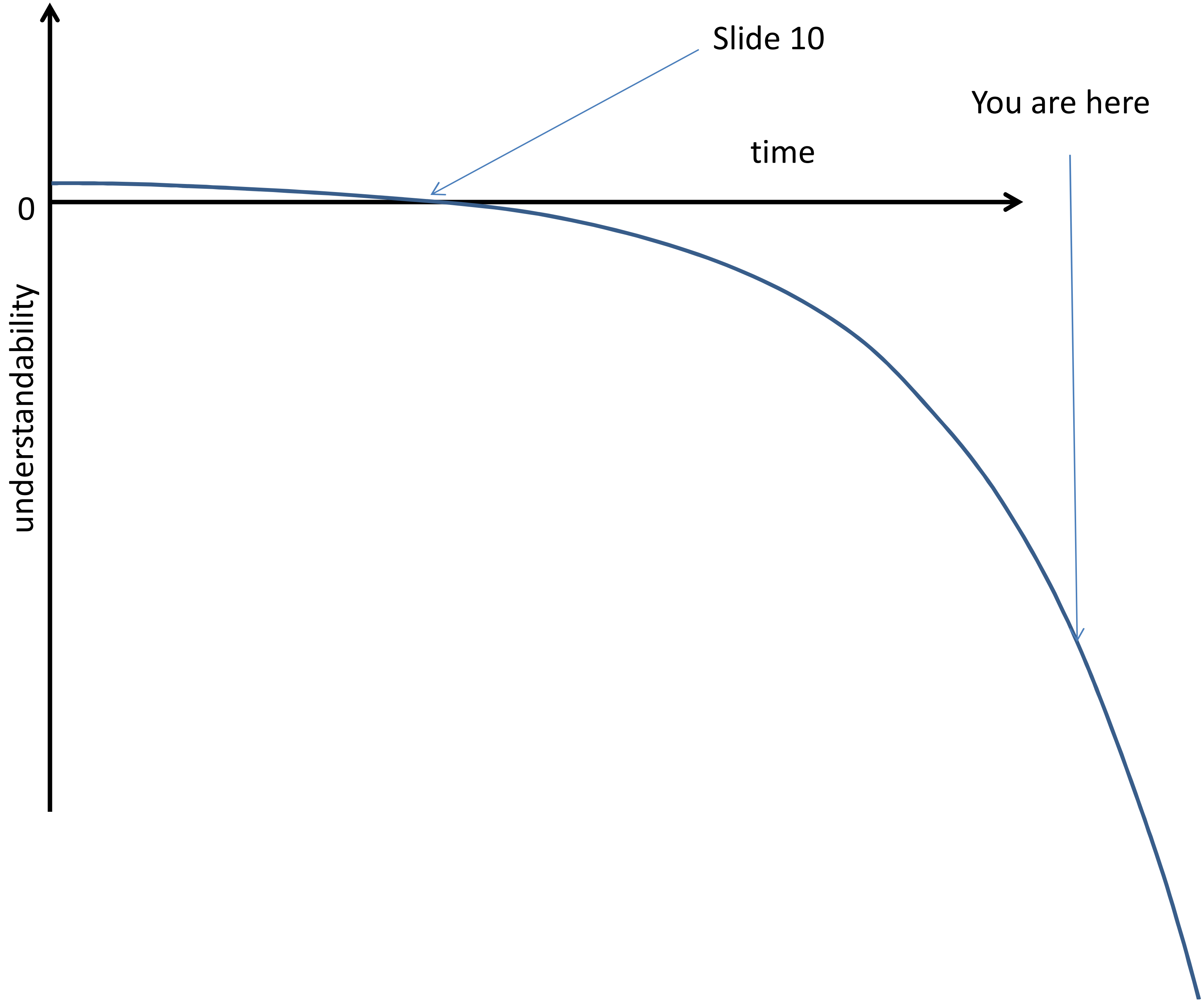
Reinforcement Learning

Spring 2019

Defining MDPs, Planning


Slide 10
You are here
time
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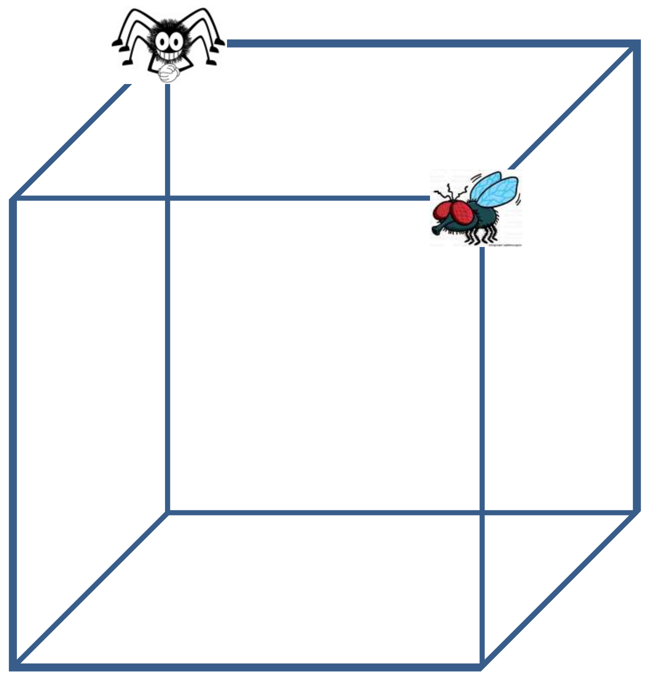
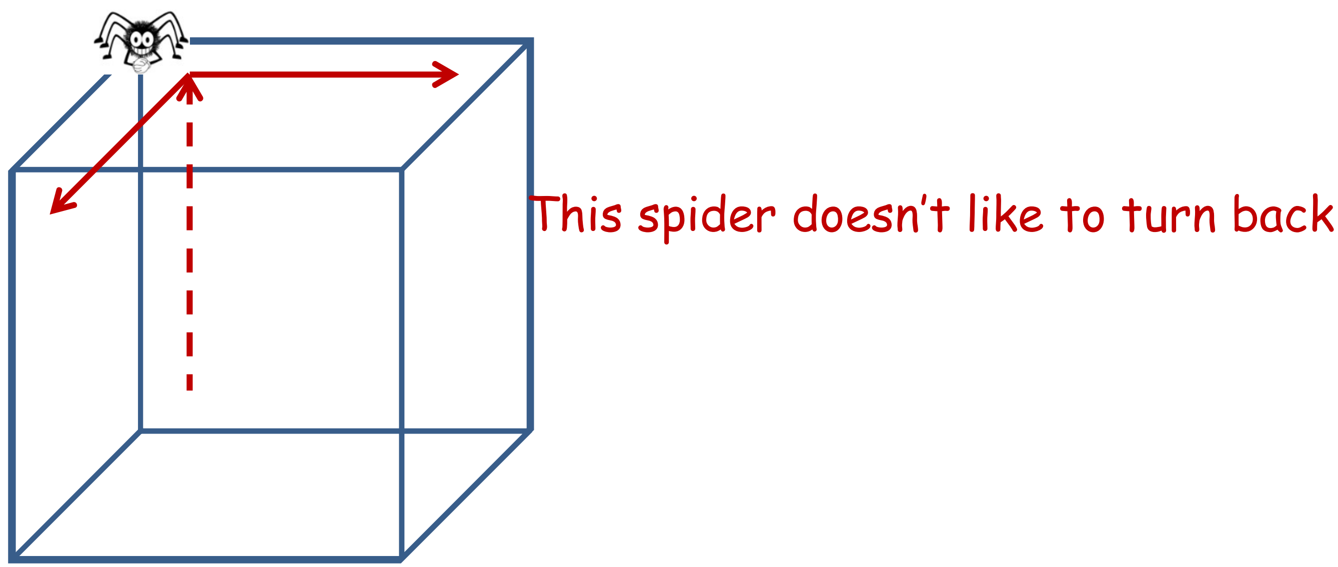
understandability

Markov Process



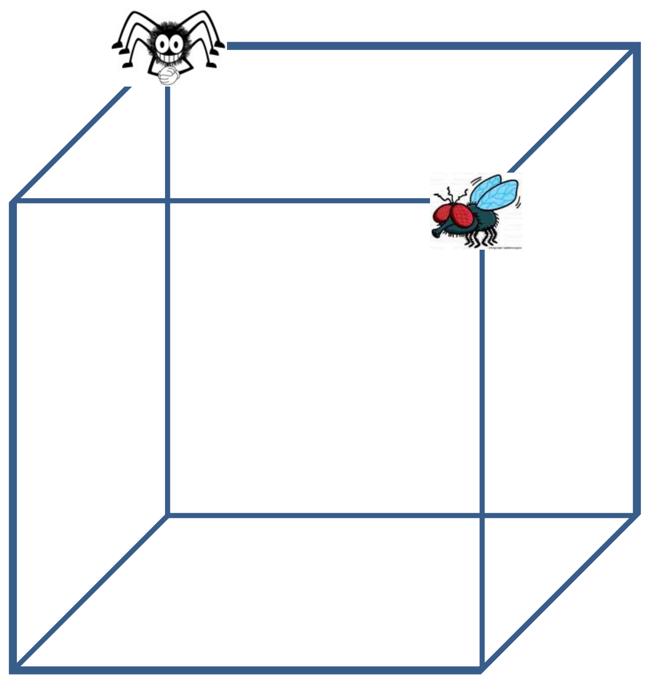
- Where you will go depends only on where you are

Markov Process: Information state



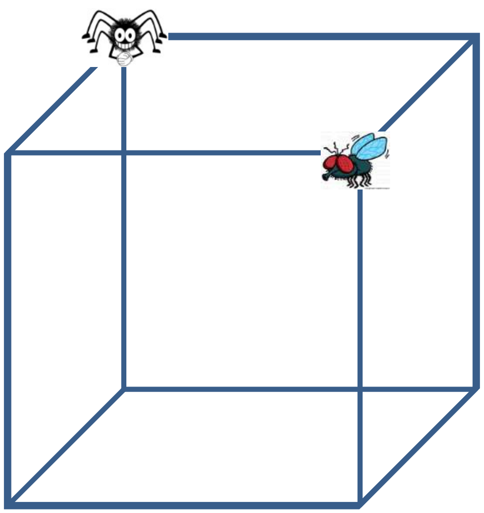
- The *information* state of a Markov process may be different from its physical state

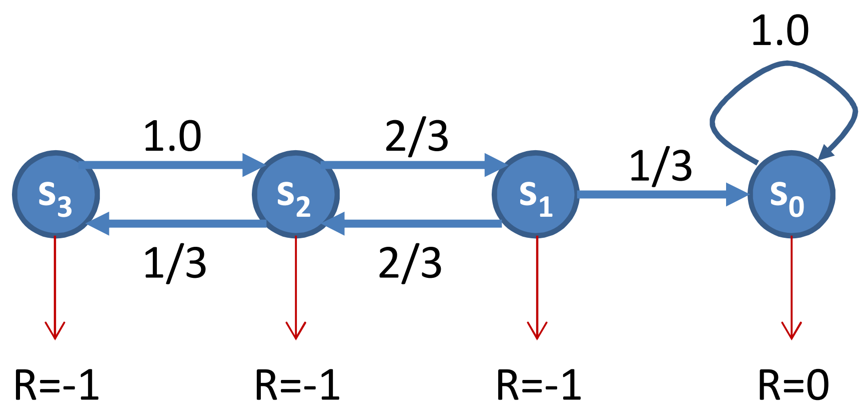
Markov Reward Process



- Random wandering through states will occasionally win you a reward

The Fly Markov Reward Process





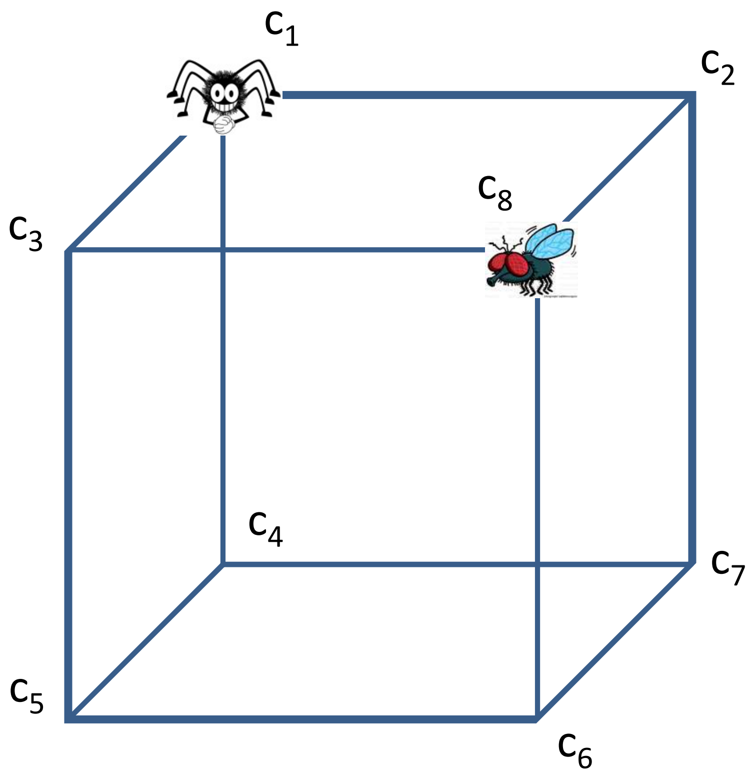
- There are, in fact, only four states, not eight
  - Manhattan distance between fly and spider = 0 (s_0)
  - Distance between fly and spider = 1 (s_1)
  - Distance between fly and spider = 2 (s_2)
  - Distance between fly and spider = 3 (s_3)
- Can, in fact, redefine the MRP entirely in terms of these 4 states

The discounted return

$$G_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k}$$

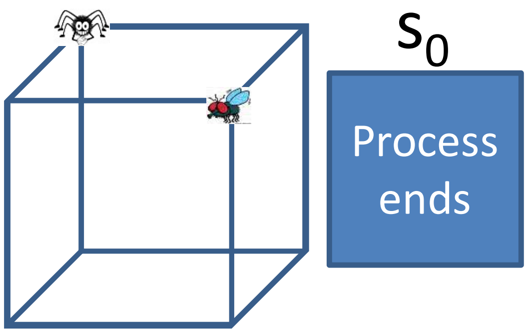
- Total *future* reward all the way to the end

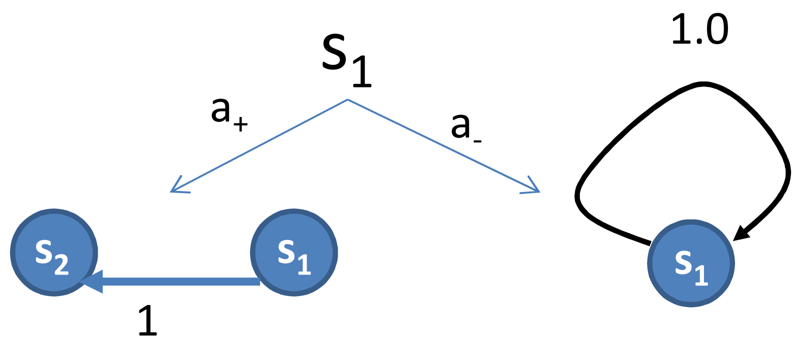
Markov Decision Process

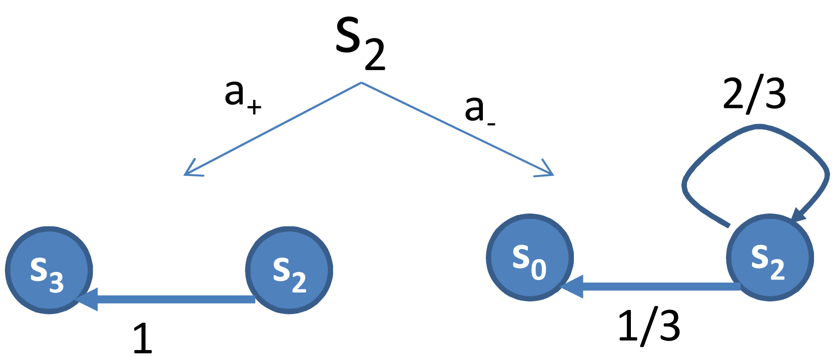


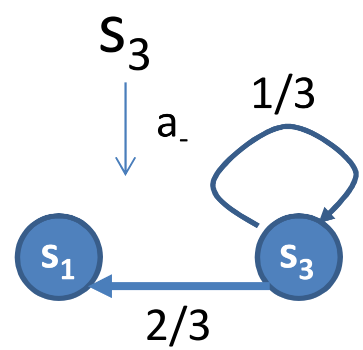
- Markov Reward Process with following change:
  - Agent has real agency
  - Agent's actions modify environment's behavior

The Fly Markov Decision Process

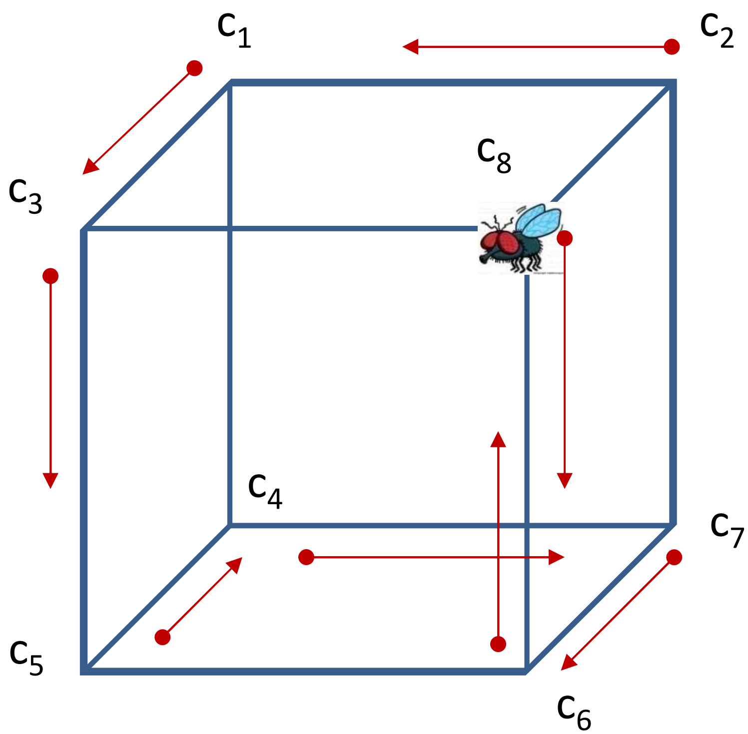








Policy



- The *policy* is the agent's choice of action in each state
  – May be stochastic

The Bellman Expectation Equations

- The Bellman expectation equation for state value function

$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(R_s^a + \gamma \sum_{s'} P_{s',s}^a \, v_\pi(s') \right)$$

- The Bellman expectation equation for action value function

$$q_\pi(s,a) = R_s^a + \gamma \sum_{s'} P_{s',s}^a \sum_{a \in \mathcal{A}} \pi(a|s') q_\pi(s',a)$$

Optimal Policies

- The optimal policy is the policy that will maximize the expected total discounted reward at every state: $E[G_t|S_t = s]$

$$= E\left[\sum_{k=0}^{\infty} \gamma^k r_{t+k} \,|S_t = s\right]$$

- **Optimal Policy Theorem**: For any MDP there exist optimal policies π_* that is better than or equal to every other policy:

$$\pi_* \geq \pi \quad \forall \pi$$

$$v_*(s) \geq v_\pi(s) \;\; \forall s$$

$$q_*(s,a) \geq q_\pi(s,a) \;\; \forall s,a$$

The optimal value function

$$\pi_*(a|s) = \begin{cases} 1 \;\; for \quad \underset{a'}{\operatorname{argmax}}\, q_*(s,a') \\ \quad\quad 0 \quad\quad otherwise \end{cases}$$

$$v_*(s) = \max_a q_*(s,a)$$

Bellman *Optimality* Equations

- Optimal value function equation

$$v_*(s) = \max_a \; R_s^a + \gamma \sum_{s\prime} P_{s\prime,s}^a \, v_*(s')$$

- Optimal action value equation

$$q_*(s,a) = R_s^a + \gamma \sum_{s\prime} P_{s\prime,s}^a \; \max_{a\prime} q_*(s',a')$$

Planning with an MDP

- Problem:
  - **Given:** an MDP $\langle \mathcal{S}, \mathcal{P}, \mathcal{A}, \mathcal{R}, \gamma \rangle$
  - **Find:** Optimal policy π_*

- Can either
  - **Value-based Solution:** Find optimal value (or action value) function, and derive policy from it OR
  - **Policy-based Solution:** Find optimal policy directly

Value-based Planning

- "Value"-based solution

- **Breakdown:**
  - **Prediction:** Given *any* policy π find value function $v_\pi(s)$
  - **Control:** Find the optimal policy

Prediction DP

- Iterate

$$v_\pi^{(k+1)}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(R_s^a + \gamma \sum_{s'} P_{s',s}^a \, v_\pi^{(k)}(s') \right)$$

Policy Iteration

- Start with any policy $\pi^{(0)}$
- Iterate ($k = 0$ … convergence):
  - Use prediction DP to find the value function $v_{\pi^{(k)}}(s)$
  - Find the greedy policy

$$\pi^{(k+1)}(\mathrm{s}) = greedy\left(v_{\pi^{(k)}}(s)\right)$$

Value iteration

$$v_*^{(k)}(s) = \max_a \; R_s^a + \gamma \sum_{s'} P_{s',s}^a \, v_*^{(k-1)}(s')$$

- Each state simply inherits the cost of its best neighbour state
  - Cost of neighbour is the value of the neighbour plus cost of getting there

Problem so far

- *Given all details of the MDP*
  - Compute optimal value function
  - Compute optimal action value function
  - *Compute optimal policy*
- This is the problem of *planning*

- **Problem:** In real life, nobody gives you the MDP
  - How do we plan???

Model-Free Methods

- AKA model-free **reinforcement learning**

- How do you find the value of a policy, without knowing the underlying MDP?
  - Model-free *prediction*
- How do you find the optimal policy, without knowing the underlying MDP?
  - Model-free *control*

Model-Free Methods

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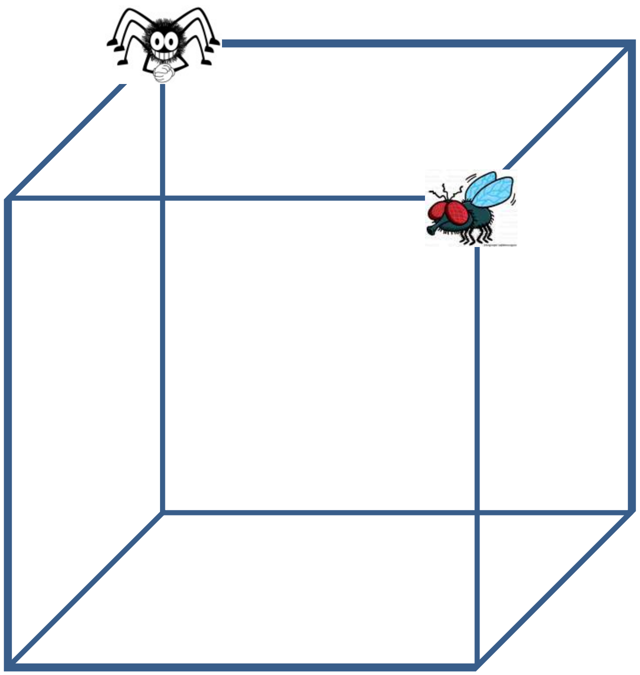
- How do you find the value of a policy, without knowing the underlying MDP?
  - Model-free *prediction*
- How do you find the optimal policy, without knowing the underlying MDP?
  - Model-free *control*

- ***Assumption:*** We can identify the states, know the *actions,* and measure rewards, but have no knowledge of the system dynamics
  - The key knowledge required to "solve" for the best policy
  - A reasonable assumption in many discrete-state scenarios
  - Can be generalized to other scenarios with infinite or unknowable state

Model-Free Assumption



- Can see the fly
- Know the distance to the fly
- Know possible actions (get closer/farther)
- But have no idea of how the fly will respond
  - Will it move, and if so, to what corner

Model-Free Methods

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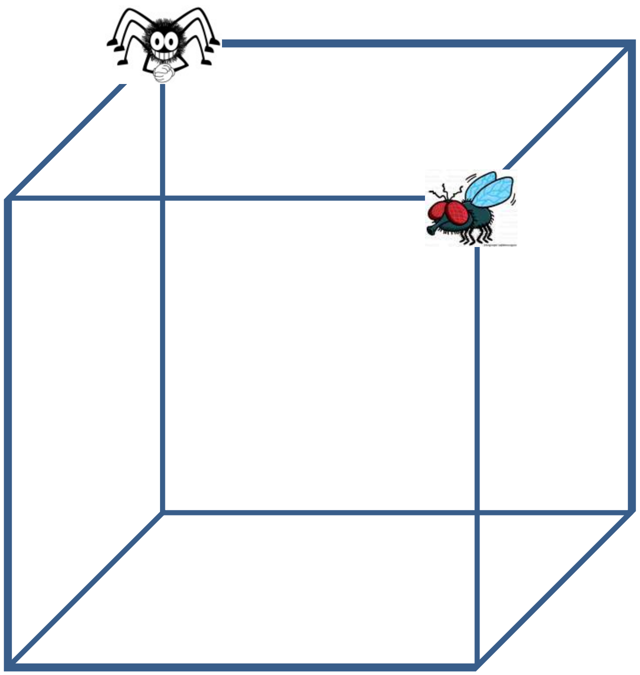
- How do you find the value of a policy, without knowing the underlying MDP?
  - Model-free *prediction*

- How do you find the optimal policy, without knowing the underlying MDP?
  - Model-free *control*

Model-Free Assumption



- Can see the fly and distance to the fly
- But have no idea of how the fly will respond to actions
  - Will it move, and if so, to what corner
- *But will always try to reduce distance to fly (have a known, fixed, policy)*
- ***What is the value of being a distance D from the fly?***

Methods

- *Monte-Carlo* Learning

- *Temporal-Difference* Learning
  - *TD(1)*
  - *TD(K)*
  - $TD(\lambda)$

Monte-Carlo learning to learn the value of a policy π

- Just "let the system run" while following the policy π and learn the value of different states
- Procedure: Record several *episodes* of the following
  - Take actions according to policy π
  - Note states visited and rewards obtained as a result
  - Record entire sequence:
  - $S_1, A_1, R_2, S_2, A_2, R_3, \dots, S_T$
  - **Assumption:** Each "episode" ends at some time
- Estimate value functions based on observations by counting

Monte-Carlo Value Estimation

- Objective: Estimate value function $v_\pi(s)$ for every state s, given recordings of the kind:

$$S_1, A_1, R_2, S_2, A_2, R_3, \dots, S_T$$

- Recall, the value function is the expected return:

$$v_\pi(s) = E[G_t | S_t = s]$$
$$= E[R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-t-1} R_T | S_t = s]$$

- To estimate this, we replace the *statistical* expectation $E[G_t | S_t = s]$ by the *empirical* average $avg[G_t | S_t = s]$

A bit of notation

- We actually record *many* episodes
  - $episode(1) = S_{11}, A_{11}, R_{12}, S_{12}, A_{12}, R_{13}, \dots, S_{1T_1}$
  - $episode(2) = S_{21}, A_{21}, R_{22}, S_{22}, A_{22}, R_{23}, \dots, S_{2T_2}$
  - …
  - Different episodes may be different lengths

Counting Returns

- For each episode, we count the returns at all times:
  - $S_{11}, A_{11}, R_{12}, S_{12}, A_{12}, R_{13}, S_{13}, A_{13}, R_{14}, \ldots, S_{1T_1}$

$G_{1,1}$

- Return at time t
  - $G_{1,1} = R_{12} + \gamma R_{13} + \cdots + \gamma^{T_1-2} R_{1T_1}$

Counting Returns

- For each episode, we count the returns at all times:
  - $S_{11}, A_{11}, R_{12}, S_{12}, A_{12}, R_{13}, S_{13}, A_{13}, R_{14}, \ldots, S_{1T_1}$

    $G_{1,2}$

- Return at time t
  - $G_{1,1} = R_{12} + \gamma R_{13} + \cdots + \gamma^{T_1-2} R_{1T_1}$
  - $G_{1,2} = R_{13} + \gamma R_{14} + \cdots + \gamma^{T_1-3} R_{1T_1}$

Counting Returns

- For each episode, we count the returns at all times:
  - $S_{11}, A_{11}, R_{12}, S_{12}, A_{12}, R_{13}, S_{13}, A_{13}, R_{14}, \ldots, S_{1T_1}$
- Return at time t
  - $G_{1,1} = R_{12} + \gamma R_{13} + \cdots + \gamma^{T_1-2} R_{1T_1}$
  - $G_{1,2} = R_{13} + \gamma R_{14} + \cdots + \gamma^{T_1-3} R_{1T_1}$
  - $\ldots$
  - $G_{1,t} = R_{1,t+1} + \gamma R_{1,t+2} + \cdots + \gamma^{T_1-t-1} R_{1T_1}$

Estimating the Value of a State

- To estimate the value of any state, identify the instances of that state in the episodes:
  - $S_{11}, A_{11}, R_{12}, S_{12}, A_{12}, R_{13}, S_{13}, A_{13}, R_{14}, \ldots, S_{1T_1}$

    $\boldsymbol{s_a}$ $\qquad s_b \qquad$ $\boldsymbol{s_a}$ $\ldots$

- Compute the average return from those instances

$$v_\pi(s_a) = avg\left(G_{1,1}, G_{1,3}, \ldots\right)$$

Estimating the Value of a State

- For every state s
  - Initialize: Count $N(s) = 0$, Total return $v_\pi(s) = 0$
  - For every episode e
    - For every time $t = 1 \dots T_e$
      - Compute G_t
      - If $(S_t == s)$
        - » $N(s) = N(s) + 1$
        - » $v_\pi(s) = v_\pi(s) + G_t$
  - $v_\pi(s) = v_\pi(s)/N(s)$

- Can be done more efficiently..

Online Version

- For all s Initialize: Count $N(s) = 0$, Total return $totv_\pi(s) = 0$
- For every episode e
  - For every time $t = 1 \ldots T_e$
    - Compute G_t
    - $N(S_t) = N(S_t) + 1$
    - tot$v_\pi(S_t) = totv_\pi(S_t) + G_t$
  - For every state s : $v_\pi(s) = totv_\pi(s)/N(s)$

- Updating values at the end of each episode
- Can be done more efficiently..

Monte Carlo estimation

- Learning from experience explicitly
- After a sufficiently large number of episodes, in which all states have been visited a sufficiently large number of times, we will obtain good estimates of the value functions of all states
- Easily extended to evaluating *action value functions*

Estimating the Action Value function

- To estimate the value of any state-action pair, identify the instances of that state-action pair in the episodes:
  - $S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \dots, S_T$

    $\boldsymbol{s_a}$ $\boldsymbol{a_x}$ s_b a_y $\boldsymbol{s_a}$ a_y ...

- Compute the average return from those instances

$$q_\pi(s_a, a_x) = avg(G_{1,1}, \dots)$$

Online Version

- For all s, a Initialize: Count $N(s,a) = 0$, Total value $totq_\pi(s,a) = 0$
- For every episode e
  - For every time $t = 1 \dots T_e$
    - Compute G_t
    - $N(S_t, A_t) = N(S_t, A_t) + 1$
    - tot$q_\pi(S_t, A_t) = totq_\pi(S_t, A_t) + G_t$
  - For every $s, a : q(s,a) = totq_\pi(s,a)/N(s,a)$
- Updating values at the end of each episode

Monte Carlo: Good and Bad

- Good:
  - Will eventually get to the right answer
  - *Unbiased* estimate

- Bad:
  - Cannot update anything until the end of an episode
    - Which may last for ever
  - High variance! Each return adds many random values
  - Slow to converge

Online methods for estimating the value of a policy: Temporal Difference Leaning (TD)

- Idea: Update your value estimates after every observation

$$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \dots, S_T$$

Update for S_1

Update for S_2

Update for S_3

  - Do not actually wait until the end of the episode

Incremental Update of Averages

- Given a sequence $x_1, x_2, x_3, \ldots$ a running estimate of their average can be computed as

$$\mu_k = \frac{1}{k}\sum_{i=1}^{k} x_i$$

- This can be rewritten as:

$$\mu_k = \frac{(k-1)\mu_{k-1} + x_k}{k}$$

- And further refined to

$$\mu_k = \mu_{k-1} + \frac{1}{k}(x_k - \mu_{k-1})$$

Incremental Update of Averages

- Given a sequence $x_1, x_2, x_3, \dots$ a running estimate of their average can be computed as

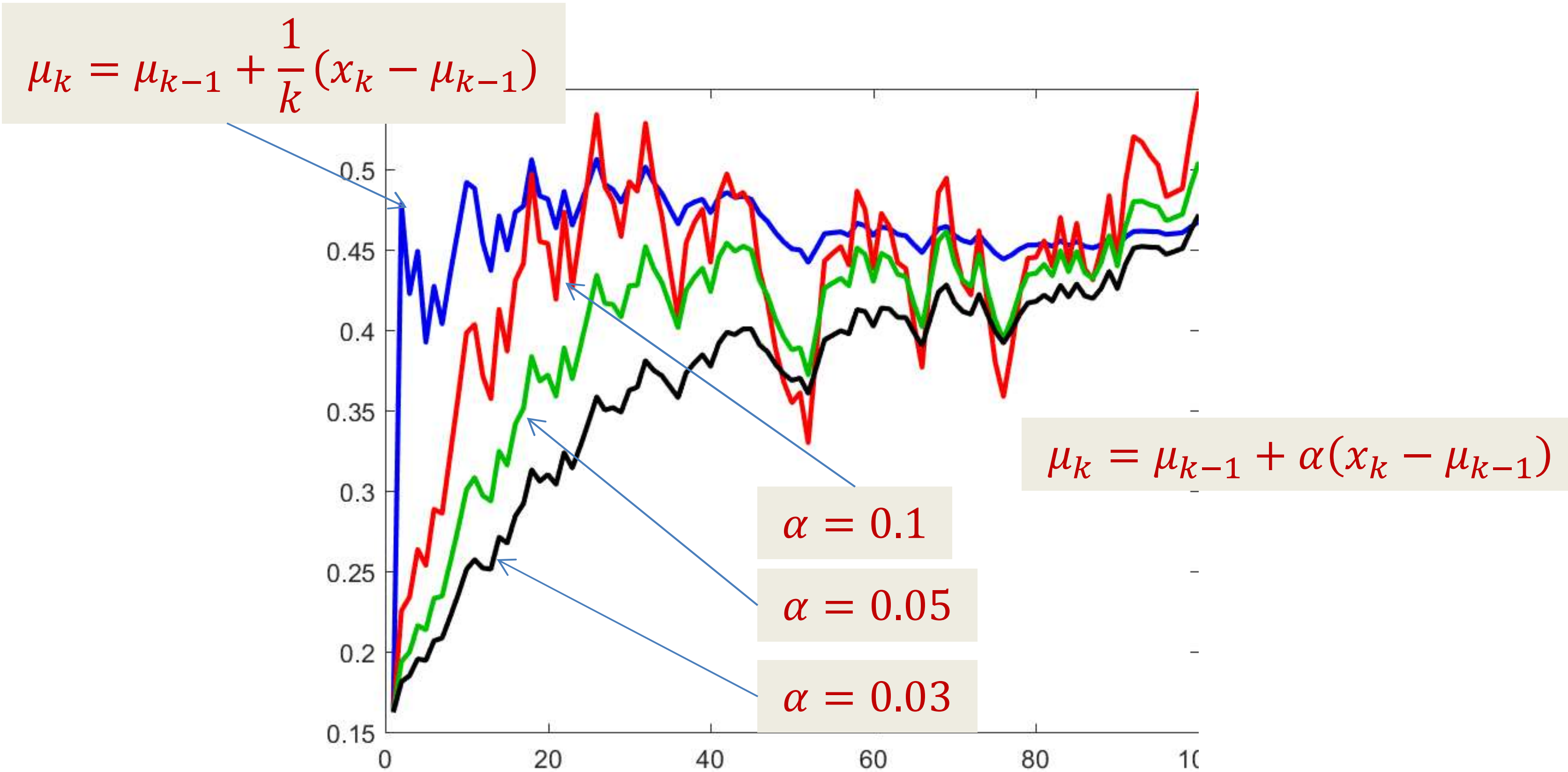
$$\mu_k = \mu_{k-1} + \frac{1}{k}(x_k - \mu_{k-1})$$

- Or more generally as

$$\mu_k = \mu_{k-1} + \alpha(x_k - \mu_{k-1})$$

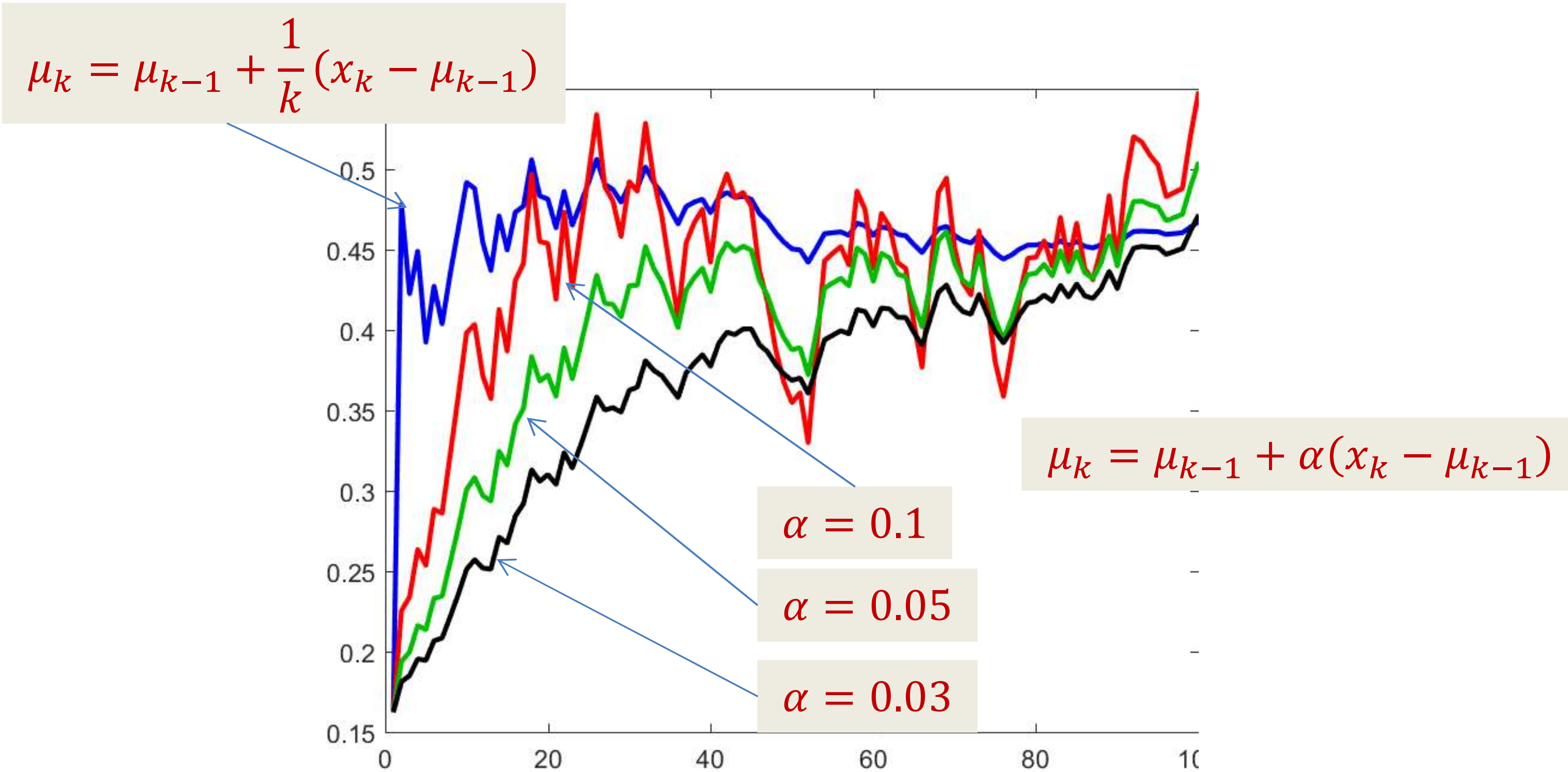
- The latter is particularly useful for non-stationary environments
- For stationary environments α must shrink with iterations, but not too fast
  - $\sum_k \alpha_k^2 < C, \quad \sum_k \alpha_k = \infty, \quad \alpha_k \geq 0$

Incremental Updates



- Example of running average of a uniform random variable

Incremental Updates



- Correct equation is *unbiased* and converges to true value
- Equation with α is *biased* (early estimates can be expected to be wrong) but *converges* to true value

Updating Value Function Incrementally

- Actual update

$$v_\pi(s) = \frac{1}{N(s)} \sum_{i=1}^{N(s)} G_{t(i)}$$

  - $N(s)$ is the total number of visits to state s across all episodes
  - $G_{t(i)}$ is the discounted return at the time instant of the i-th visit to state s

Online update

- Given any episode

$$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \dots, S_T$$

- Update the value of each state visited

$$N(S_t) = N(S_t) + 1$$

$$v_\pi(S_t) = v_\pi(S_t) + \frac{1}{N(S_t)}\big(G_t - v_\pi(S_t)\big)$$

- Incremental version

$$v_\pi(S_t) = v_\pi(S_t) + \alpha\big(G_t - v_\pi(S_t)\big)$$

- Still an unrealistic rule
  - Requires the entire track until the end of the episode to compute Gt

Online update

- Given any episode

$$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \ldots, S_T$$

- Update the value of each state visited

$$N(S_t) = N(S_t) + 1$$

$$v_\pi(S_t) = v_\pi(S_t) + \frac{1}{N(S_t)}\big(G_t - v_\pi(S_t)\big)$$

Problem

- Incremental version

$$v_\pi(S_t) = v_\pi(S_t) + \alpha\big(G_t - v_\pi(S_t)\big)$$

- Still an unrealistic rule
  - Requires the entire track until the end of the episode to compute Gt

TD solution

$$v_\pi(S_t) = v_\pi(S_t) + \alpha\big(G_t - v_\pi(S_t)\big)$$

Problem

- But

$$G_t = R_{t+1} + \gamma G_{t+1}$$

- We can approximate G_{t+1} by the *expected* return at the next state S_{t+1}

Counting Returns

- For each episode, we count the returns at all times:
  - $S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \ldots, S_T$
- Return at time t
  - $G_1 = R_2 + \gamma R_3 + \cdots + \gamma^{T-2} R_T$
  - $G_2 = R_3 + \gamma R_4 + \cdots + \gamma^{T-3} R_T$
  - …
  - $G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-t-2} R_T$
- **Can rewrite as**
  - $G_1 = R_2 + \gamma G_2$
- Or
  - $G_1 = R_2 + \gamma R_3 + \gamma^2 G_3$
  - …
  - $G_t = R_{t+1} + \sum_{i=1}^{N} \gamma^i R_{t+1+i} + \gamma^{N+1} G_{t+1+N}$

TD solution

$$v_\pi(S_t) = v_\pi(S_t) + \alpha\big(G_t - v_\pi(S_t)\big)$$

Problem

- But

$$G_t = R_{t+1} + \gamma G_{t+1}$$

- We can approximate G_{t+1} by the *expected* return at the next state $S_{t+1} \approx v_\pi(S_{t+1})$

$$G_t \approx R_{t+1} + \gamma v_\pi(S_{t+1})$$

- We don't know the real value of $v_\pi(S_{t+1})$ but we can "bootstrap" it by its current estimate

TD(1) true online update

$$v_\pi(S_t) = v_\pi(S_t) + \alpha\big(G_t - v_\pi(S_t)\big)$$

- Where

$$G_t \approx R_{t+1} + \gamma v_\pi(S_{t+1})$$

- Giving us
  - $v_\pi(S_t) = v_\pi(S_t) + \alpha\big(R_{t+1} + \gamma v_\pi(S_{t+1}) - v_\pi(S_t)\big)$

TD(1) true online update

$$v_\pi(S_t) = v_\pi(S_t) + \alpha\delta_t$$

- Where

$$\delta_t = R_{t+1} + \gamma v_\pi(S_{t+1}) - v_\pi(S_t)$$

- δ_t is the TD *error*
  - The error between an (estimated) *observation* of G_t and the current estimate $v_\pi(S_t)$

TD(1) true online update

- For all s Initialize: $v_\pi(s) = 0$
- For every episode e
  - For every time $t = 1 \dots T_e$
    - $v_\pi(S_t) = v_\pi(S_t) + \alpha\left(R_{t+1} + \gamma v_\pi(S_{t+1}) - v_\pi(S_t)\right)$

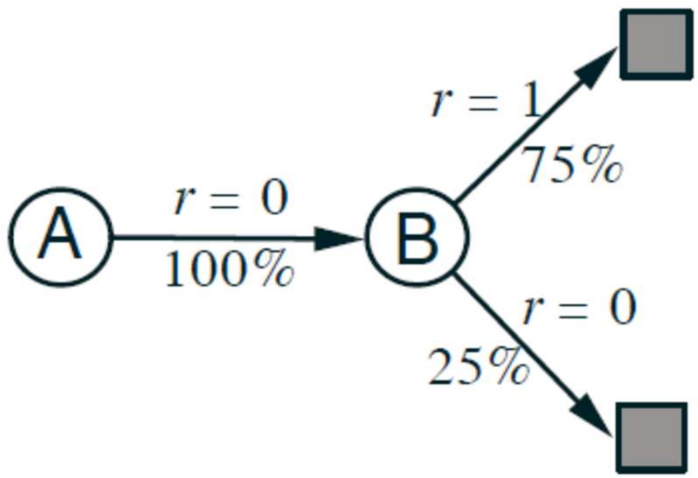
- There's a "lookahead" of one state, to know which state the process arrives at at the next time
- But is otherwise online, with continuous updates

TD(1)

- Updates continuously – improve estimates as soon as you observe a state (and its successor)
- Can work even with *infinitely long* processes that never terminate
- Guaranteed to converge to the true values eventually
  - Although initial values will be biased as seen before
  - Is actually lower variance than MC!!
    - Only incorporates one RV at any time
- TD can give correct answers when MC goes wrong
  - Particularly when TD is allowed to *loop* over all learning episodes

TD vs MC

A, 0, B, 0
B, 1
B, 1
B, 1
B, 1
B, 1
B, 1
B, 0



- What are $v(A)$ and $v(B)$
  - Using MC
  - Using TD(1), where you are allowed to repeatedly go over the data

TD – look ahead further?

- TD(1) has a look ahead of 1 time step

$$G_t \approx R_{t+1} + \gamma v_\pi(S_{t+1})$$

- But we can look ahead further out
  - $G_t(2) = R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2})$
  - $\ldots$
  - $G_t(N) = R_{t+1} \sum_{i=1}^{N} \gamma^i R_{t+1+i} + \gamma^{N+1} v_\pi(S_{t+N})$

TD(N) with lookahead

$$v_\pi(S_t) = v_\pi(S_t) + \alpha\delta_t(N)$$

- Where

$$\delta_t(N) = R_{t+1} + \sum_{i=1}^{N} \gamma^i R_{t+1+i} + \gamma^{N+1} v_\pi(S_{t+N}) - v_\pi(S_t)$$

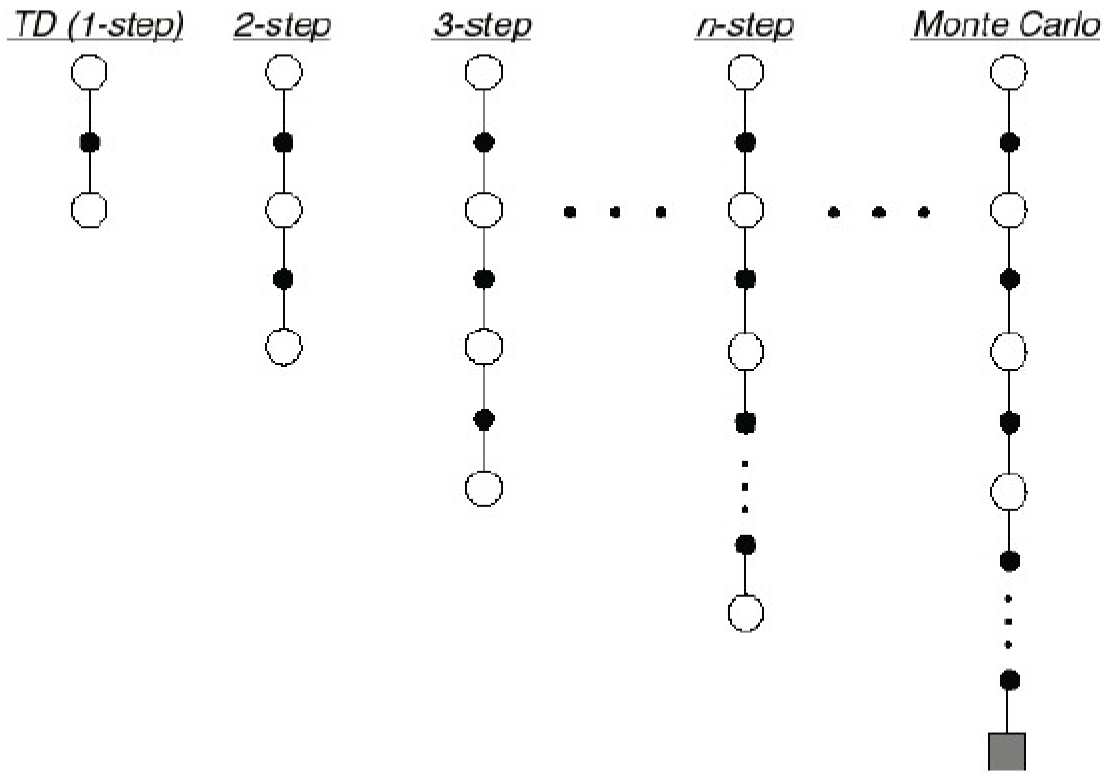
- $\delta_t(N)$ is the TD *error* with *N* step lookahead

Lookahead is good

- Good: The further you look ahead, the better your estimates get
- Problems:
  - But you also get more variance
  - At infinite lookahead, you're back at MC
- Also, you have to wait to update your estimates
  - A lag between observation and estimate
- So how much lookahead must you use

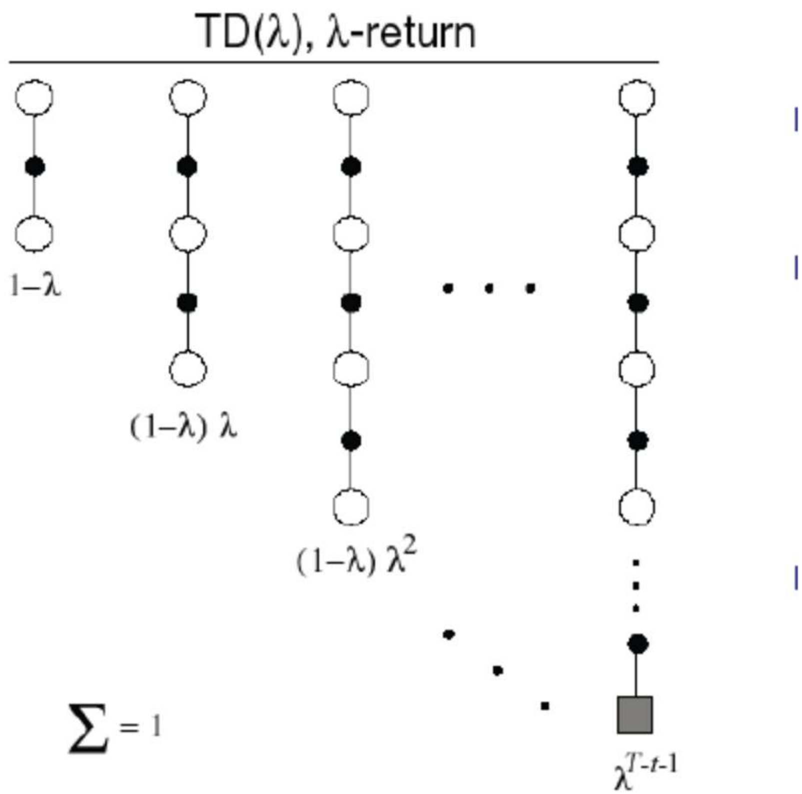
Looking Into The Future

- Let TD target look n steps into the future



- How much various TDs look into the future
- Which do we use?

Solution: Why choose?



- Each lookahead provides an estimate of G_t
- Why not just combine the lot with discounting?

TD(λ)

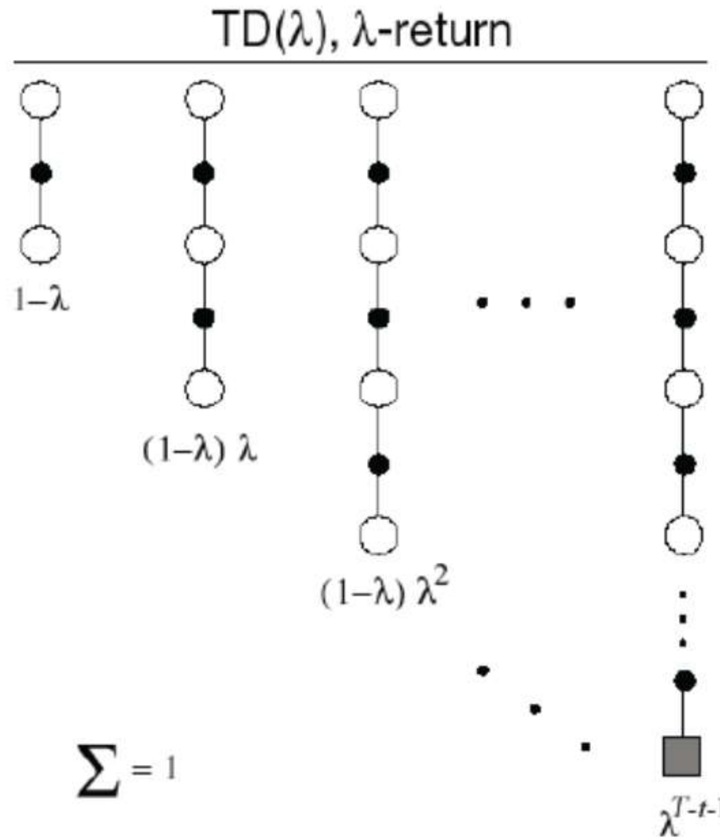
$$G_t^{\lambda} = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t(n)$$

- Combine the predictions from all lookaheads with an exponentially falling weight
  - Weights sum to 1.0

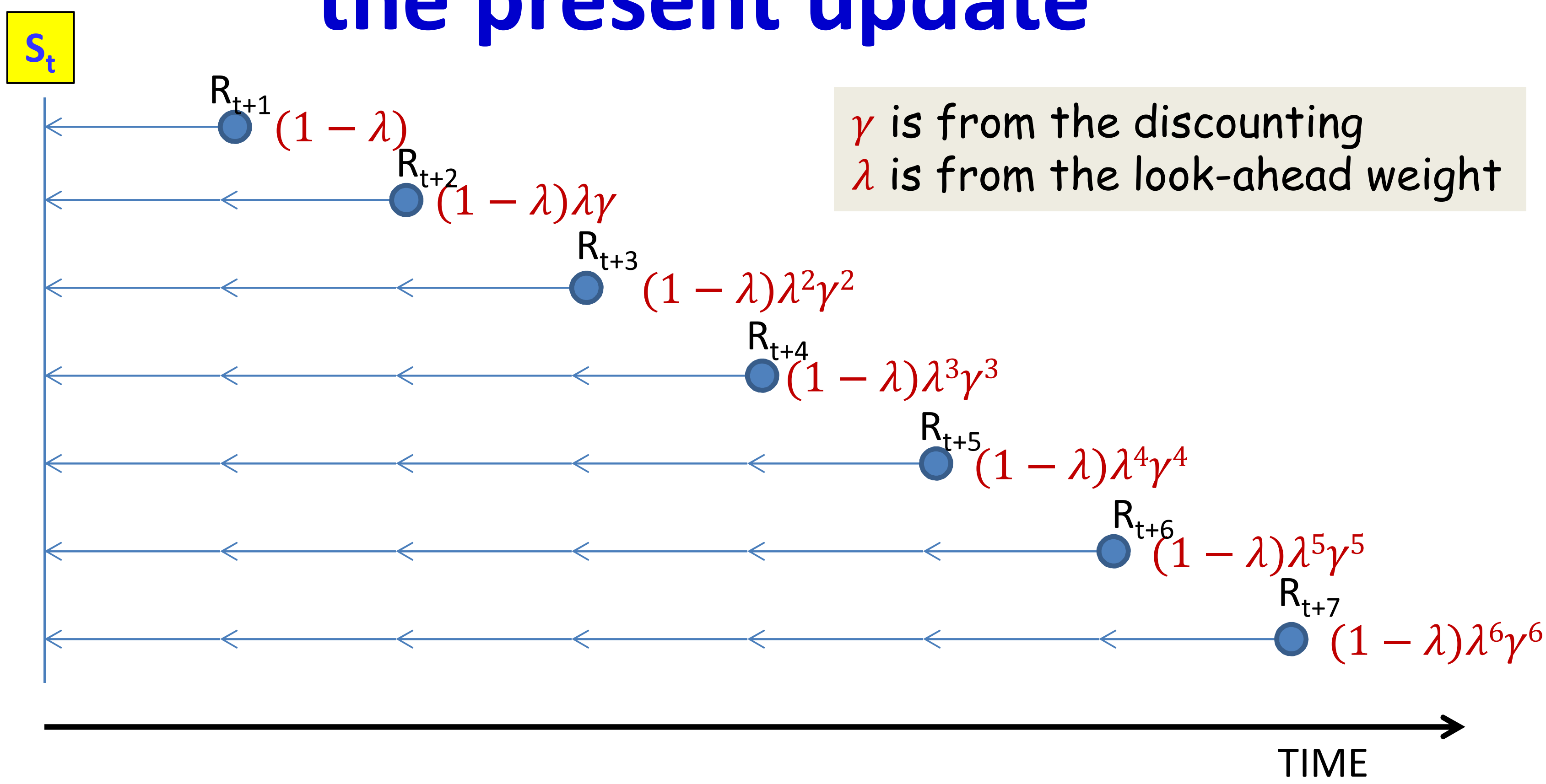
$$V(S_t) \leftarrow V(S_t) + \alpha\left(G_t^{\lambda} - V(S_t)\right)$$

Something magical just happened

- TD(λ) looks into the infinite future
  - I.e. we must have all the rewards of the future to compute our updates
  - How does that help?

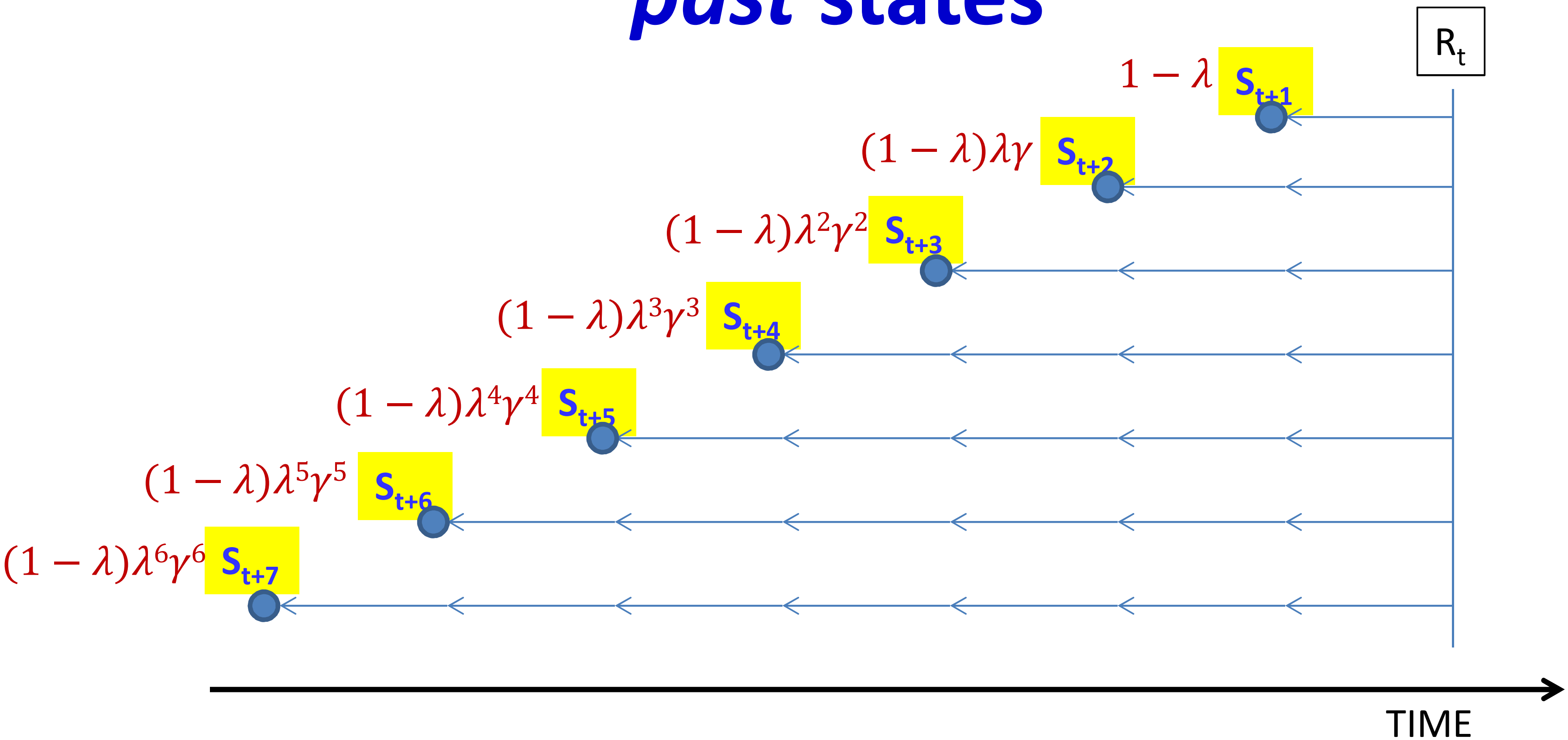


The contribution of future rewards to the present update



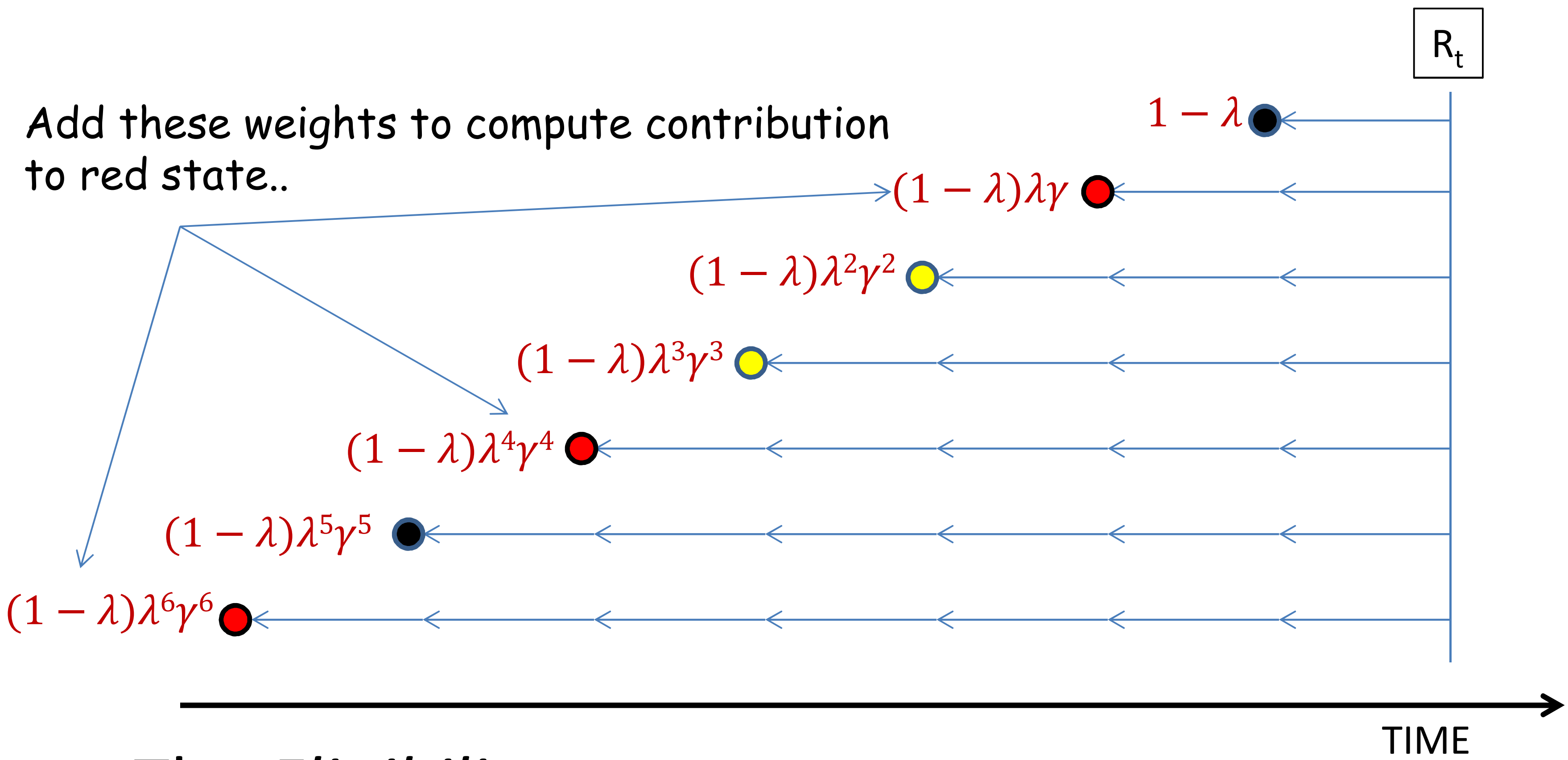
- All future rewards contribute to the update of the value of the current state

The contribution of current reward to *past* states



- All current reward contributes to the update of the value of all past states!

TD(λ) backward view



- The *Eligibility* trace:
  - Keeps track of *total* weight for any state
    - Which may have occurred at multiple times in the past

TD(λ)

- Maintain an eligibility trace for *every* state

$$E_0(s) = 0$$
$$E_t(s) = \lambda\gamma E_{t-1}(s) + 1(S_t = s)$$

- Computes total weight for the state until the present time

TD(λ)

- At every time, update the value of *every state* according to its eligibility trace

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$

- Any state that was visited will be updated
  - Those that were not will not be, though

The magic of TD(λ)

- Managed to get the effect of inifinite lookahead, by performing infinite *lookbehind*
  - Or at least look behind to the beginning

- Every reward updates the value of *all states* leading to the reward!
  - E.g., in a chess game, if we win, we want to increase the value of all game states we visited, not just the final move
  - But early states/moves must gain much less than later moves

- When $\lambda = 1$ this is exactly equivalent to MC

Story so far

- Want to compute the *values* of all states, given a policy, but no knowledge of dynamics
- Have seen monte-carlo and temporal difference solutions
  - TD is quicker to update, and in many situations the better solution
  - TD(λ) actually emulates an infinite lookahead
    - But we must choose good values of α and λ

Optimal Policy: Control

- We learned how to estimate the state value functions for an MDP whose transition probabilities are unknown ***for a given policy***

- *How do we find the optimal policy?*

Value vs. Action Value

- The solution we saw so far only computes the *value functions* of states

- Not sufficient – to compute the optimal policy from value functions alone, we will need extra information, namely transition probabilities
  - Which we do not have

- Instead, we can use the same method to compute *action value* functions
  - Optimal policy in any state : Choose the action that has the largest *optimal* action value

Value vs. Action value

- Given only value functions, the optimal policy must be estimated as:

$$\pi'(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}} \; \mathcal{R}_s^a + \mathcal{P}_{ss'}^a V(s')$$

  - Needs knowledge of transition probabilities
- Given action value functions, we can find it as:

$$\pi'(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}} \, Q(s, a)$$

- This is *model free* (no need for knowledge of model parameters)

Problem of optimal control

- From a series of episodes of the kind:
  $$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \ldots, S_T$$
- Find the optimal action value function $q_*(s, a)$
  - The optimal policy can be found from it
- Ideally do this *online*
  - So that we can continuously improve our policy from *ongoing experience*

Exploration vs. Exploitation

- Optimal policy search happens while gathering experience *while following a policy*
- For fastest learning, we will follow an estimate of the optimal policy
- Risk: We run the risk of positive feedback
  - Only learn to evaluate our current policy
  - Will never learn about alternate policies that may turn out to be better
- Solution: We will follow our current optimal policy $1 - \epsilon$ of the time
  - But choose a random action ϵ of the time
  - The “epsilon-greedy” policy

GLIE Monte Carlo

- **Greedy in the limit with infinite exploration**
- Start with some random initial policy π
- Start the process at the initial state, and follow an action according to initial policy π
- Produce the episode

$$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \dots, S_T$$

- Process the episode using the following online update rules:

$$N(S_t, A_t) \leftarrow N(S_t, A_t) + 1$$

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{N(S_t, A_t)}(G_t - Q(S_t, A_t))$$

- Compute the ϵ-greedy policy for each state

$$\pi(a|s) = \begin{cases} 1 - \epsilon & for\ a = arg\max_{a\prime} Q(s, a') \\ \dfrac{\epsilon}{N_a - 1} & otherwise \end{cases}$$

- Repeat

GLIE Monte Carlo

- **Greedy in the limit with infinite exploration**
- Start with some random initial policy π
- Start the process at the initial state, and follow an action according to initial policy π
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$$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \dots, S_T$$

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- Compute the ϵ-greedy policy for each state

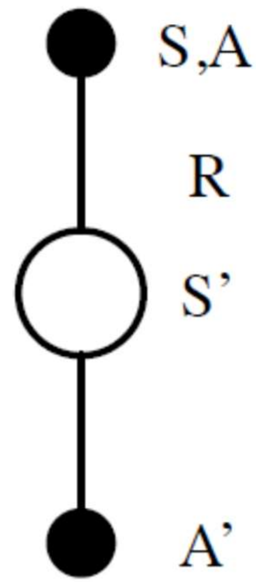
$$\pi(a|s) = \begin{cases} 1 - \epsilon & for\ a = arg\max_{a'} Q(s, a') \\ \dfrac{\epsilon}{N_a - 1} & otherwise \end{cases}$$

- Repeat

On-line version of GLIE: SARSA

- Replace G_t with an estimate
- TD(1) or TD(λ)
  - Just as in the prediction problem

- **TD(1) → SARSA**



$$Q(S, A) \leftarrow Q(S, A) + \alpha(R + \gamma Q(S', A') - Q(S, A))$$

SARSA

- Initialize $Q(s,a)$ for all s,a
- Start at initial state S_1
- Select an initial action A_1
- For t = 1.. Terminate
  - Get reward R_t
  - Let system transition to new state S_{t+1}
  - Draw A_{t+1} according to ϵ -greedy policy

$$\pi(a|s) = \begin{cases} 1-\epsilon & for\ a = arg\max_{a'} Q(s,a') \\ \dfrac{\epsilon}{N_a - 1} & otherwise \end{cases}$$

  - Update

$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha\big(R_t + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\big)$$

SARSA(λ)

- Again, the TD(1) estimate can be replaced by a TD(λ) estimate
- Maintain an eligibility trace for every state-action pair:

$$E_0(s,a) = 0$$
$$E_t(s,a) = \lambda\gamma E_{t-1}(s,a) + 1(S_t = s, A_t = a)$$

- Update every state-action pair visited so far

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$

$$Q(s,a) \leftarrow Q(s,a) + \alpha\delta_t E_t(s,a)$$

SARSA(λ)

- For all s, a initialize $\mathrm{Q}(s, a)$
- For each episode e
  - For all s, a initialize $E(s, a) = 0$
  - Initialize S_1, A_1
  - For $t = 1 \dots Termination$
    - Observe R_{t+1}, S_{t+1}
    - Choose action A_{t+1} using policy obtained from Q
    - $\delta = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$
    - $E(S_t, A_t) += 1$
    - For all s, a
      - $Q(s, a) = Q(s, a) + \alpha\delta E(s, a)$
      - $E(s, a) = \gamma\lambda E(s, a)$

On-policy vs. Off-policy

- SARSA assumes you're following the same policy that you're learning
- Its possible to follow one policy, while learning from others
  - E.g. learning by observation
- The policy for learning is the whatif policy

$$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \ldots, S_T$$

$$\hat{A}_2 \qquad \hat{A}_3$$

hypothetical

- Modifies learning rule

$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha\big(R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\big)$$

- to

$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha\left(R_{t+1} + \gamma Q\big(S_{t+1}, \hat{A}_{t+1}\big) - Q(S_t, A_t)\right)$$

- Q will actually represent the action value function of the *hypothetical policy*

SARSA: Suboptimality

- SARSA: From any state-action (S, A), accept reward (R), transition to next state (S'), choose next action (A')

- Use TD rules to update:

$$\delta = R + \gamma Q(S', A') - Q(S, A)$$

- Problem: which policy do we use to choose A'

SARSA: Suboptimality

- SARSA: From any state-action (S, A), accept reward (R), transition to next state (S'), choose next action (A')

- Problem: which policy do we use to choose A'
- If we choose the *current judgment of the best action* at S' we will become too greedy
  - Never explore
- If we choose a *sub-optimal* policy to follow, we will never find the best policy

Solution: Off-policy learning

- The policy for learning is the whatif policy

$$S_1, A_1, R_2, S_2, A_2, R_3, S_3, A_3, R_4, \dots, S_T$$

$$\hat{A}_2 \qquad \hat{A}_3 \quad \text{hypothetical}$$

- Use the *best* action for S_{t+1} as your hypothetical off-policy action
- But actually follow an *epsilon-greedy* action
  - The hypothetical action is guaranteed to be better than the one you actually took
  - But you still explore (non-greedy)

Q-Learning

- From any state-action pair S, A
  - Accept reward R
  - Transition to S'
  - Find the *best action* A' for S'
  - Use it to update $Q(S, A)$
  - But then *actually perform an epsilon-greedy action* A'' from S'

Q-Learning (TD(1) version)

- For all s, a initialize $\mathrm{Q}(s, a)$
- For each episode e
  - Initialize S_1, A_1
  - For $t = 1 \dots Termination$
    - Observe R_{t+1}, S_{t+1}
    - Choose action A_{t+1} at S_{t+1} using epsilon-greedy policy obtained from Q
    - Choose action $\hat{A}_{t+1}$ at S_{t+1} as $\hat{A}_{t+1} = arg\max_a Q(S_{t+1}, a)$
    - $\delta = R_{t+1} + \gamma Q\left(S_{t+1}, \hat{A}_{t+1}\right) - Q(S_t, A_t)$
    - $Q(S_t, A_t) = Q(S_t, A_t) + \alpha\delta$

Q-Learning (TD(λ) version)

- For all s, a initialize $\mathrm{Q}(s, a)$
- For each episode e
  - For all s, a initialize $E(s, a) = 0$
  - Initialize S_1, A_1
  - For $t = 1 \ldots Termination$
    - Observe R_{t+1}, S_{t+1}
    - Choose action A_{t+1} at S_{t+1} using epsilon-greedy policy obtained from Q
    - Choose action $\hat{A}_{t+1}$ at S_{t+1} as $\hat{A}_{t+1} = arg\max_a Q(S_{t+1}, a)$
    - $\delta = R_{t+1} + \gamma Q\left(S_{t+1}, \hat{A}_{t+1}\right) - Q(S_t, A_t)$
    - $E(S_t, A_t) \mathrel{+}= 1$
    - For all s, a
      - $Q(s, a) = Q(s, a) + \alpha\delta E(s, a)$
      - $E(s, a) = \gamma\lambda E(s, a)$

What about the actual policy?

- Optimal greedy policy:

$$\pi(a|s) = \begin{cases} 1 \quad for\ a = arg\max_{a'} Q(s, a') \\ 0\ otherwise \end{cases}$$

- Exploration policy

$$\pi(a|s) = \begin{cases} 1 - \epsilon \quad for\ a = arg\max_{a'} Q(s, a') \\ \dfrac{\epsilon}{N_a - 1}\ otherwise \end{cases}$$

- Ideally ϵ should decrease with time

Q-Learning

- Currently most-popular RL algorithm
- Topics not covered:
  - Value function approximation
  - Continuous state spaces
  - Deep-Q learning
  - Action replay
  - Application to real problem..