, 1983.