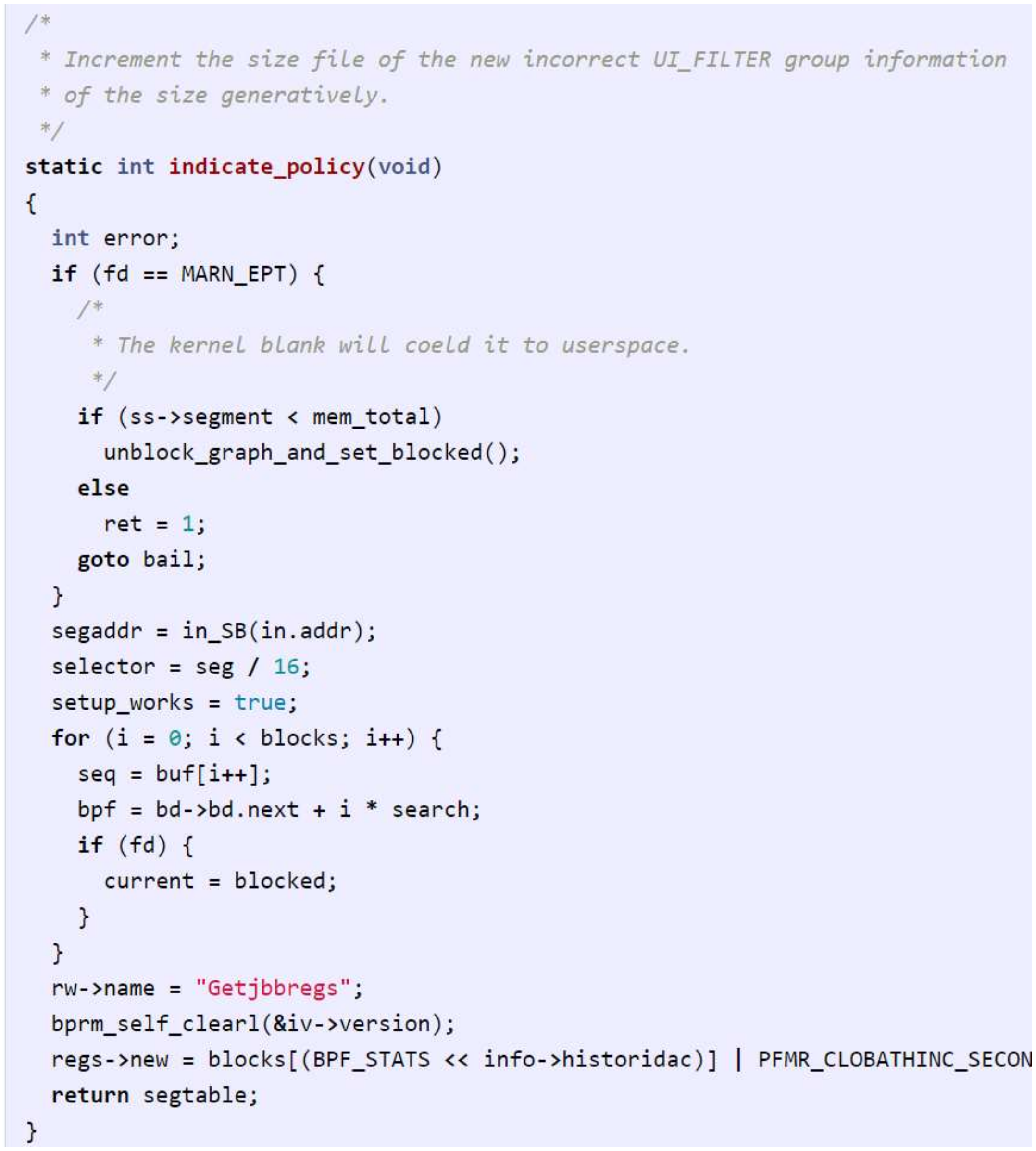
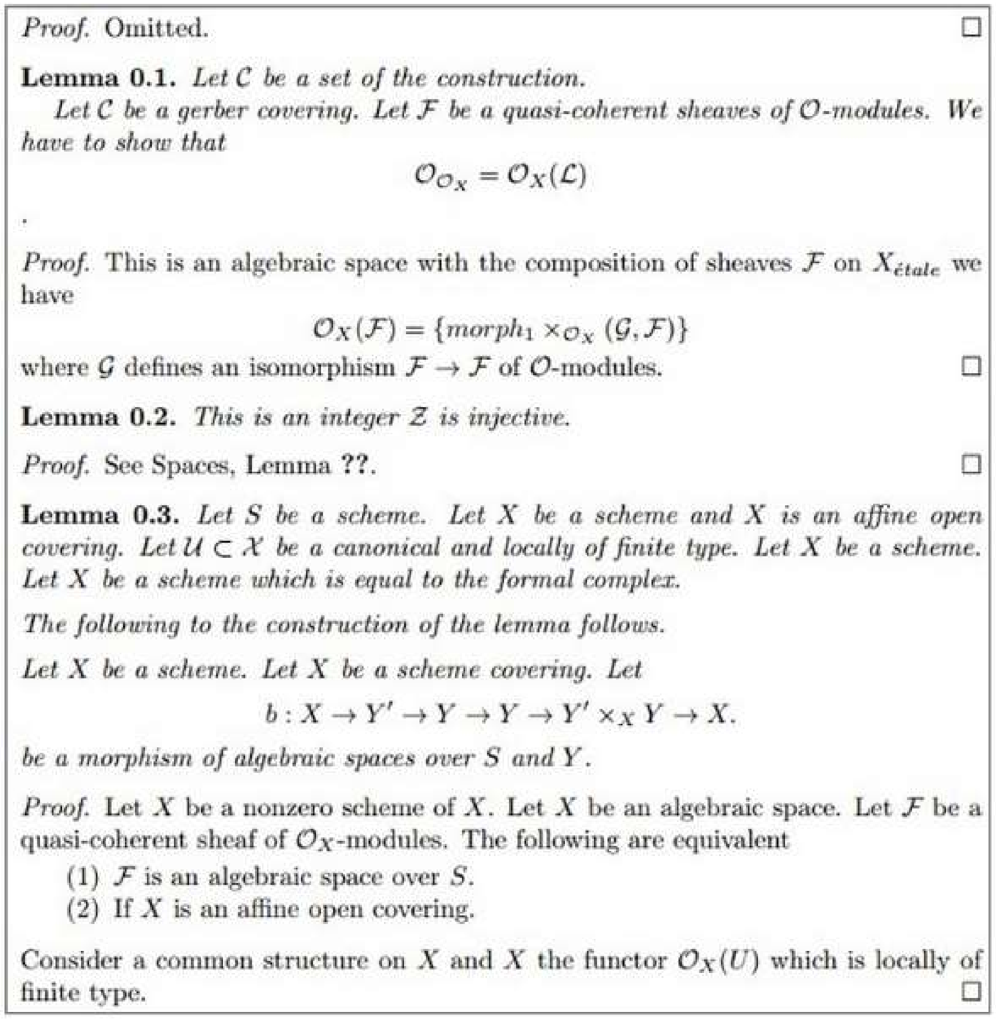
Deep Learning
Recurrent Networks: Stability analysis and LSTMs

Which open source project?



```
/*
 * Increment the size file of the new incorrect UI_FILTER group information
 * of the size generatively.
 */
static int indicate_policy(void)
{
  int error;
  if (fd == MARN_EPT) {
    /*
     * The kernel blank will coeld it to userspace.
     */
    if (ss->segment < mem_total)
      unblock_graph_and_set_blocked();
    else
      ret = 1;
    goto bail;
  }
  segaddr = in_SB(in.addr);
  selector = seg / 16;
  setup_works = true;
  for (i = 0; i < blocks; i++) {
    seq = buf[i++];
    bpf = bd->bd.next + i * search;
    if (fd) {
      current = blocked;
    }
  }
  rw->name = "Getjbbregs";
  bprm_self_clearl(&iv->version);
  regs->new = blocks[(BPF_STATS << info->historidac)] | PFMR_CLOBATHINC_SECON
  return segtable;
}
```

Related math. What is it talking about?



Proof. Omitted. □

Lemma 0.1. *Let $\mathcal{C}$ be a set of the construction.*

Let $\mathcal{C}$ be a gerber covering. Let $\mathcal{F}$ be a quasi-coherent sheaves of $\mathcal{O}$-modules. We have to show that

$$\mathcal{O}_{\mathcal{O}_X} = \mathcal{O}_X(\mathcal{L})$$

.

Proof. This is an algebraic space with the composition of sheaves $\mathcal{F}$ on $X_{étale}$ we have

$$\mathcal{O}_X(\mathcal{F}) = \{morph_1 \times_{\mathcal{O}_X} (\mathcal{G}, \mathcal{F})\}$$

where $\mathcal{G}$ defines an isomorphism $\mathcal{F} \to \mathcal{F}$ of $\mathcal{O}$-modules. □

Lemma 0.2. *This is an integer $\mathcal{Z}$ is injective.*

Proof. See Spaces, Lemma ??. □

Lemma 0.3. *Let S be a scheme. Let X be a scheme and X is an affine open covering. Let $\mathcal{U} \subset \mathcal{X}$ be a canonical and locally of finite type. Let X be a scheme. Let X be a scheme which is equal to the formal complex.*

The following to the construction of the lemma follows.

Let X be a scheme. Let X be a scheme covering. Let

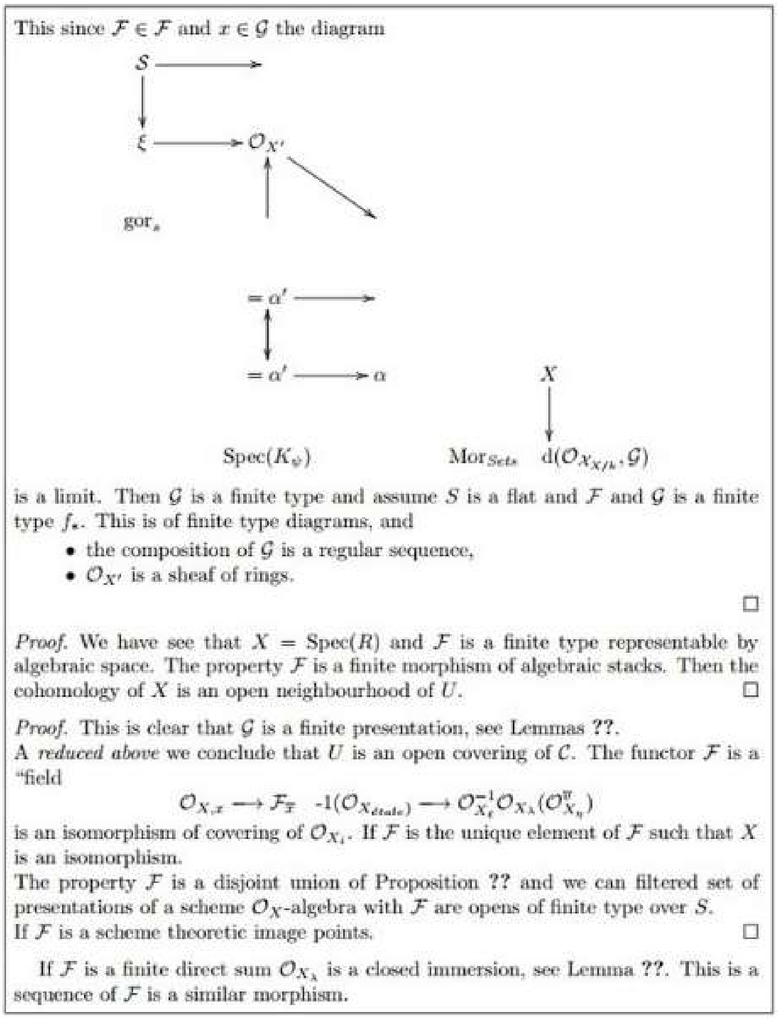
$$b : X \to Y' \to Y \to Y \to Y' \times_X Y \to X.$$

be a morphism of algebraic spaces over S and Y.

Proof. Let X be a nonzero scheme of X. Let X be an algebraic space. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. The following are equivalent

(1) $\mathcal{F}$ is an algebraic space over S.
(2) If X is an affine open covering.

Consider a common structure on X and X the functor $\mathcal{O}_X(U)$ which is locally of finite type. □



This since $\mathcal{F} \in \mathcal{F}$ and $x \in \mathcal{G}$ the diagram

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & {} & & & \\
\xi \arrow[r] & \mathcal{O}_{X'} \arrow[dr] & & & \\
\mathrm{gor}_* & {} \arrow[u] & {} & & \\
& =\alpha' \arrow[r] \arrow[d, leftrightarrow] & {} & & \\
& =\alpha' \arrow[r] & \alpha & & X \arrow[d] \\
& \mathrm{Spec}(K_\psi) & & \mathrm{Mor}_{Sets} & \mathrm{d}(\mathcal{O}_{X_{X/k}}, \mathcal{G})
\end{tikzcd}$$

is a limit. Then $\mathcal{G}$ is a finite type and assume S is a flat and $\mathcal{F}$ and $\mathcal{G}$ is a finite type f_*. This is of finite type diagrams, and

- the composition of $\mathcal{G}$ is a regular sequence,
- $\mathcal{O}_{X'}$ is a sheaf of rings.

□

Proof. We have see that $X = \mathrm{Spec}(R)$ and $\mathcal{F}$ is a finite type representable by algebraic space. The property $\mathcal{F}$ is a finite morphism of algebraic stacks. Then the cohomology of X is an open neighbourhood of U. □

Proof. This is clear that $\mathcal{G}$ is a finite presentation, see Lemmas ??.
A *reduced above* we conclude that U is an open covering of $\mathcal{C}$. The functor $\mathcal{F}$ is a "field

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{F}_{\overline{x}} \;\; \text{-}1(\mathcal{O}_{X_{étale}}) \longrightarrow \mathcal{O}_{X_\ell}^{-1}\mathcal{O}_{X_\lambda}(\mathcal{O}_{X_\eta}^{\overline{v}})$$

is an isomorphism of covering of $\mathcal{O}_{X_i}$. If $\mathcal{F}$ is the unique element of $\mathcal{F}$ such that X is an isomorphism.
The property $\mathcal{F}$ is a disjoint union of Proposition ?? and we can filtered set of presentations of a scheme $\mathcal{O}_X$-algebra with $\mathcal{F}$ are opens of finite type over S.
If $\mathcal{F}$ is a scheme theoretic image points. □

If $\mathcal{F}$ is a finite direct sum $\mathcal{O}_{X_\lambda}$ is a closed immersion, see Lemma ??. This is a sequence of $\mathcal{F}$ is a similar morphism.

And a Wikipedia page explaining it all

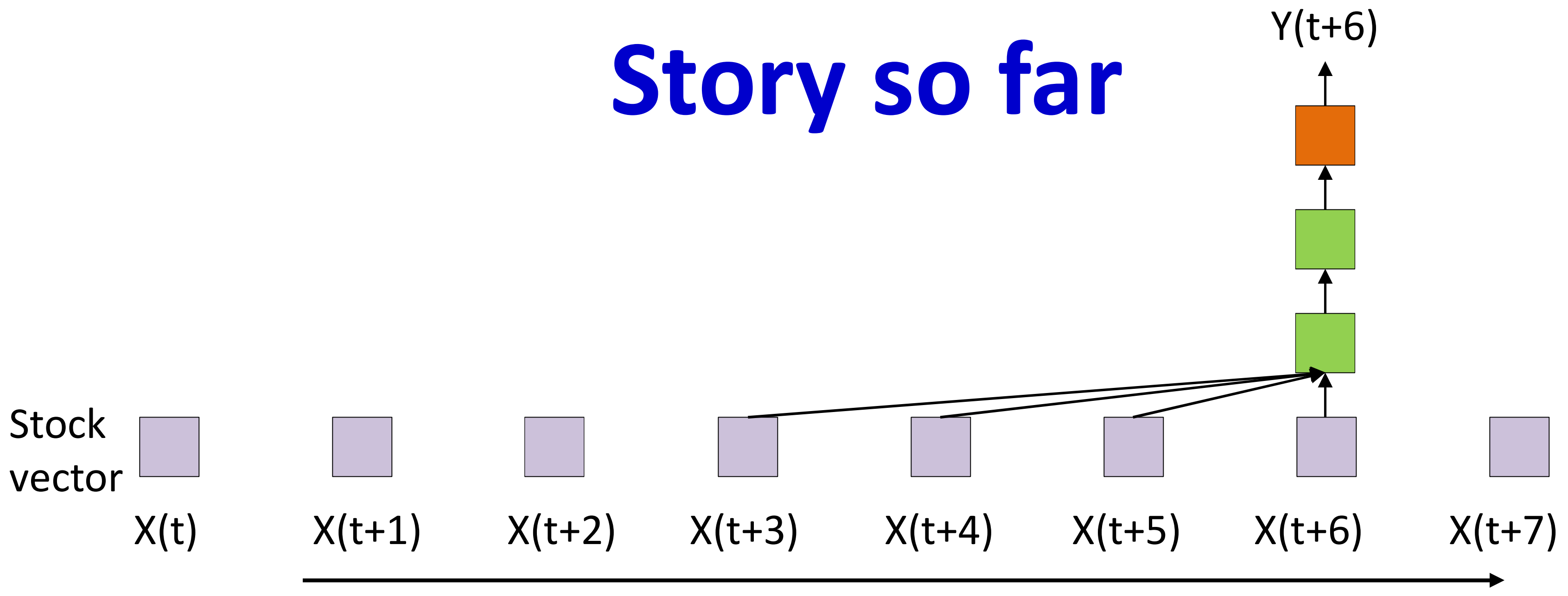
```
Naturalism and decision for the majority of Arab countries' capitalide was grounded
by the Irish language by [[John Clair]], [[An Imperial Japanese Revolt]], associated
with Guangzham's sovereignty. His generals were the powerful ruler of the Portugal
in the [[Protestant Immineners]], which could be said to be directly in Cantonese
Communication, which followed a ceremony and set inspired prison, training. The
emperor travelled back to [[Antioch, Perth, October 25|21]] to note, the Kingdom
of Costa Rica, unsuccessful fashioned the [[Thrales]], [[Cynth's Dajoard]], known
in western [[Scotland]], near Italy to the conquest of India with the conflict.
Copyright was the succession of independence in the slop of Syrian influence that
was a famous German movement based on a more popular servicious, non-doctrinal
and sexual power post. Many governments recognize the military housing of the
[[Civil Liberalization and Infantry Resolution 265 National Party in Hungary]],
that is sympathetic to be to the [[Punjab Resolution]]
(PJS)[http://www.humah.yahoo.com/guardian.
cfm/7754800786d17551963s89.htm Official economics Adjoint for the Nazism, Montgomery
was swear to advance to the resources for those Socialism's rule,
was starting to signing a major tripad of aid exile.]]
```

The unreasonable effectiveness of recurrent neural networks..

- All previous examples were *generated* blindly by a *recurrent* neural network..
- http://karpathy.github.io/2015/05/21/rnn-effectiveness/

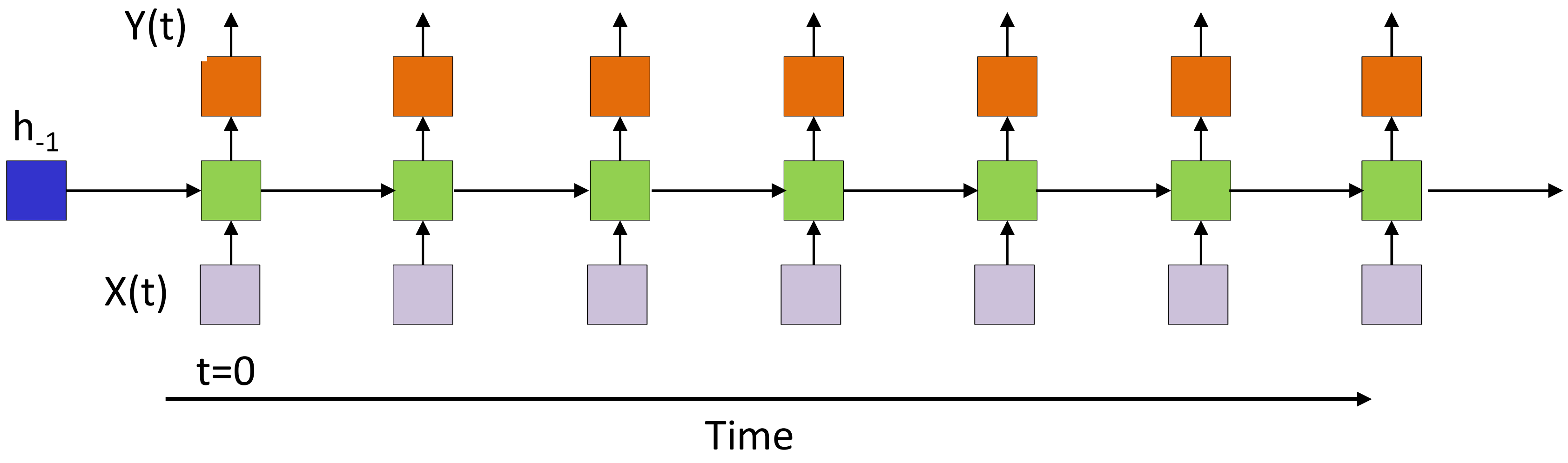
- Examples of models that analyze (or in this case, generate) time-series data

Story so far



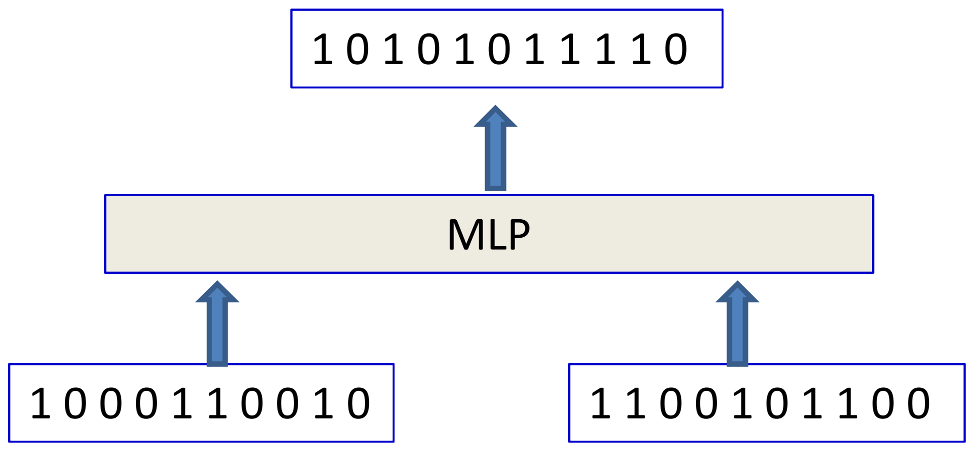
- ***Iterated structures*** are good for analyzing time series data with short-time dependence on the past
  - These are "***Time delay***" neural nets, AKA ***convnets***

Story so far



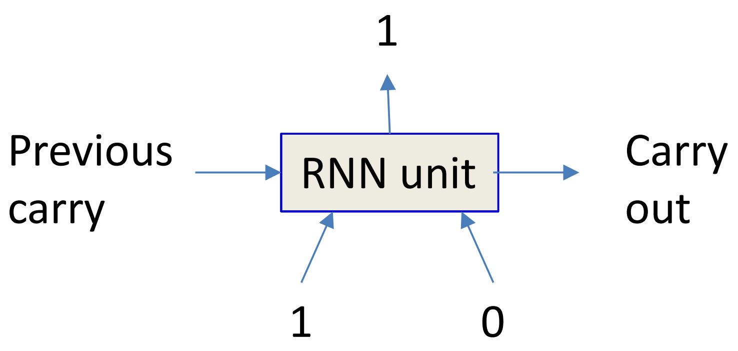
- Iterated structures are good for analyzing time series data with short-time dependence on the past
  - These are "Time delay" neural nets, AKA convnets
- ***Recurrent structures*** are good for analyzing time series data with ***long-term*** dependence on the past
  - These are ***recurrent*** neural networks

Recurrent structures can do what static structures cannot



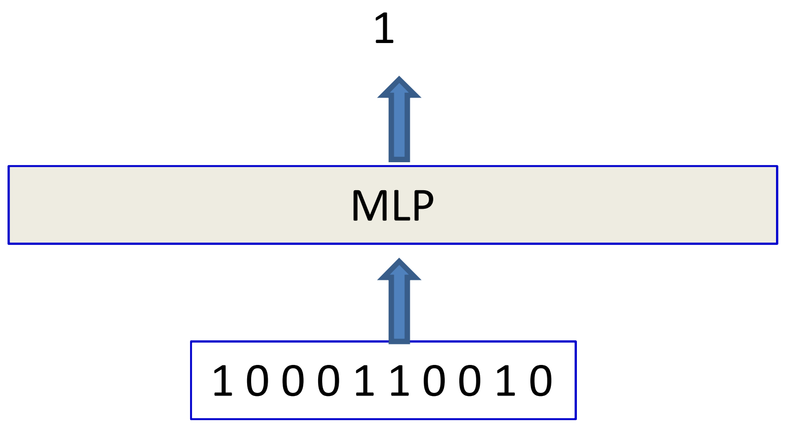
- The addition problem: Add two N-bit numbers to produce a N+1-bit number
  - Input is binary
  - Will require large number of training instances
    - Output must be specified for every pair of inputs
    - Weights that generalize will make errors
  - Network trained for N-bit numbers will not work for N+1 bit numbers

MLPs vs RNNs



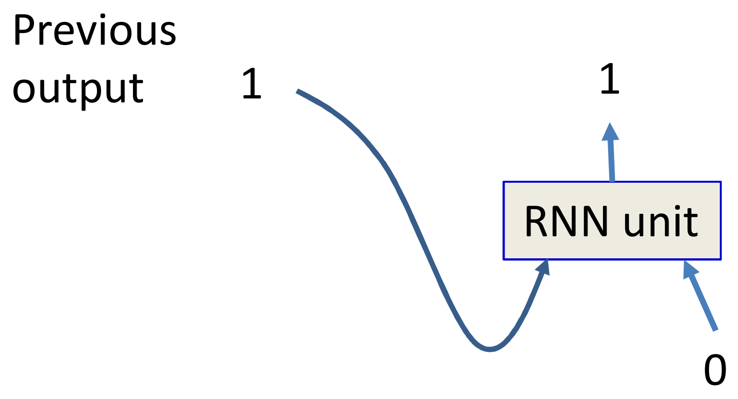
- The addition problem: Add two N-bit numbers to produce a N+1-bit number
- **RNN solution:** Very simple, can add two numbers of any size

MLP: The parity problem



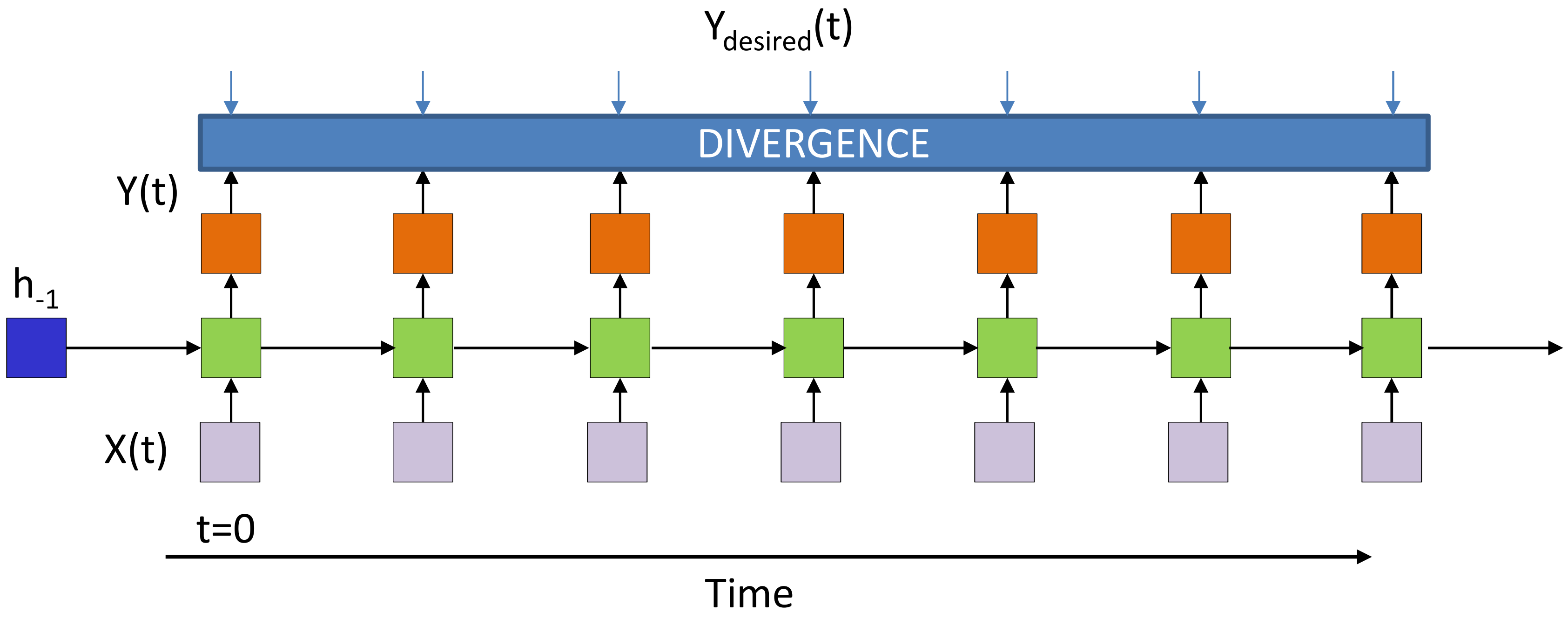
- Is the number of "ones" even or odd
- Network must be complex to capture all patterns
  - XOR network, quite complex
  - Fixed input size

RNN: The parity problem



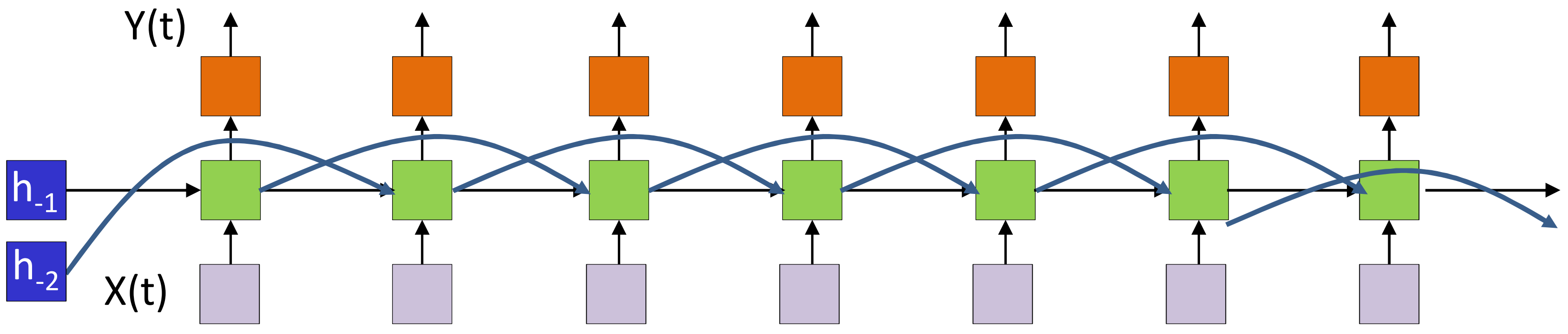
- Trivial solution
- Generalizes to input of any size

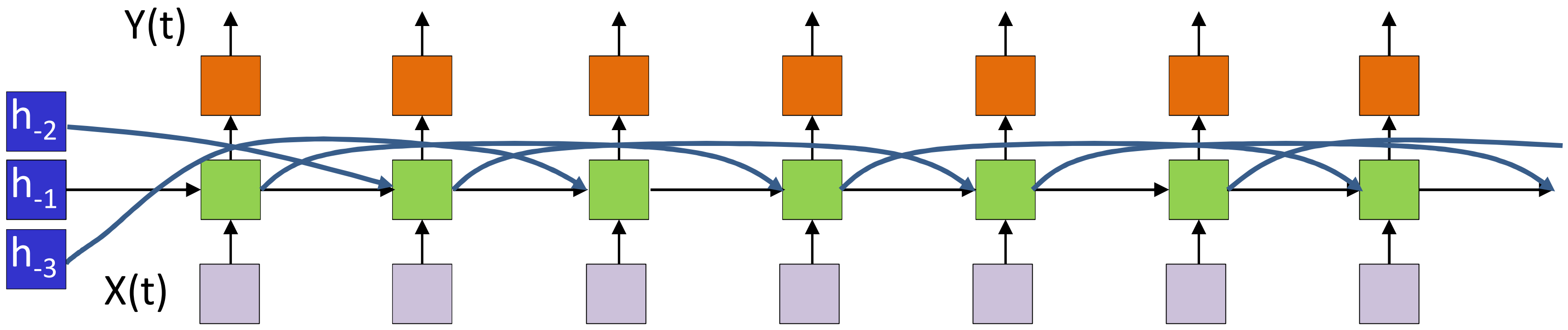
Story so far



- Recurrent structures can be trained by minimizing the divergence between the ***sequence*** of outputs and the ***sequence*** of desired outputs
  – Through gradient descent and backpropagation

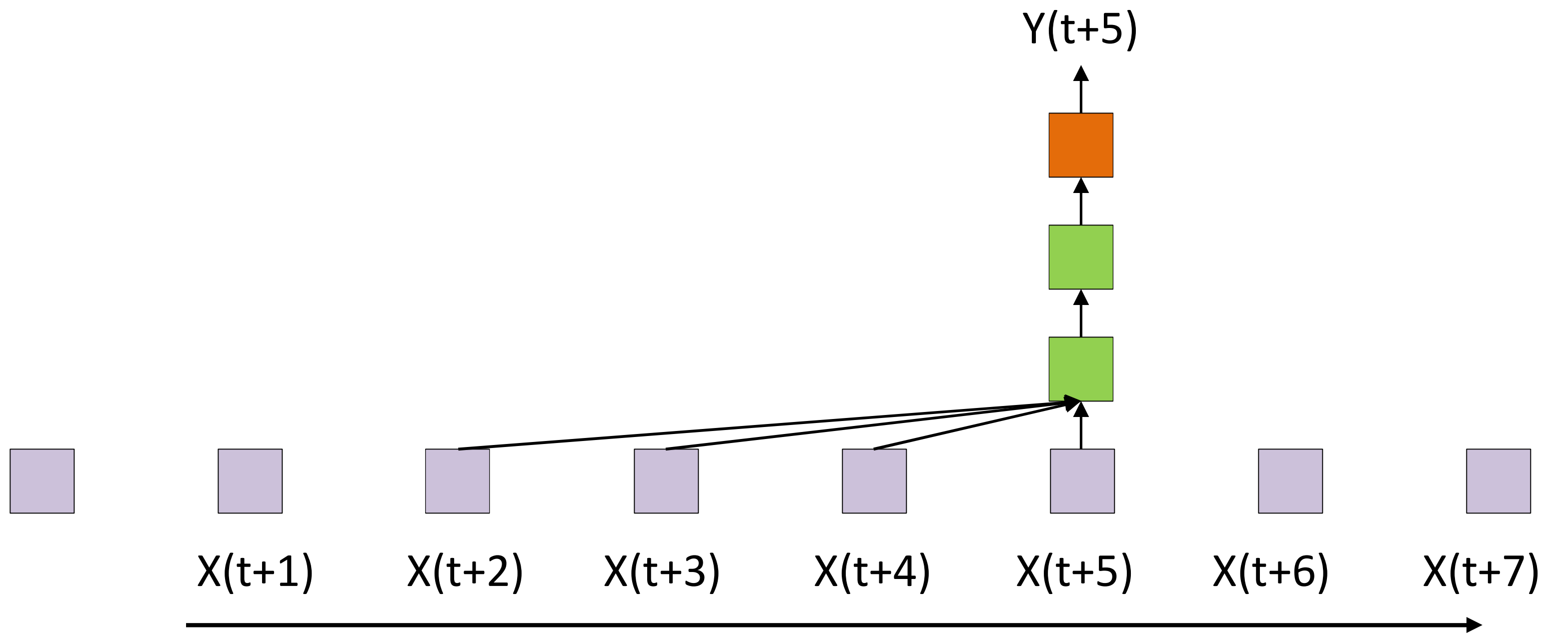
Types of recursion





- Nothing special about a one step recursion

The behavior of recurrence..

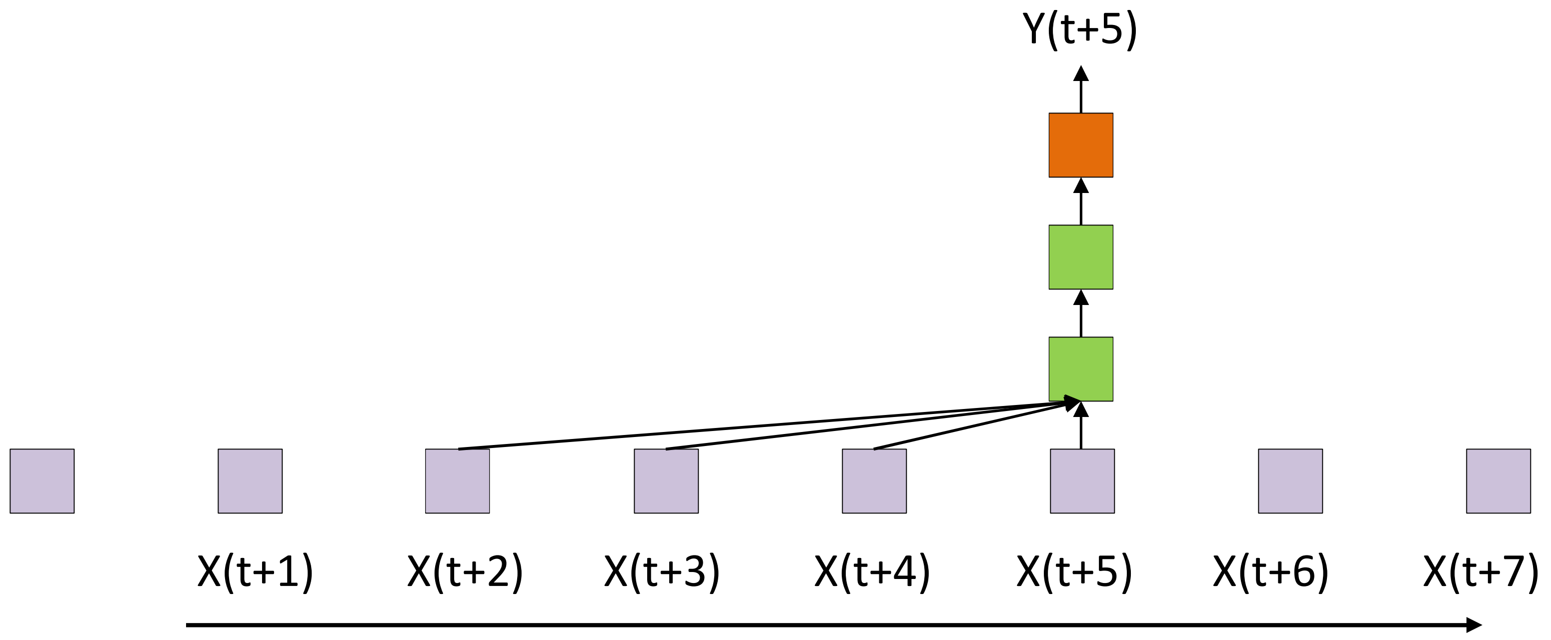


- Returning to an old model..

$$Y(t) = f(X(t-i), i = 1..K)$$

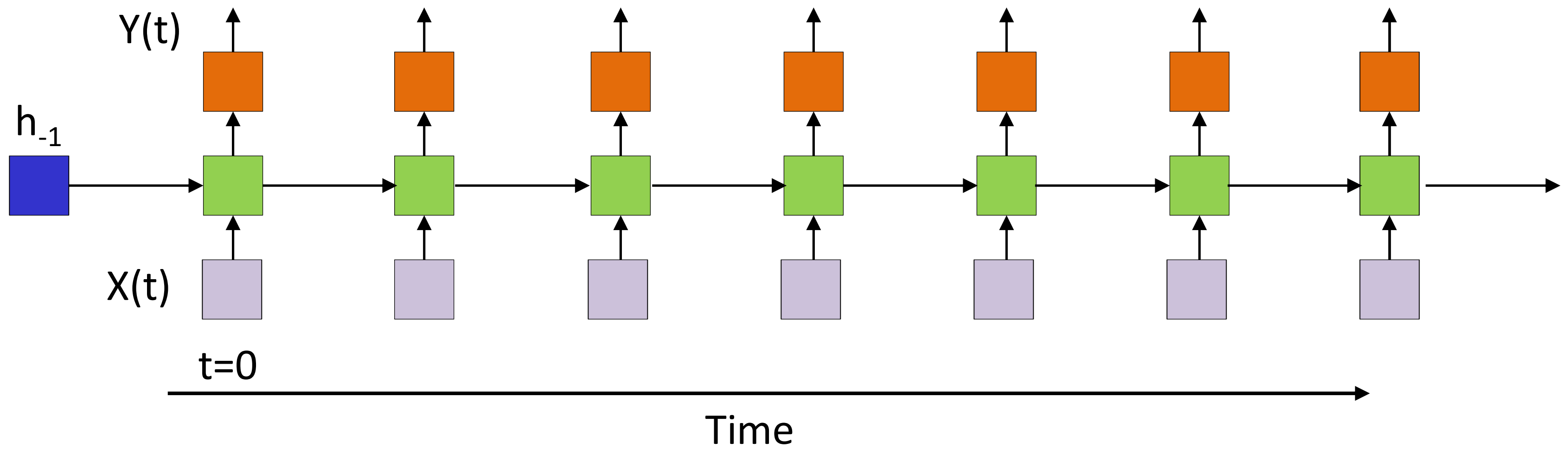
- When will the output “blow up”?

"BIBO" Stability



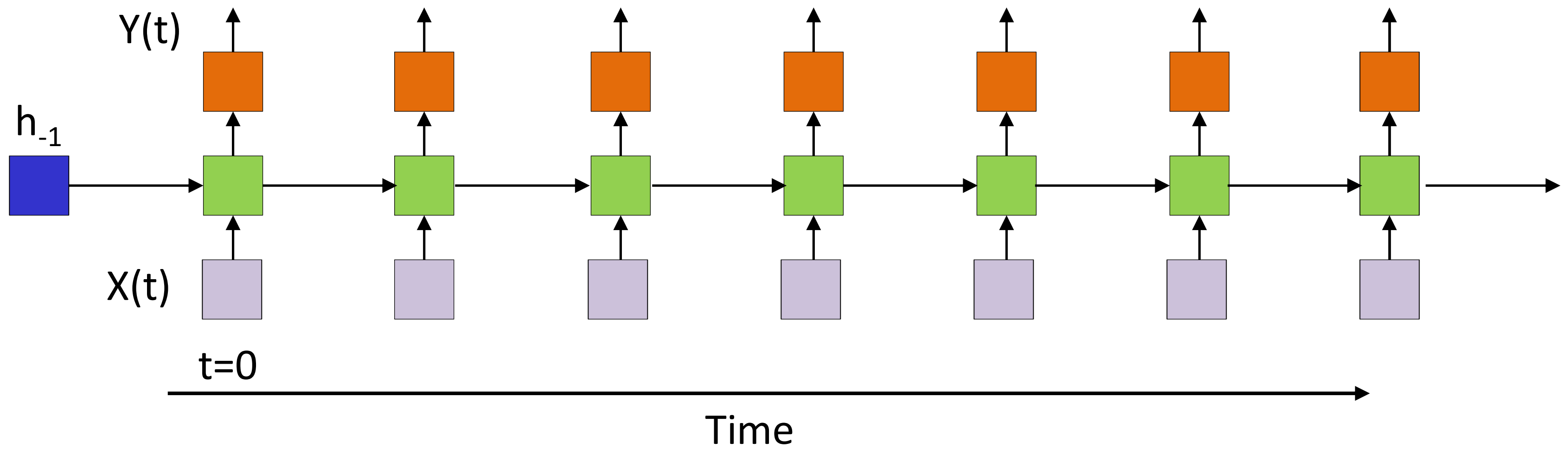
- Time-delay structures have bounded output if
  - The function $f()$ has bounded output for bounded input
    - Which is true of almost every activation function
  - $X(t)$ is bounded
- "Bounded Input Bounded Output" stability
  - This is a highly desirable characteristic

Is this BIBO?



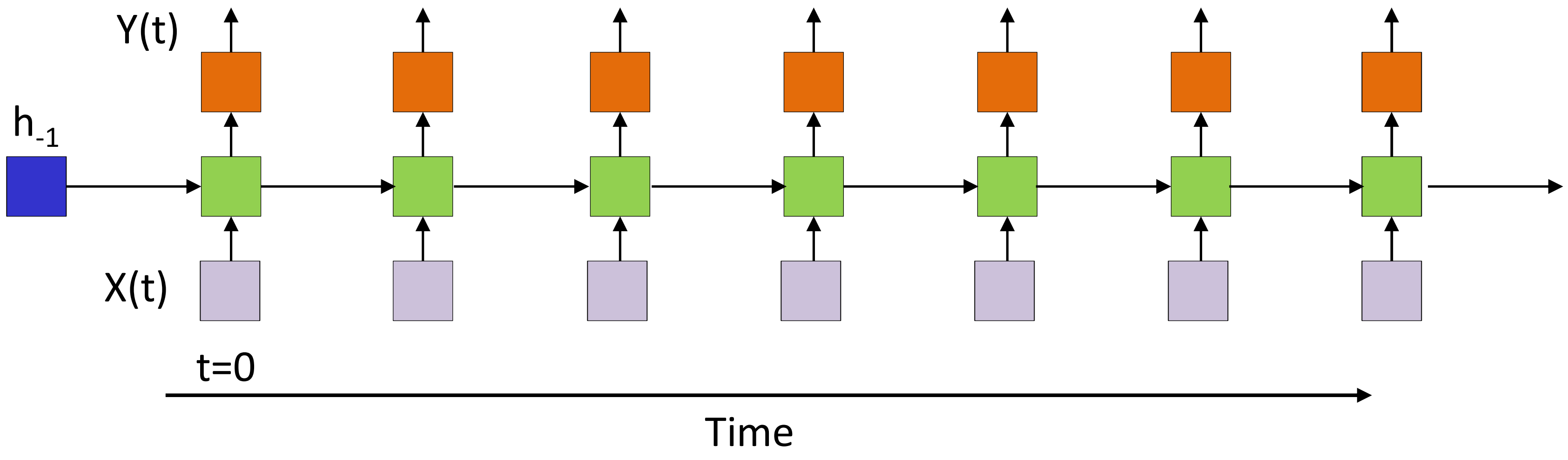
- Will this necessarily be BIBO?

Is this BIBO?



- Will this necessarily be BIBO?
  - Guaranteed if output and hidden activations are bounded
    - But will it *saturate* (and where)
  - What if the activations are linear?

Analyzing recurrence



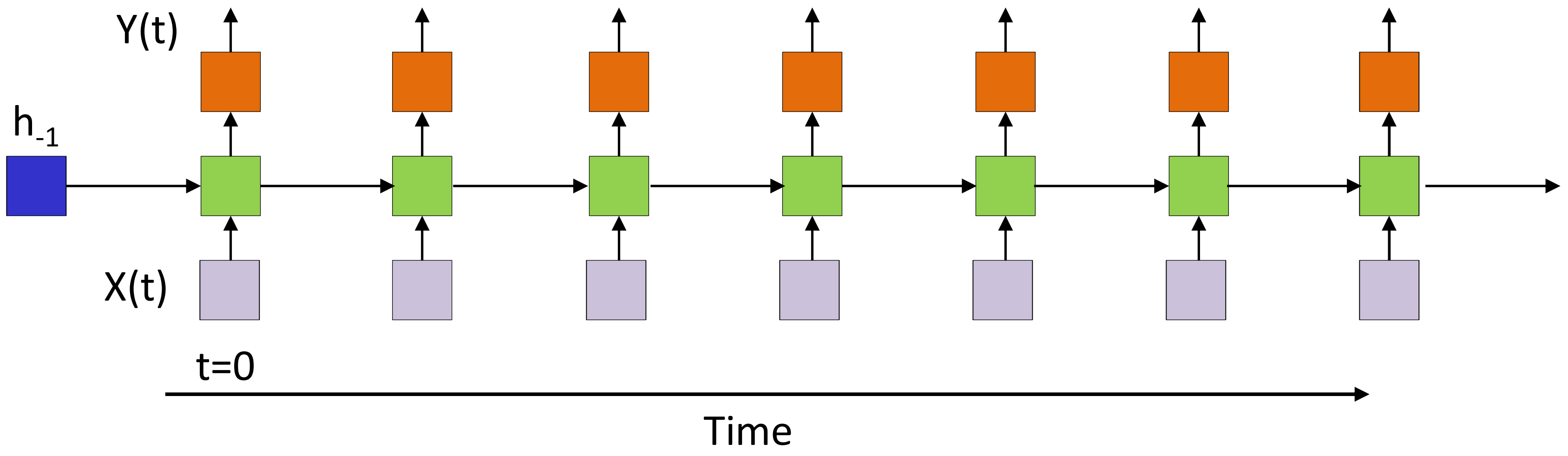
- Sufficient to analyze the behavior of the hidden layer h_k since it carries the relevant information
  – Will assume only a single hidden layer for simplicity

Analyzing Recursion

The streetlight effect is a type of observational bias where people only look for whatever they are searching by looking where it is easiest

"I'm searching for my keys."

Streetlight effect



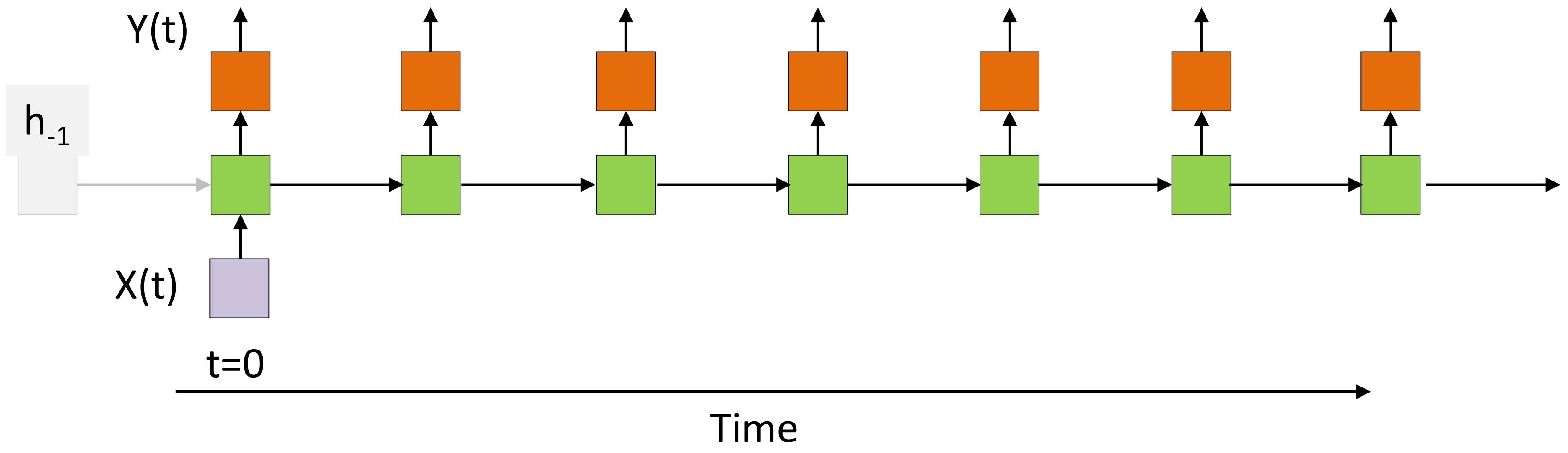
- Easier to analyze *linear* systems
  - Will attempt to extrapolate to non-linear systems subsequently
- All activations are identity functions
  - $z_k = W_h h_{k-1} + W_x x_k, \qquad h_k = z_k$

Linear systems

- $h_k = W_h h_{k-1} + W_x x_k$
  - $h_{k-1} = W_h h_{k-2} + W_x x_{k-1}$
- $h_k = W_h^2 h_{k-2} + W_h W_x x_{k-1} + W_x x_k$
- $h_k = W_h^{k+1} h_{-1} + W_h^k W_x x_0 + W_h^{k-1} W_x x_1 + W_h^{k-2} W_x x_2 + \cdots$
- $h_k = H_k(h_{-1}) + H_k(x_0) + H_k(x_1) + H_k(x_2) + \cdots$
  - $= h_{-1} H_k(1_{-1}) + x_0 H_k(1_0) + x_1 H_k(1_1) + x_2 H_k(1_2) + \cdots$

- Where $H_k(1_t)$ is the hidden response at time k when the input is $[0\ 0\ 0\ \ \ldots 1\ 0\ ..0]$ (where the 1 occurs in the t-th position) with 0 initial condition
  - The initial condition may be viewed as an input of h_{-1} at $t = -1$

Streetlight effect

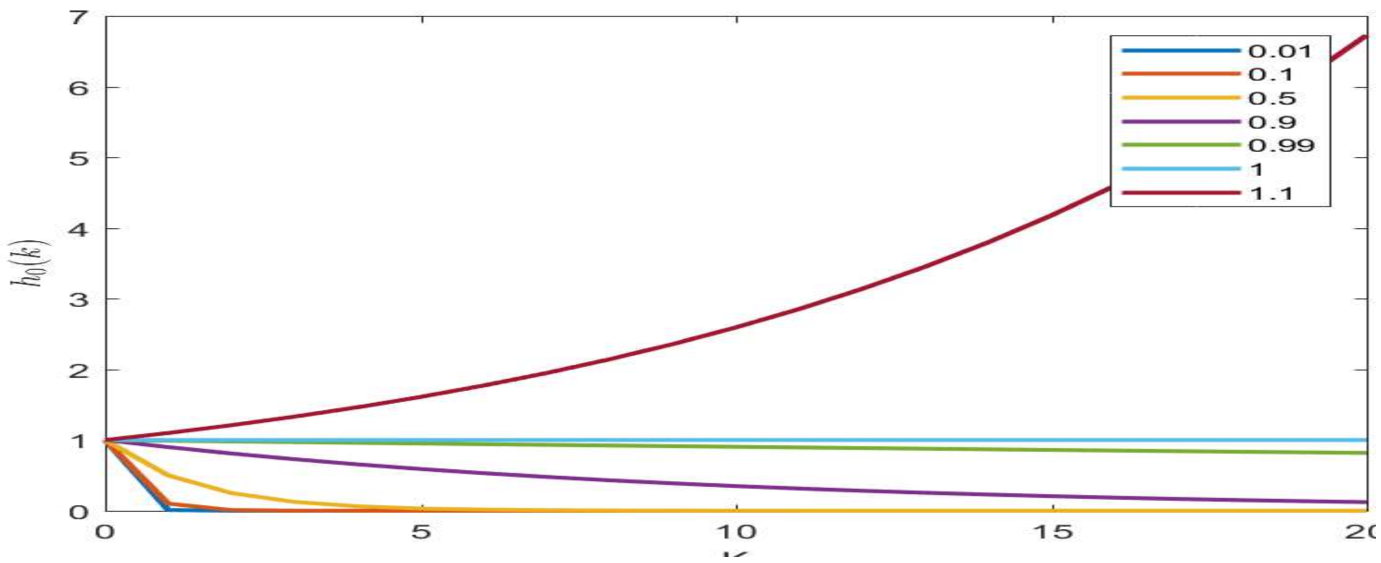


- Sufficient to analyze the response to a single input at $t = 0$
  - Principle of superposition in linear systems:

$$h_k = h_{-1}H_k(1_{-1}) + x_0H_k(1_0) + x_1H_k(1_1) + x_2H_k(1_2) + \cdots$$

Linear recursions

- Consider simple, scalar, linear recursion (note change of notation)
  - $h(t) = wh(t-1) + cx(t)$
  - $h_0(t) = w^t cx(0)$
    - Response to a single input at 0



Linear recursions: Vector version

- Vector linear recursion (note change of notation)
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = W^t Cx(0)$
    - Length of response vector to a single input at 0 is $|h_0(t)|$

- We can write $W = U\Lambda U^{-1}$
  - $Wu_i = \lambda_i u_i$
  - For any vector h we can write
    - $h = a_1u_1 + a_2u_2 + \cdots + a_nu_n$
    - $Wh = a_1\lambda_1u_1 + a_2\lambda_2u_2 + \cdots + a_n\lambda_nu_n$
    - $W^th = a_1\lambda_1^tu_1 + a_2\lambda_2^tu_2 + \cdots + a_n\lambda_n^tu_n$
  - $\lim_{t\to\infty}|W^th| = a_m\lambda_m^tu_m$ where $m = \operatorname{argmax}_j \lambda_j$

Linear recursions: Vector version

- Vector linear recursion (note change of notation)
  - $h(t) = Wh(t-1) + Cx(t)$
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- We can write $W = U\Lambda U^{-1}$
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For any input, for large t the length of the hidden vector will expand or contract according to the t −th power of the largest eigen value of the hidden-layer weight matrix

    - $W^t h = a_1\lambda_1^t u_1 + a_2\lambda_2^t u_2 + \cdots + a_n\lambda_n^t u_n$
  - $\lim_{t\to\infty} |W^t h| = a_m\lambda_m^t u_m$ where $m = \operatorname{argmax}_j \lambda_j$

Linear recursions: Vector version

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    - Length of response vector to a single input at 0 is $|h_0(t)|$

For any input, for large t the length of the hidden vector will expand or contract according to the t –th power of the largest eigen value of the hidden-layer weight matrix

Unless it has no component along the eigen vector corresponding to the largest eigen value. In that case it will grow according to the *second* largest Eigen value..

And so on..

    - $Wh = a_1\lambda_1 u_1 + a_2\lambda_2 u_2 + \cdots + a_n\lambda_n u_n$
    - $W^t h = a_1\lambda_1^t u_1 + a_2\lambda_2^t u_2 + \cdots + a_n\lambda_n^t u_n$
  - $\lim_{t\to\infty} |W^t h| = a_m\lambda_m^t u_m$ where $m = \operatorname{argmax}_j \lambda_j$

Linear recursions: Vector version

- Vector linear recursion (note change of notation)

If $|\lambda_{max}| > 1$ it will blow up, otherwise it will contract and shrink to 0 rapidly

  - Length of response vector to a single input at 0 is $|h_0(t)|$

For any input, for large t the length of the hidden vector will expand or contract according to the t th power of the largest eigen value of the hidden-layer weight matrix

Unless it has no component along the eigen vector corresponding to the largest eigen value. In that case it will grow according to the *second* largest Eigen value..

And so on..

  - $W^t h = a_1\lambda_1^t u_1 + a_2\lambda_2^t u_2 + \cdots + a_n\lambda_n^t u_n$

– $\lim_{t\to\infty} |W^t h| = a_m\lambda_m^t u_m$ where $m = \underset{j}{\operatorname{argmax}}\, \lambda_j$

Linear recursions: Vector version

What about at middling values of t? It will depend on the other eigen values

of notation)

If $|\lambda_{max}| > 1$ it will blow up, otherwise it will contract and shrink to 0 rapidly

For any input, for large t the length of the hidden vector will expand or contract according to the t th power of the largest eigen value of the hidden-layer weight matrix

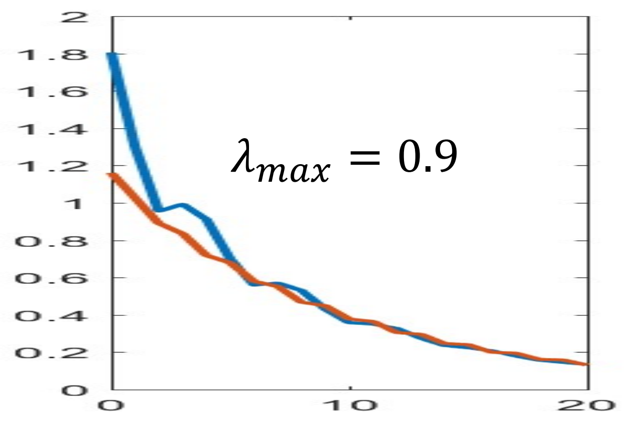
Unless it has no component along the eigen vector corresponding to the largest eigen value. In that case it will grow according to the *second* largest Eigen value..

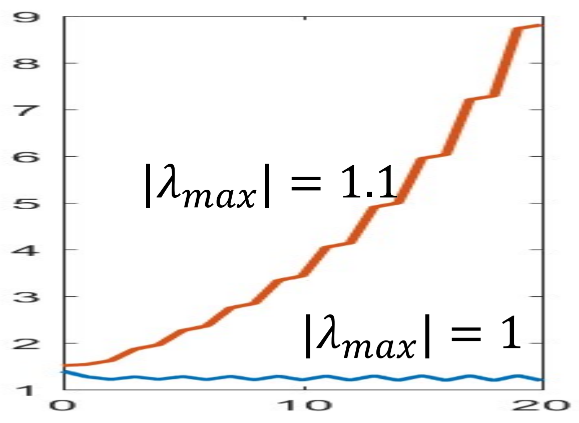
And so on..

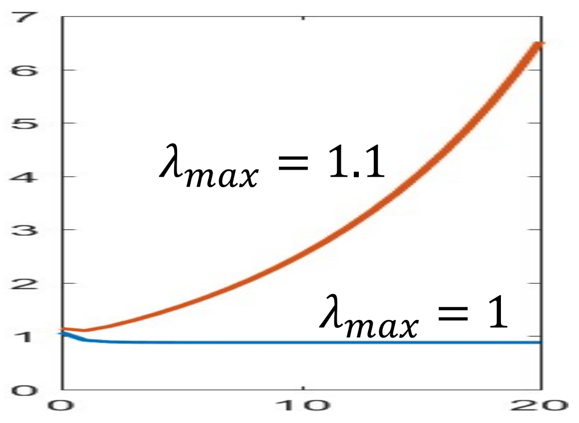
- $W^t h = a_1 \lambda_1^t u_1 + a_2 \lambda_2^t u_2 + \cdots + a_n \lambda_n^t u_n$
  - $\lim_{t\to\infty} |W^t h| = a_m \lambda_m^t u_m$ where $m = \underset{j}{\operatorname{argmax}}\, \lambda_j$

Linear recursions

- Vector linear recursion
  - $h(t) = Wh(t-1) + Cx(t)$
  - $h_0(t) = w^t cx(0)$
    - Response to a single input [1 1 1 1] at 0







Linear recursions

- Vector linear recursion
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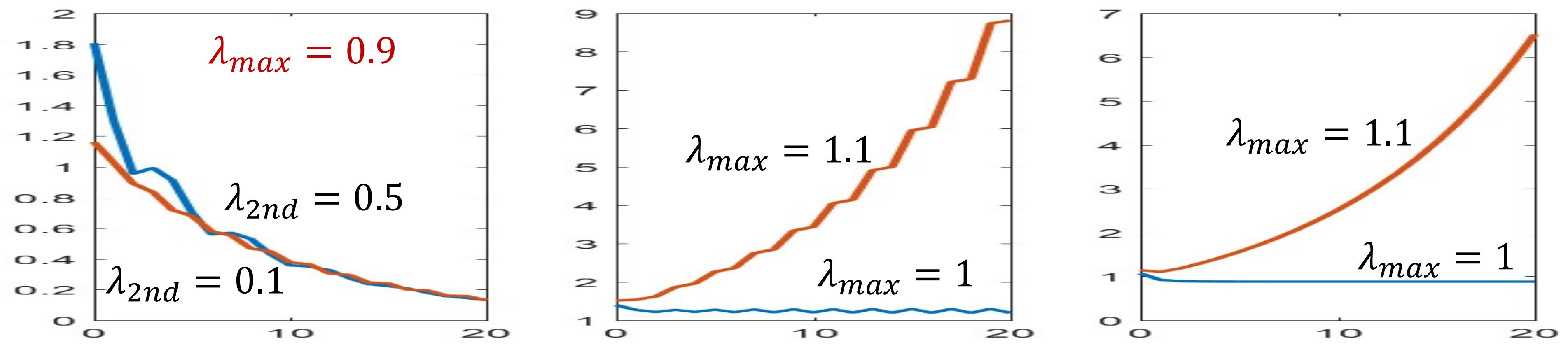


Complex Eigenvalues

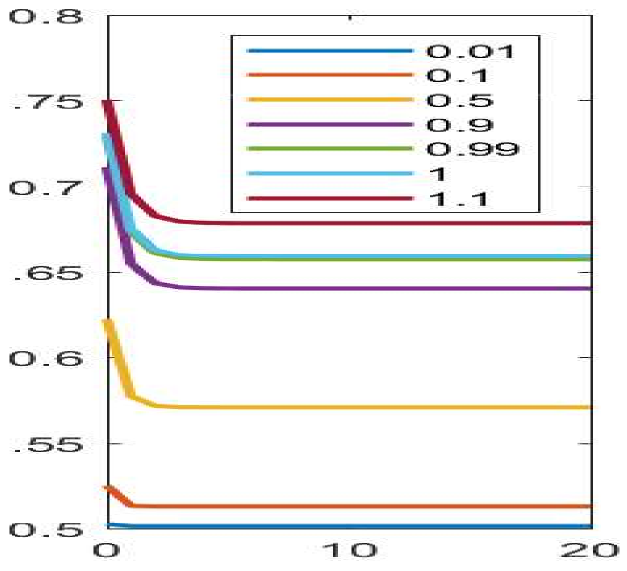
Lesson..

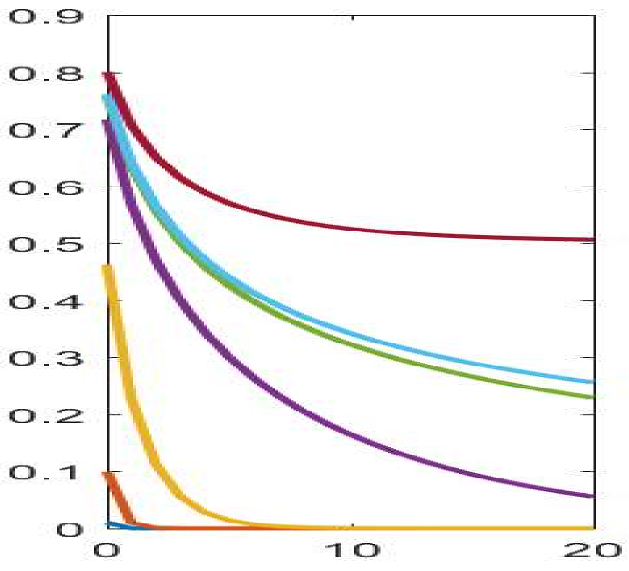
- In linear systems, long-term behavior depends entirely on the eigenvalues of the hidden-layer weights matrix
  - If the largest Eigen value is greater than 1, the system will “blow up”
  - If it is lesser than 1, the response will “vanish” very quickly
  - Complex Eigen values cause oscillatory response
    - Which we may or may not want
    - For smooth behavior, must force the weights matrix to have real Eigen values
      - Symmetric weight matrix

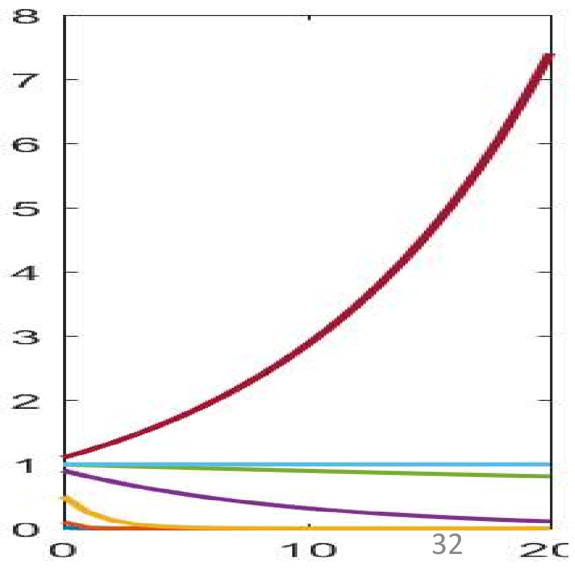
How about non-linearities (scalar)

$$h(t) = f(wh(t-1) + cx(t))$$

- The behavior of scalar non-linearities
- Left: Sigmoid, Middle: Tanh, Right: Relu
  - Sigmoid: Saturates in a limited number of steps, regardless of w
  - Tanh: Sensitive to w, but eventually saturates
    - "Prefers" weights close to 1.0
  - Relu: Sensitive to w, can blow up



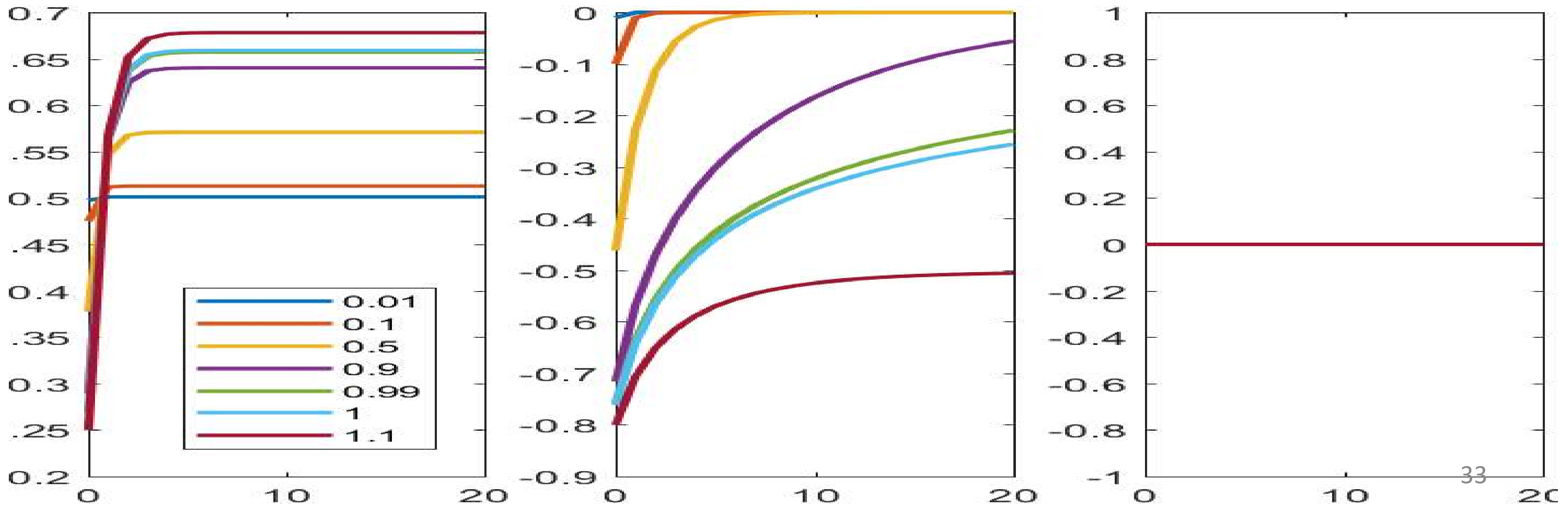




How about non-linearities **(scalar)**

$$h(t) = f(wh(t-1) + cx(t))$$

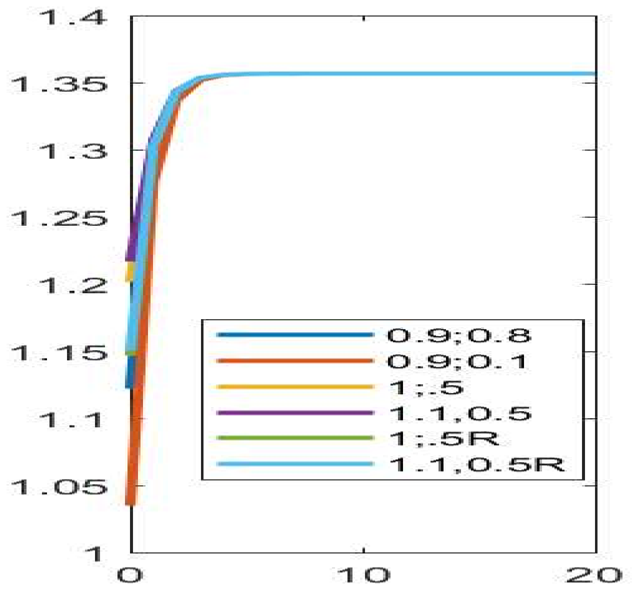
- With a negative start
- Left: Sigmoid, Middle: Tanh, Right: Relu
  - Sigmoid: Saturates in a limited number of steps, regardless of w
  - Tanh: Sensitive to w, but eventually saturates
  - Relu: For negative starts, has no response



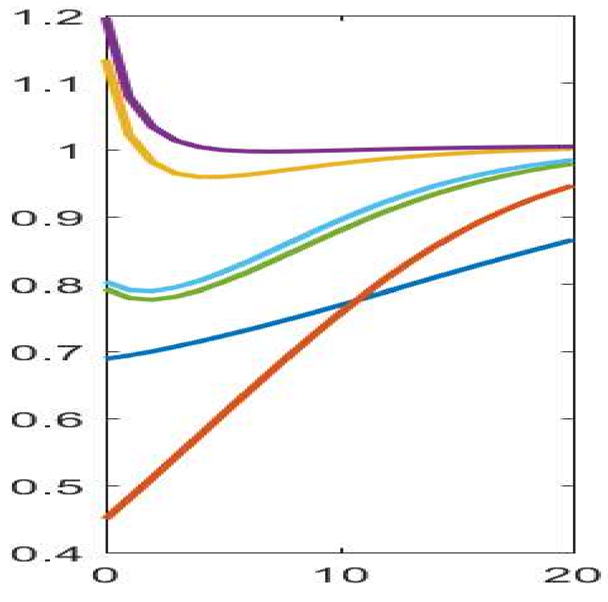
Vector Process

$$h(t) = f(Wh(t-1) + Cx(t))$$

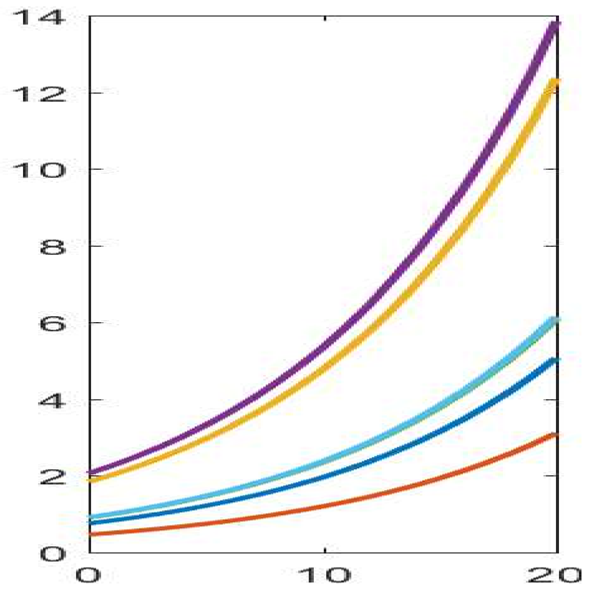
- Assuming a uniform unit vector initialization
  - $[1,1,1,\ldots]/\sqrt{N}$
  - Behavior similar to scalar recursion
- Eigenvalues less than 1.0 retain the most "memory"



sigmoid



tanh

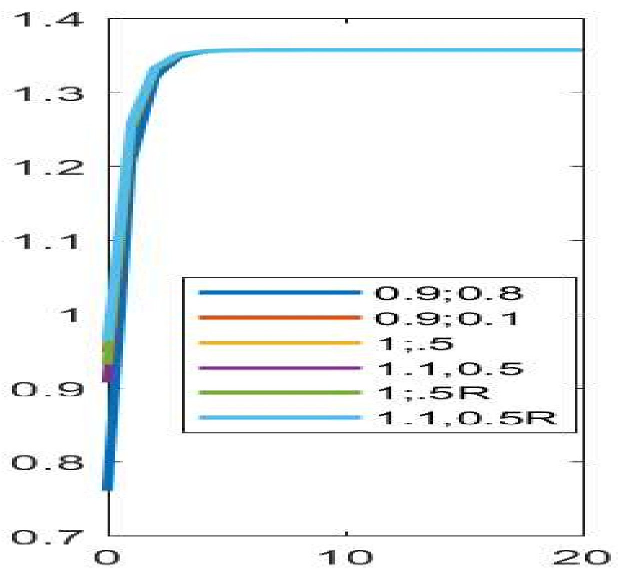


relu

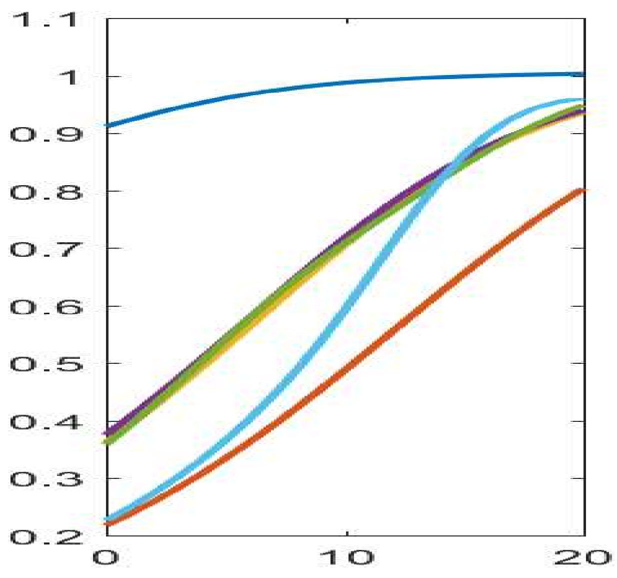
Vector Process

$$h(t) = f(Wh(t-1) + Cx(t))$$

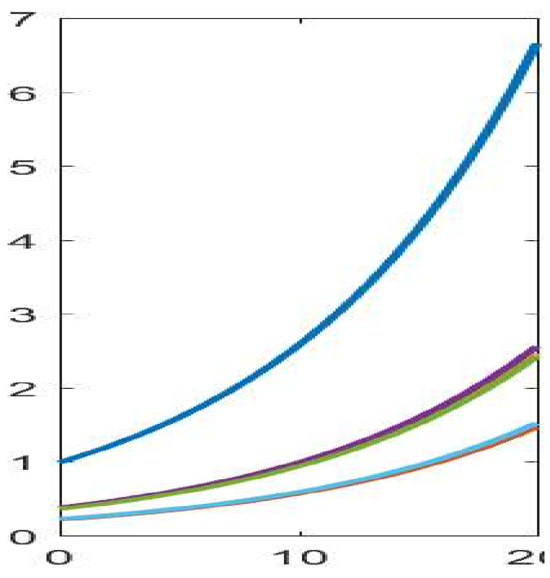
- Assuming a uniform unit vector initialization
  - $[-1, -1, -1, \ldots]/\sqrt{N}$
  - Behavior similar to scalar recursion



sigmoid



tanh



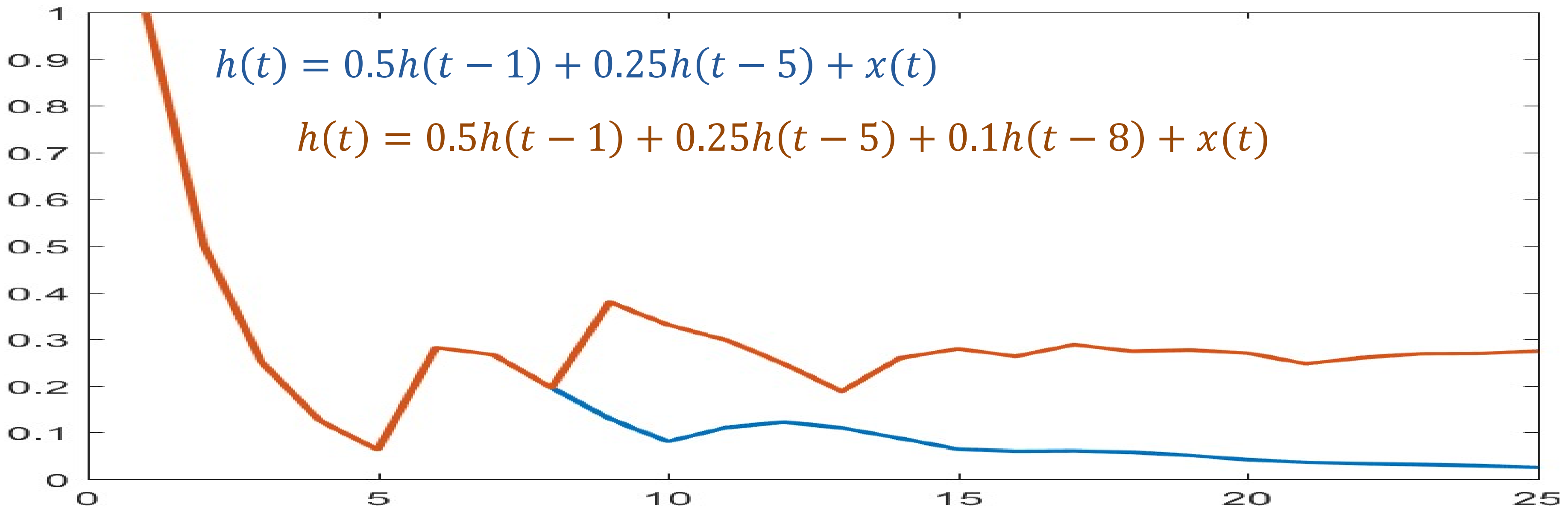
relu

Stability Analysis

- Formal stability analysis considers convergence of "Lyapunov" functions
  - Alternately, Routh's criterion and/or pole-zero analysis
  - Positive definite functions evaluated at h
  - Conclusions are similar: only the $\tanh$ activation gives us any reasonable behavior
    - And still has very short "memory"
- Lessons:
  - Bipolar activations (e.g. tanh) have the best memory behavior
  - Still sensitive to Eigenvalues of W
  - Best case memory is short
  - *Exponential memory behavior*
    - *"Forgets" in exponential manner*

How about deeper recursion

- Consider simple, scalar, linear recursion
  - Adding more "taps" adds more "modes" to memory in somewhat non-obvious ways



Stability Analysis

- Similar analysis of vector functions with non-linear activations is relatively straightforward
  - *Linear systems:* Routh's criterion
    - And pole-zero analysis (involves tensors)
      - On board?
  - Non-linear systems: Lyapunov functions
- Conclusions do not change

Story so far

- Recurrent networks retain information from the infinite past in principle
- In practice, they tend to blow up or forget
  - If the largest Eigen value of the recurrent weights matrix is greater than 1, the network response may blow up
  - If its less than one, the response dies down very quickly
- The "memory" of the network also depends on the activation of the hidden units
  - Sigmoid activations saturate and the network becomes unable to retain new information
  - RELU activations blow up or vanish rapidly
  - Tanh activations are the most effective at storing memory
    - But still, for not very long

RNNs..

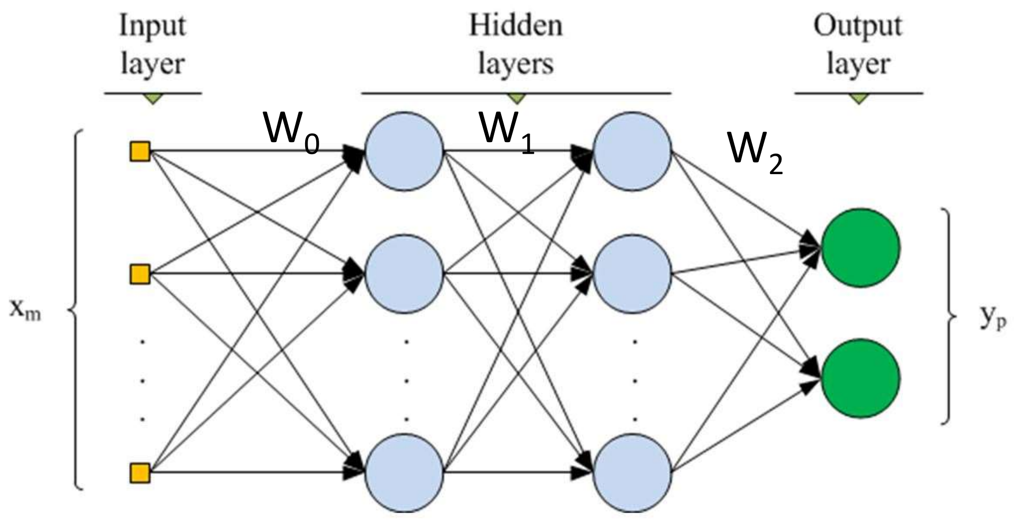
- Excellent models for time-series analysis tasks
  - Time-series prediction
  - Time-series classification
  - Sequence prediction..
  - They can even simplify problems that are difficult for MLPs

- But the memory isn’t all that great..
  - Also..

The vanishing gradient problem

- A particular problem with training deep networks..
  - (Any deep network, not just recurrent nets)
  - The gradient of the error with respect to weights is unstable..

Some useful preliminary math: The problem with training deep networks



- A multilayer perceptron is a nested function

$$Y = f_N\left(W_{N-1} f_{N-1}\left(W_{N-2} f_{N-2}(\dots W_0 X)\right)\right)$$

- W_k is the weights *matrix* at the kth layer
- The *error* for X can be written as

$$Div(X) \ = D\left(f_N\left(W_{N-1} f_{N-1}\left(W_{N-2} f_{N-2}(\dots W_0 X)\right)\right)\right)$$

Training deep networks

- Vector derivative chain rule: for any $f\big(Wg(X)\big)$:

$$\frac{df\big(Wg(X)\big)}{dX} = \frac{df\big(Wg(X)\big)}{dWg(X)}\frac{dWg(X)}{dg(X)}\frac{dg(X)}{dX}$$

Poor notation

$$\text{Let } Z = Wg(X)$$
$$\nabla_X f = \nabla_Z f . W . \nabla_X g$$

- Where
  - $\nabla_Z f$ is the *jacobian* ***matrix*** of $f(Z)$w.r.t Z
    - Using the notation $\nabla_Z f$ instead of $J_f(z)$ for consistency

Training deep networks

- For

$$Div(X) = D\left(f_N\left(W_{N-1}f_{N-1}\left(W_{N-2}f_{N-2}(\ldots W_0X)\right)\right)\right)$$

- We get:

$$\nabla_{f_k}Div = \nabla D.\nabla f_N.W_{N-1}.\nabla f_{N-1}.W_{N-2}\ldots\nabla f_{k+1}W_k$$

- Where
  - $\nabla_{f_k}Div$ is the gradient $Div(X)$ of the error w.r.t the output of the kth layer of the network
    - Needed to compute the gradient of the error w.r.t W_{k-1}
  - ∇f_n is *jacobian* of $f_N()$ w.r.t. to its current input
  - All blue terms are matrices
  - All function derivatives are w.r.t. the (entire, affine) argument of the function

Training deep networks

- For

$$Div(X) = D\left(f_N\left(W_{N-1}f_{N-1}\left(W_{N-2}f_{N-2}(\ldots W_0X)\right)\right)\right)$$

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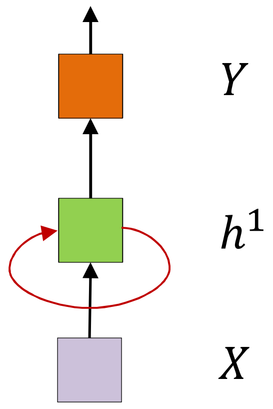
  - ∇f_n is *jacobian* of $f_N()$ w.r.t. to its current input
  - All blue terms are matrices

Lets consider these Jacobians for an RNN (or more generally for any network)

The Jacobian of the hidden layers for an RNN



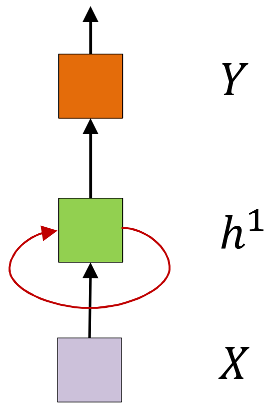
$$\nabla f_t(z_i) = \begin{bmatrix} f'_{t,1}(z_1) & 0 & \cdots & 0 \\ 0 & f'_{t,2}(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'_{t,N}(z_N) \end{bmatrix}$$

$$h_i^{(1)}(t) = f_1\left(z_i^{(1)}(t)\right)$$

- $\nabla f_t()$ is the derivative of the output of the (layer of) hidden recurrent neurons with respect to their input
  - For vector activations: A full matrix
  - For scalar activations: A matrix where the diagonal entries are the derivatives of the *activation* of the recurrent hidden layer

$$h_i^{(1)}(t) = f_1\left(z_i^{(1)}(t)\right)$$

The Jacobian

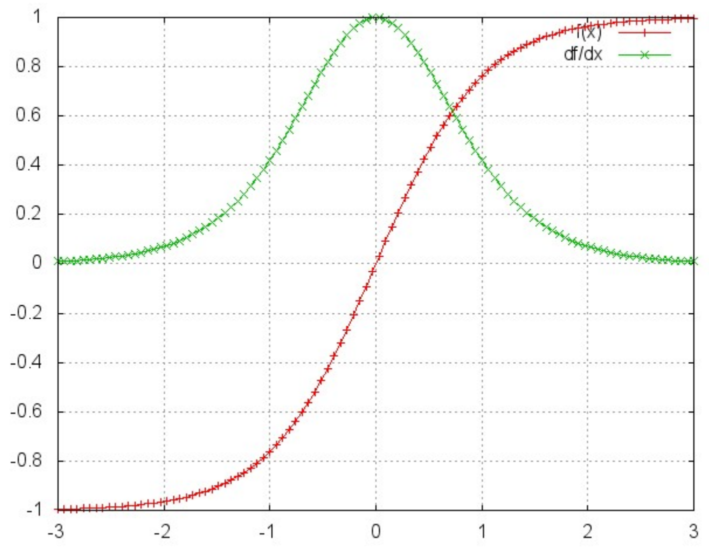


$$\nabla f_t(z_i) = \begin{bmatrix} f'_{t,1}(z_1) & 0 & \cdots & 0 \\ 0 & f'_{t,2}(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'_{t,N}(z_N) \end{bmatrix}$$

- The derivative (or subgradient) of the activation function is always bounded
  – The diagonals (or singular values) of the Jacobian are bounded
- There is a limit on how much multiplying a vector by the Jacobian will scale it

The derivative of the hidden state activation

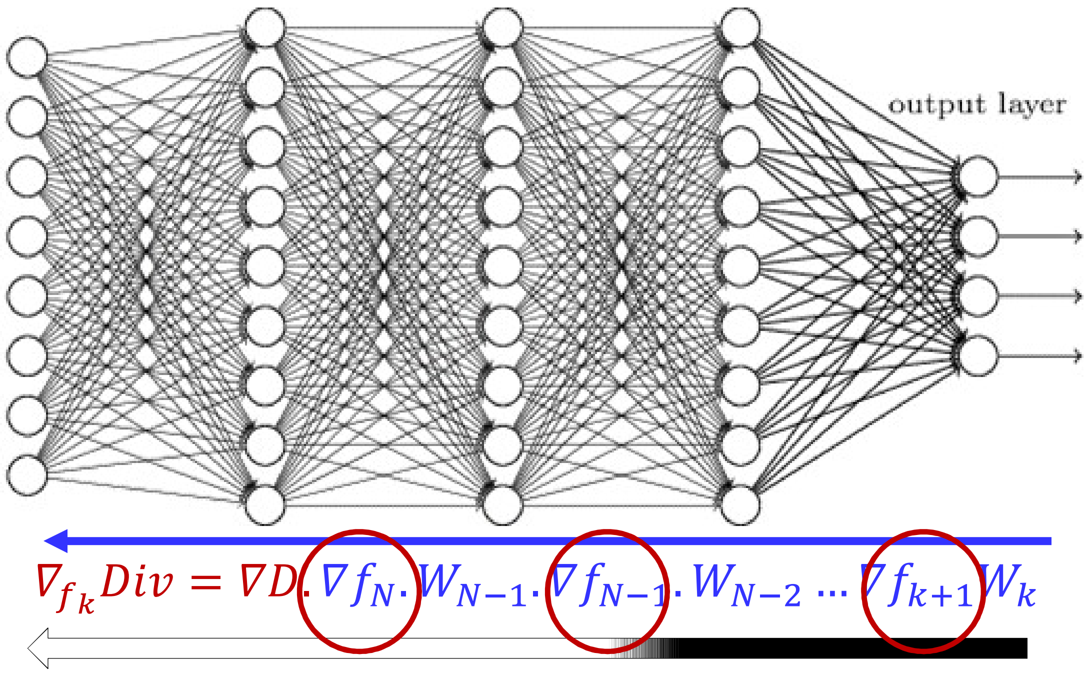
$$\nabla f_t(z_i) = \begin{bmatrix} f'_{t,1}(z_1) & 0 & \cdots & 0 \\ 0 & f'_{t,2}(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f'_{t,N}(z_N) \end{bmatrix}$$



- Most common activation functions, such as sigmoid, tanh() and RELU have derivatives that are always less than 1
- The most common activation for the hidden units in an RNN is the tanh()
  - The derivative of $\tanh()$is never greater than 1 (and mostly less than 1)

- **Multiplication by the Jacobian is always a *shrinking* operation**

Training deep networks



$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \dots \nabla f_{k+1} W_k$$

- As we go back in layers, the Jacobians of the activations constantly *shrink* the derivative
  - After a few layers the derivative of the divergence at any time is totally “forgotten”

What about the weights

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . \boxed{W_{N-1}} . \nabla f_{N-1} . \boxed{W_{N-2}} \ldots \nabla f_{k+1} \boxed{W_k}$$

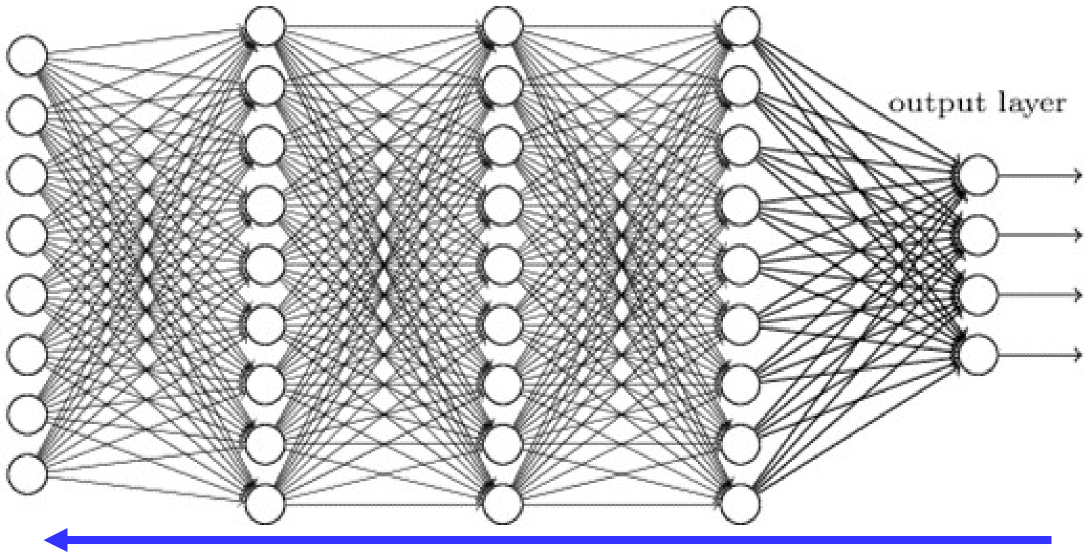
- In a single-layer RNN, the weight matrices are identical
  - The conclusion below holds for any deep network, though
- The chain product for $\nabla_{f_k} Div$ will
  - *Expand* ∇D along directions in which the singular values of the weight matrices are greater than 1
  - *Shrink* ∇D in directions where the singular values are less than 1
  - Repeated multiplication by the weights matrix will result in **Exploding** or **vanishing** gradients

Exploding/Vanishing gradients

$$\nabla_{f_k} Div = \nabla D . \underline{\nabla f_N . W_{N-1}} . \underline{\nabla f_{N-1} . W_{N-2}} \ldots \underline{\nabla f_{k+1} W_k}$$

- Every blue term is a matrix
- ∇D is proportional to the actual error
  - Particularly for L_2 and KL divergence
- The chain product for $\nabla_{f_k} Div$ will
  - *Expand* ∇D in directions where each stage has singular values greater than 1
  - *Shrink* ∇D in directions where each stage has singular values less than 1

Gradient problems in deep networks

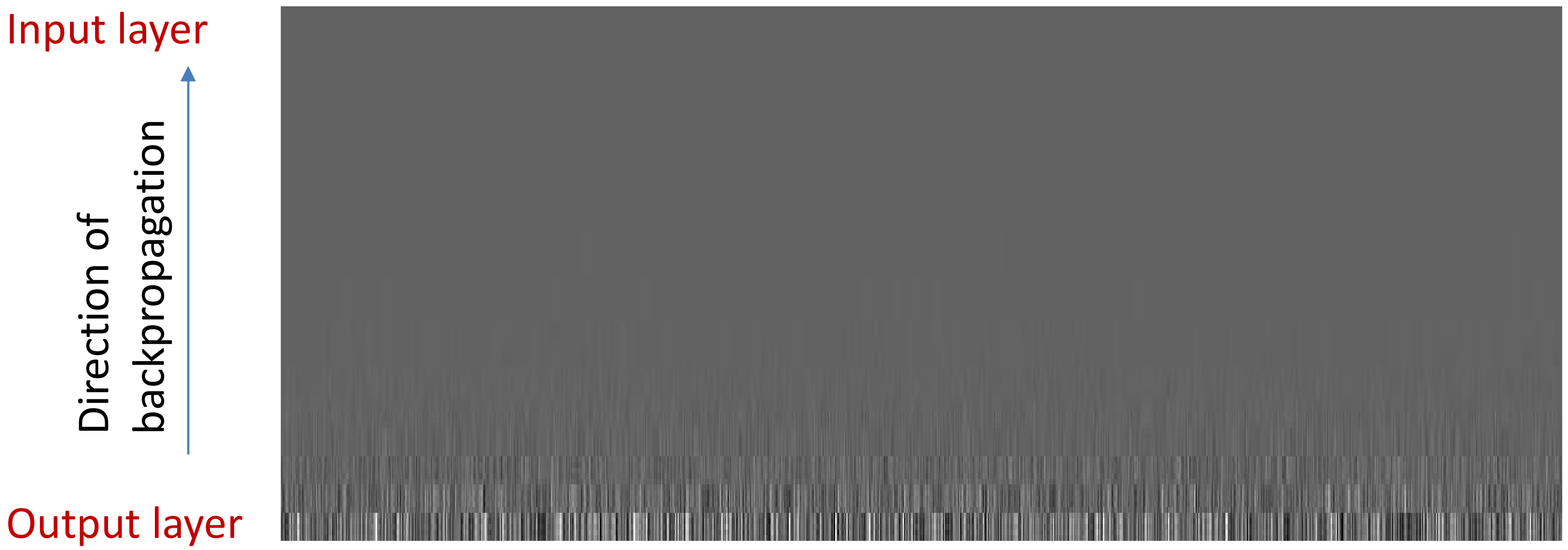


$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \ldots \nabla f_{k+1} W_k$$

- The gradients in the lower/earlier layers can *explode* or *vanish*
  - Resulting in insignificant or unstable gradient descent updates
  - Problem gets worse as network depth increases

Vanishing gradient examples..

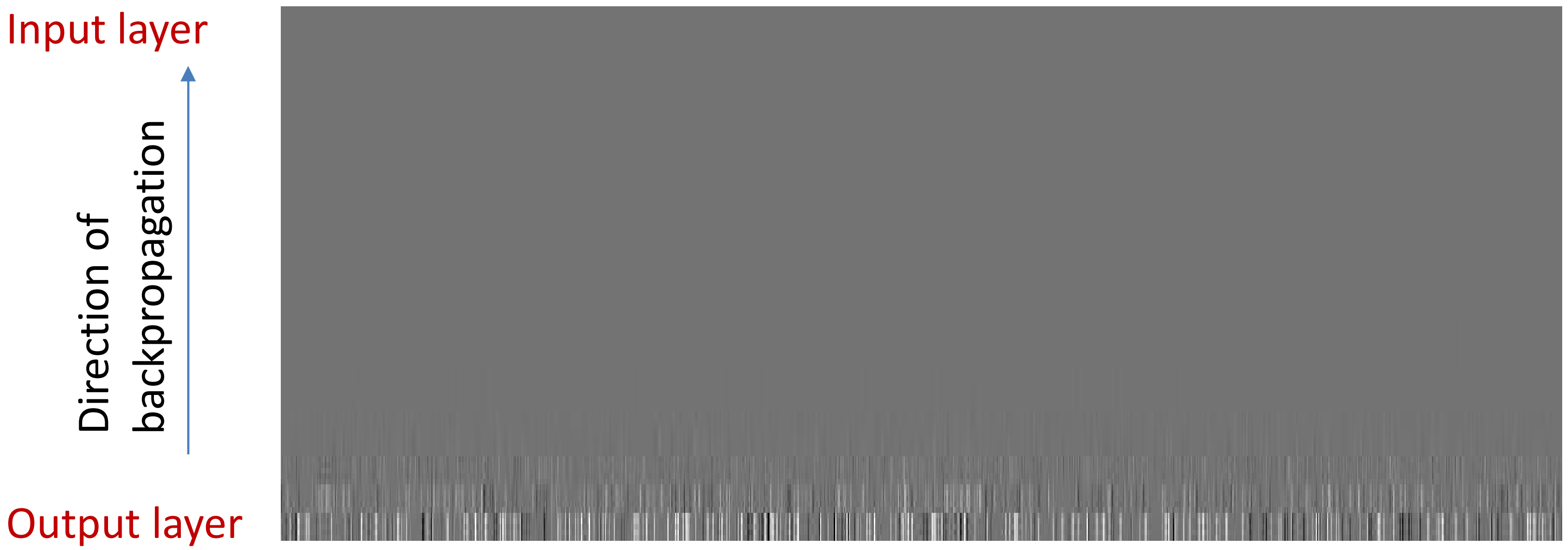
ELU activation, Batch gradients



- 19 layer MNIST model
  - Different activations: Exponential linear units, RELU, sigmoid, tanh
  - Each layer is 1024 units wide
  - Gradients shown at initialization
    - Will actually *decrease* with additional training
- Figure shows $\log\left|\nabla_{W_{neuron}} Div\right|$ where W_{neuron} is the vector of incoming weights to each neuron
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradient examples..

RELU activation, Batch gradients



- 19 layer MNIST model
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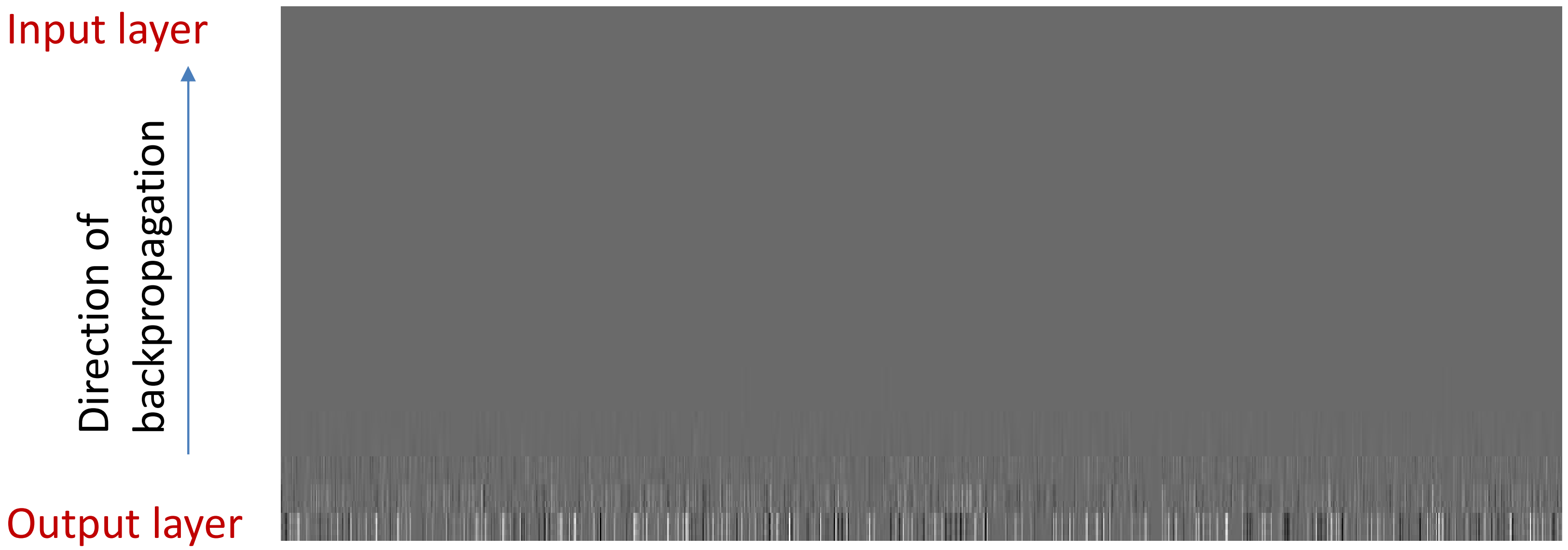
- Figure shows $\log\left|\nabla_{W_{neuron}} Div\right|$ where W_{neuron} is the vector of incoming weights to each neuron
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

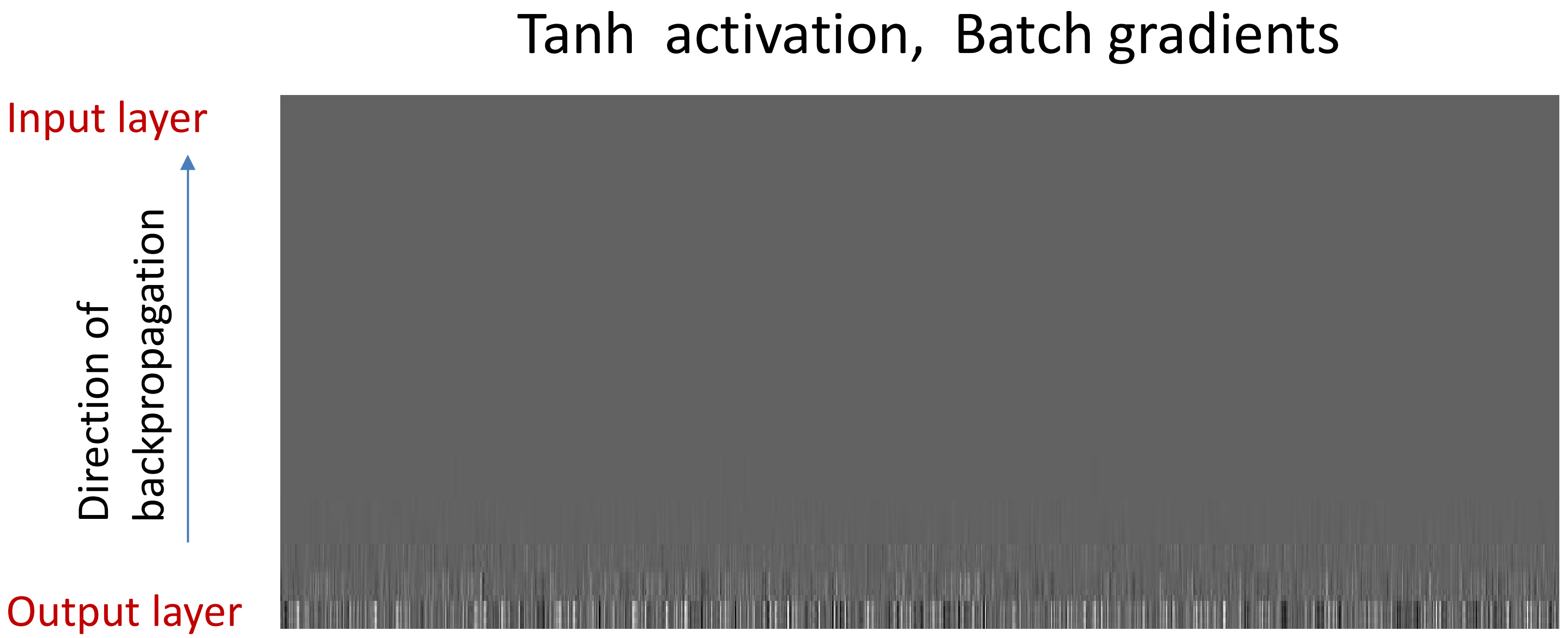
Vanishing gradient examples..

Sigmoid activation, Batch gradients



- 19 layer MNIST model
  - Different activations: Exponential linear units, RELU, sigmoid, tanh
  - Each layer is 1024 units wide
  - Gradients shown at initialization
    - Will actually *decrease* with additional training
- Figure shows $\log\left|\nabla_{W_{neuron}} Div\right|$ where W_{neuron} is the vector of incoming weights to each neuron
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradient examples..



- 19 layer MNIST model
  - Different activations: Exponential linear units, RELU, sigmoid, tanh
  - Each layer is 1024 units wide
  - Gradients shown at initialization
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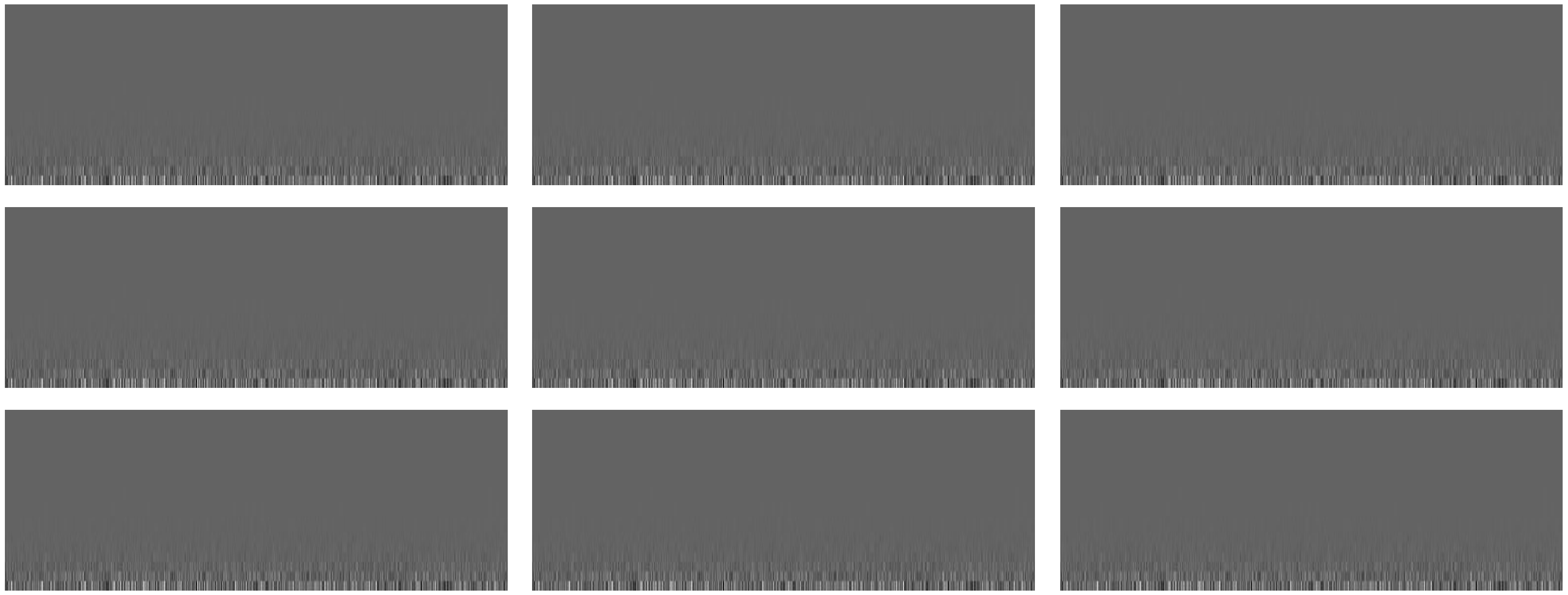
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradient examples..

ELU activation, Individual instances



- 19 layer MNIST model
  - Different activations: Exponential linear units, RELU, sigmoid, tanh
  - Each layer is 1024 units wide
  - Gradients shown at initialization
    - Will actually *decrease* with additional training
- Figure shows $\log\left|\nabla_{W_{neuron}} Div\right|$ where W_{neuron} is the vector of incoming weights to each neuron
  - I.e. the gradient of the loss w.r.t. the entire set of weights to each neuron

Vanishing gradients

- ELU activations maintain gradients longest
- But in all cases gradients effectively vanish after about 10 layers!
  - Your results may vary

- Both batch gradients and gradients for individual instances disappear
  - In reality a tiny number will actually blow up.

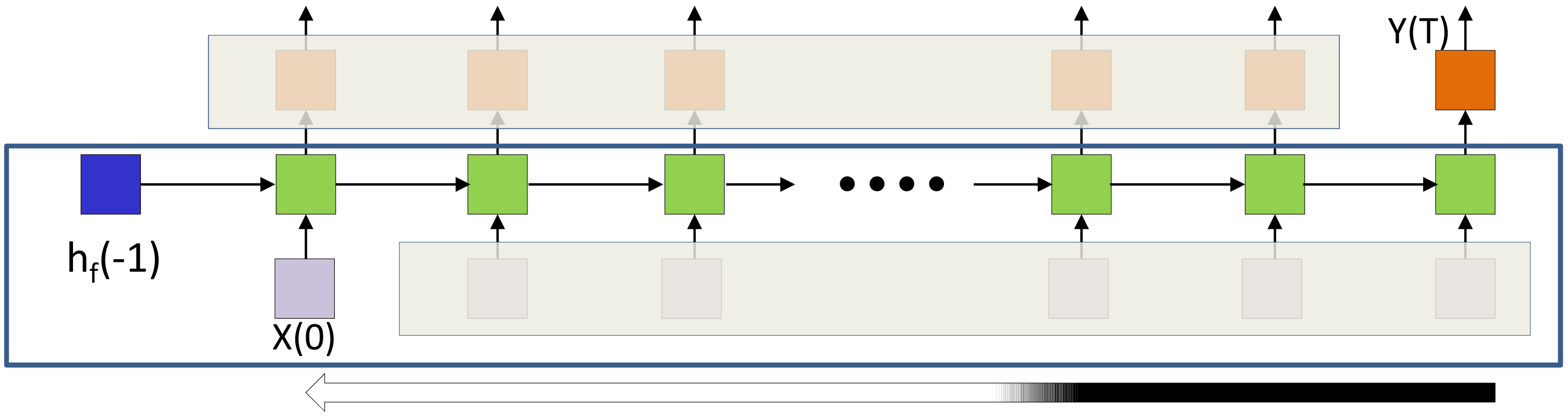
Story so far

- Recurrent networks retain information from the infinite past in principle

- In practice, they are poor at memorization
  - The hidden outputs can blow up, or shrink to zero depending on the Eigen values of the recurrent weights matrix
  - The memory is also a function of the activation of the hidden units
    - Tanh activations are the most effective at retaining memory, but even they don’t hold it very long

- Deep networks also suffer from a “vanishing or exploding gradient” problem
  - The gradient of the error at the output gets concentrated into a small number of parameters in the earlier layers, and goes to zero for others

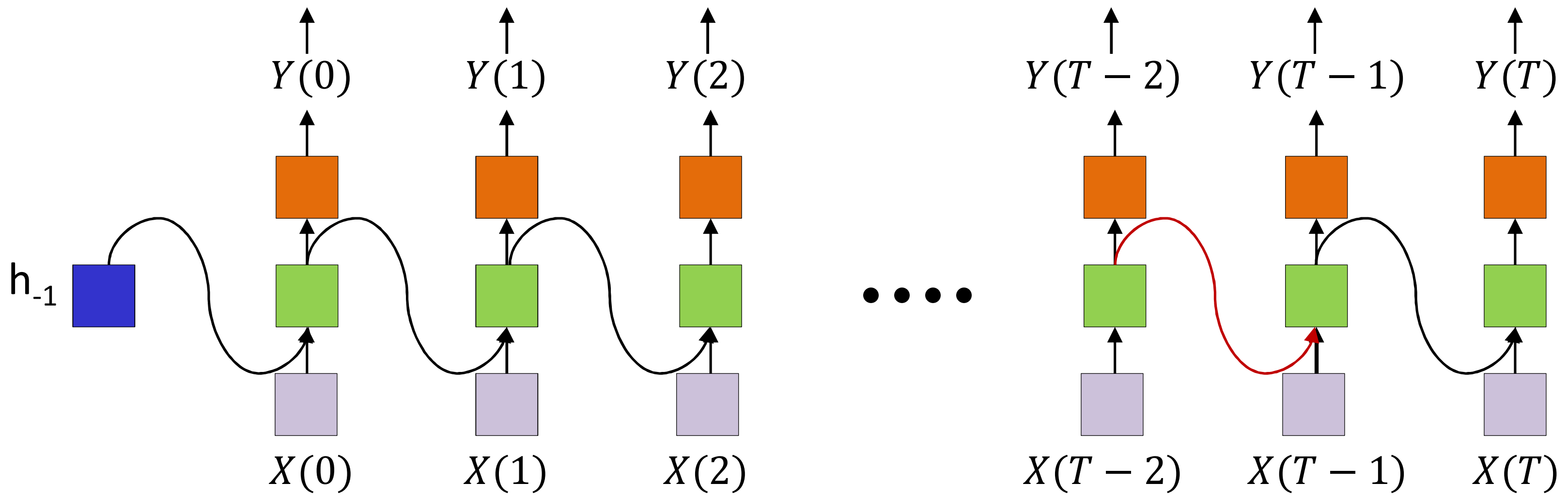
Recurrent nets are very deep nets

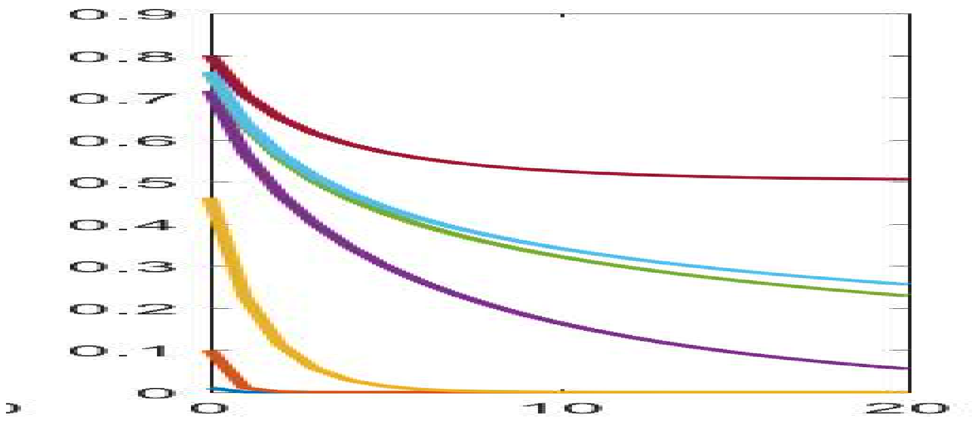


$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \ldots \nabla f_{k+1} W_k$$

- The relation between $X(0)$ and $Y(T)$ is one of a very deep network
  - Gradients from errors at t $= T$ will vanish by the time they're propagated to $t = 0$

Recall: Vanishing stuff..





- Stuff gets forgotten in the forward pass too
  - Each weights matrix and activation can shrink components of the input

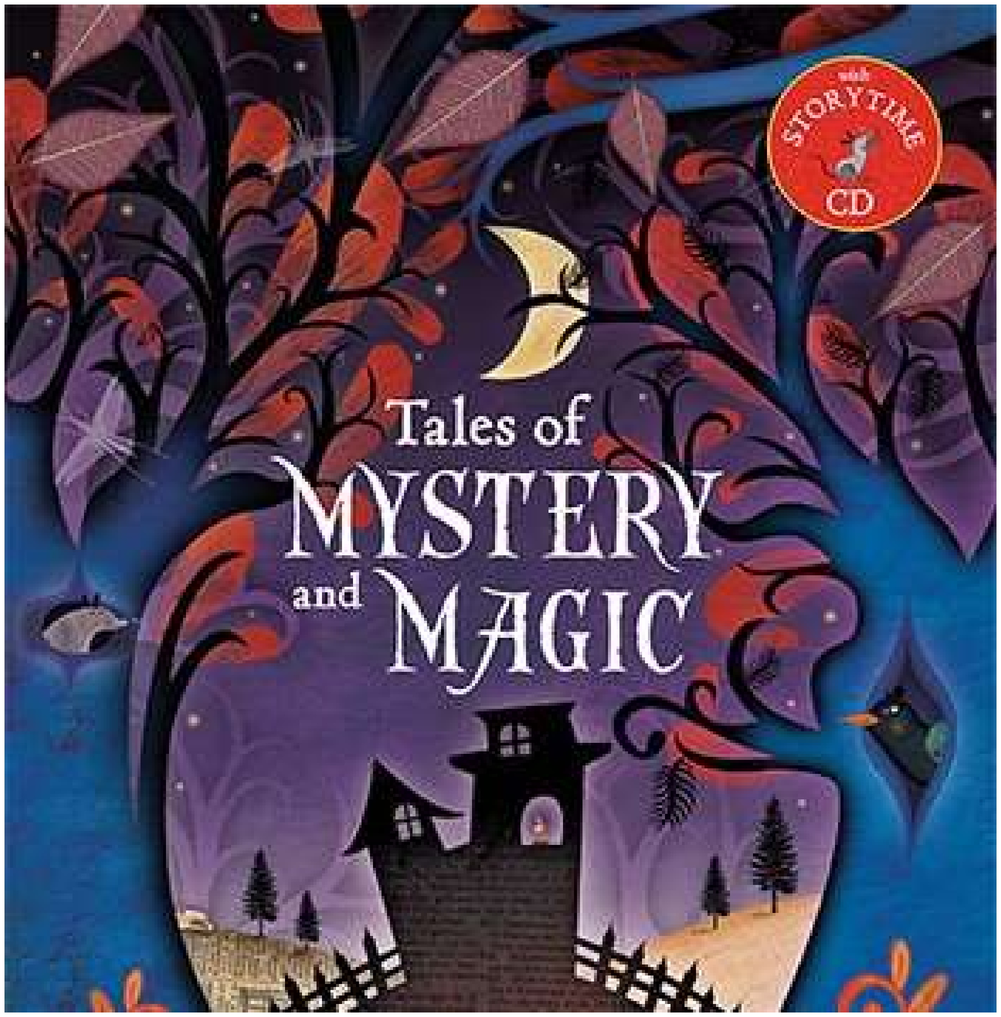
The long-term dependency problem

PATTERN1 [……………………….....] PATTERN 2

Jane had a quick lunch in the bistro. Then *she..*

- Any other pattern of any length can happen between pattern 1 and pattern 2
  - RNN will "forget" pattern 1 if intermediate stuff is too long
  - "Jane" → the next pronoun referring to her will be "she"
- Must know to "remember" for extended periods of time and "recall" when necessary
  - Can be performed with a multi-tap recursion, but how many taps?
  - Need an alternate way to "remember" stuff

And now we enter the domain of..



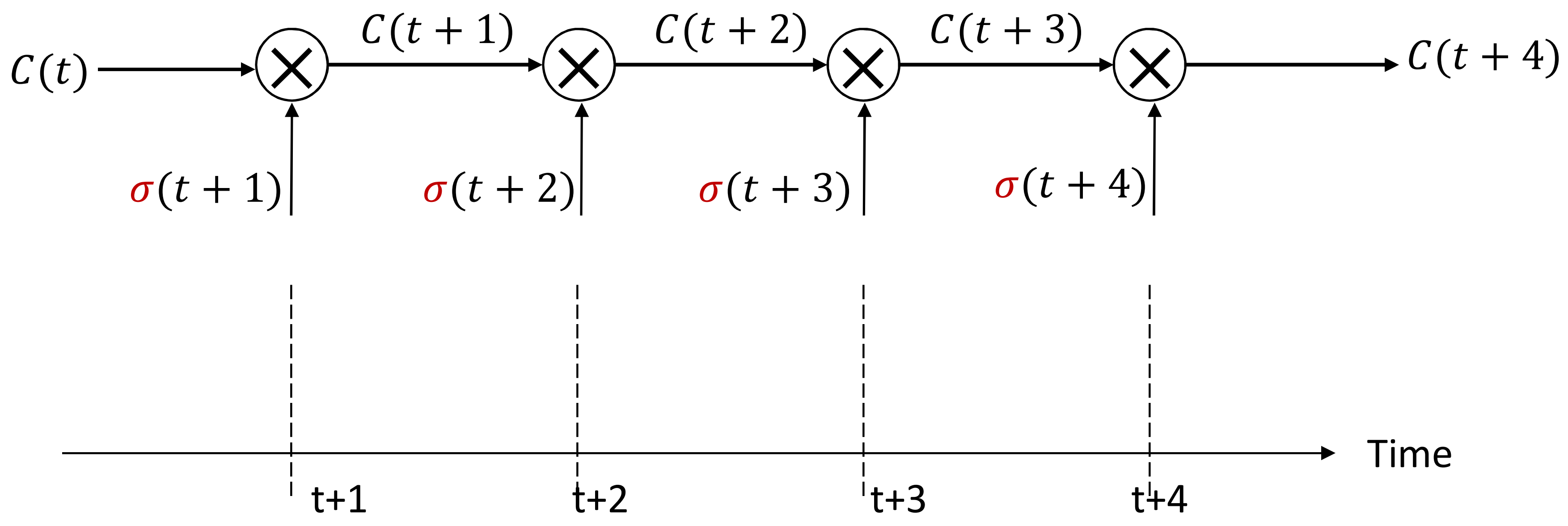
Exploding/Vanishing gradients

$$\nabla_{f_k} Div = \nabla D . \nabla f_N . W_{N-1} . \nabla f_{N-1} . W_{N-2} \underline{\ldots \nabla f_{k+1} W_k}$$

- Can we replace this with something that doesn't fade or blow up?

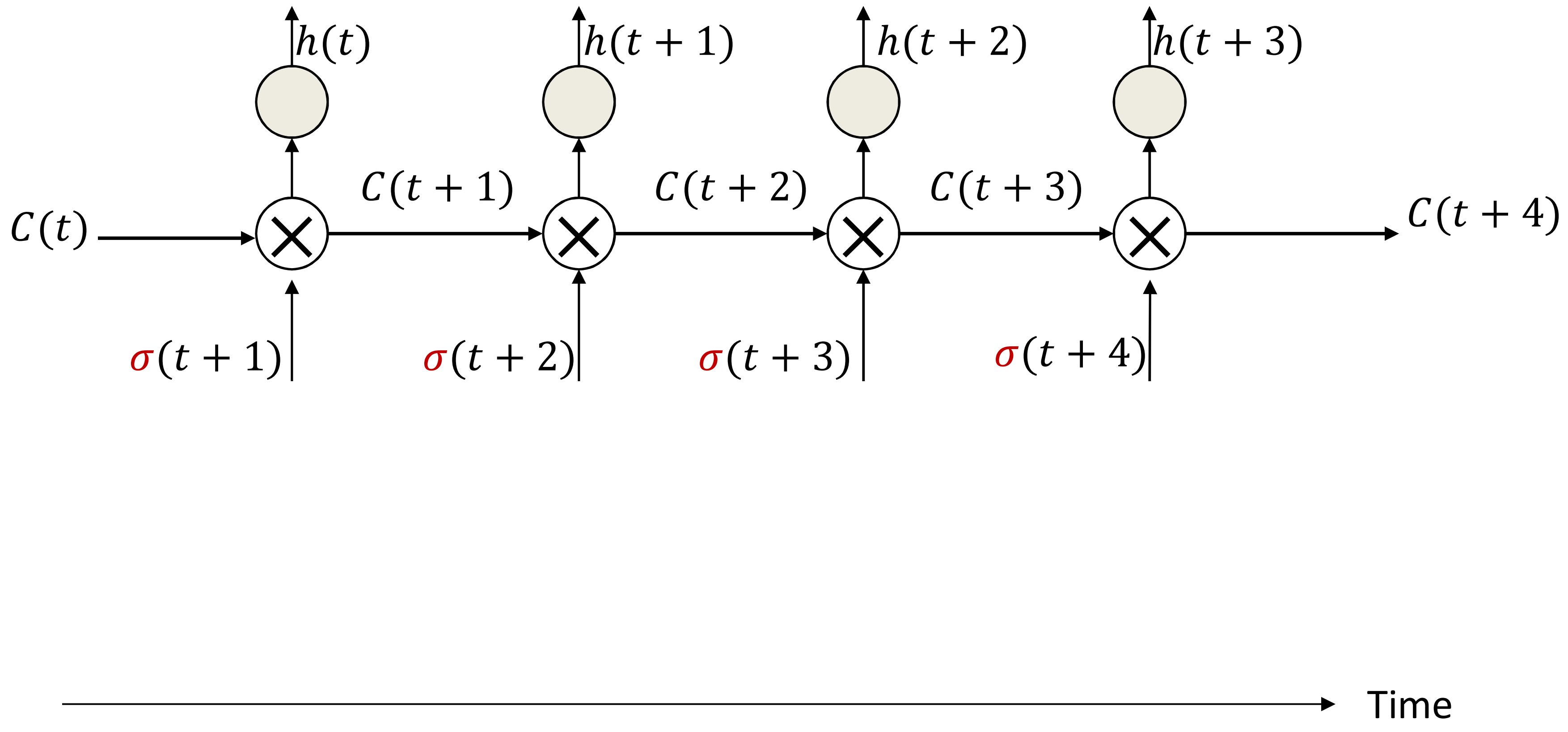
- Can we have a network that just "remembers" arbitrarily long, to be recalled on demand?
  - Not be directly dependent on vagaries of network parameters, but rather on input-based determination of *whether it must be remembered*
    - Retain memoris until a switch *based on the input* flags them as ok to forget
      - Or remember less
  - $Memory(k) \approx C\sigma_k C\sigma_{k-1} C \ldots \sigma_1$
  - $\nabla_{f_k} Div \approx \nabla D C\sigma'_N C\sigma'_{N-1} C \ldots \sigma'_k$

Enter – the constant error carousel



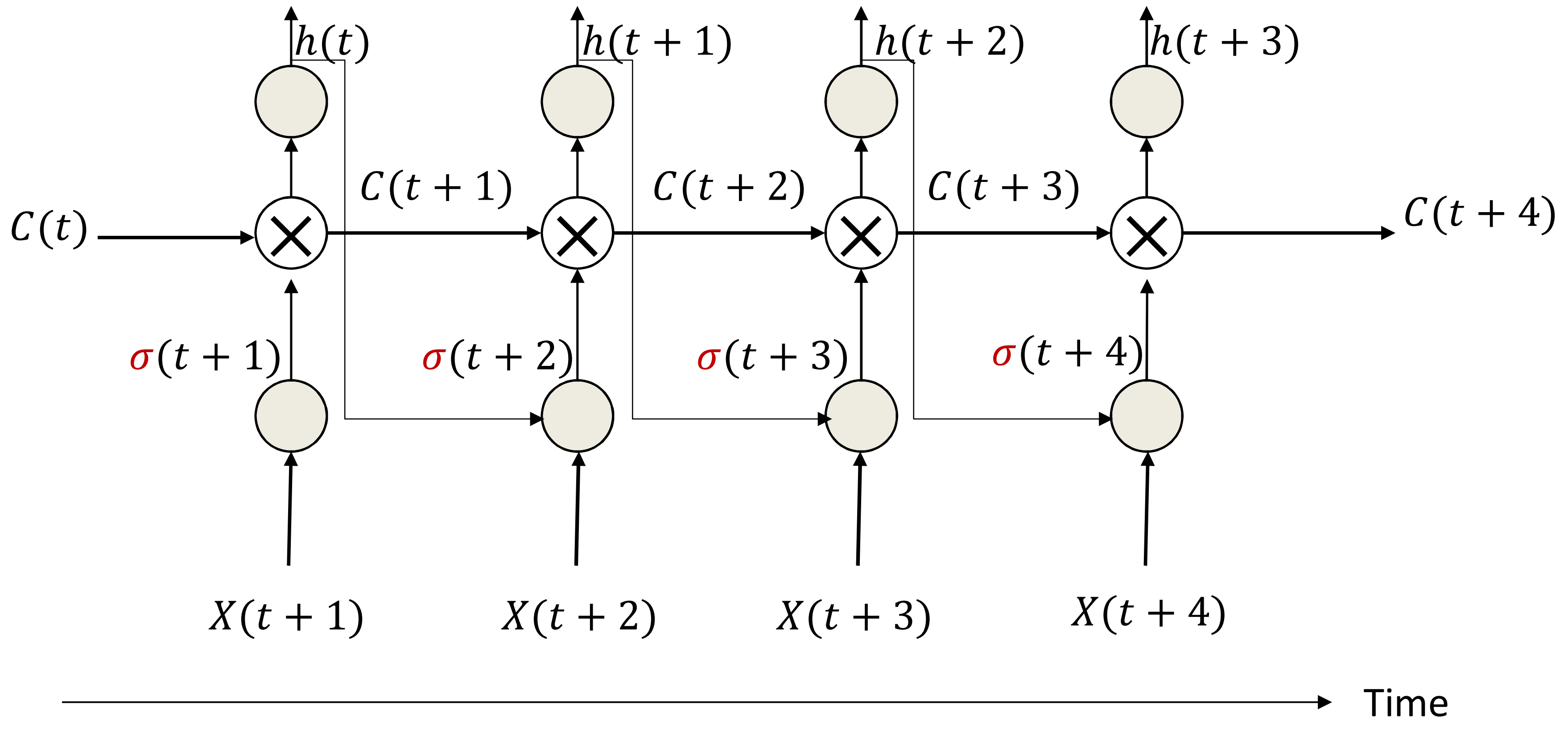
- History is carried through uncompressed
  - No weights, no nonlinearities
  - Only scaling is through the σ “gating” term that captures other triggers
  - E.g. “Have I seen Pattern2”?

Enter – the constant error carousel



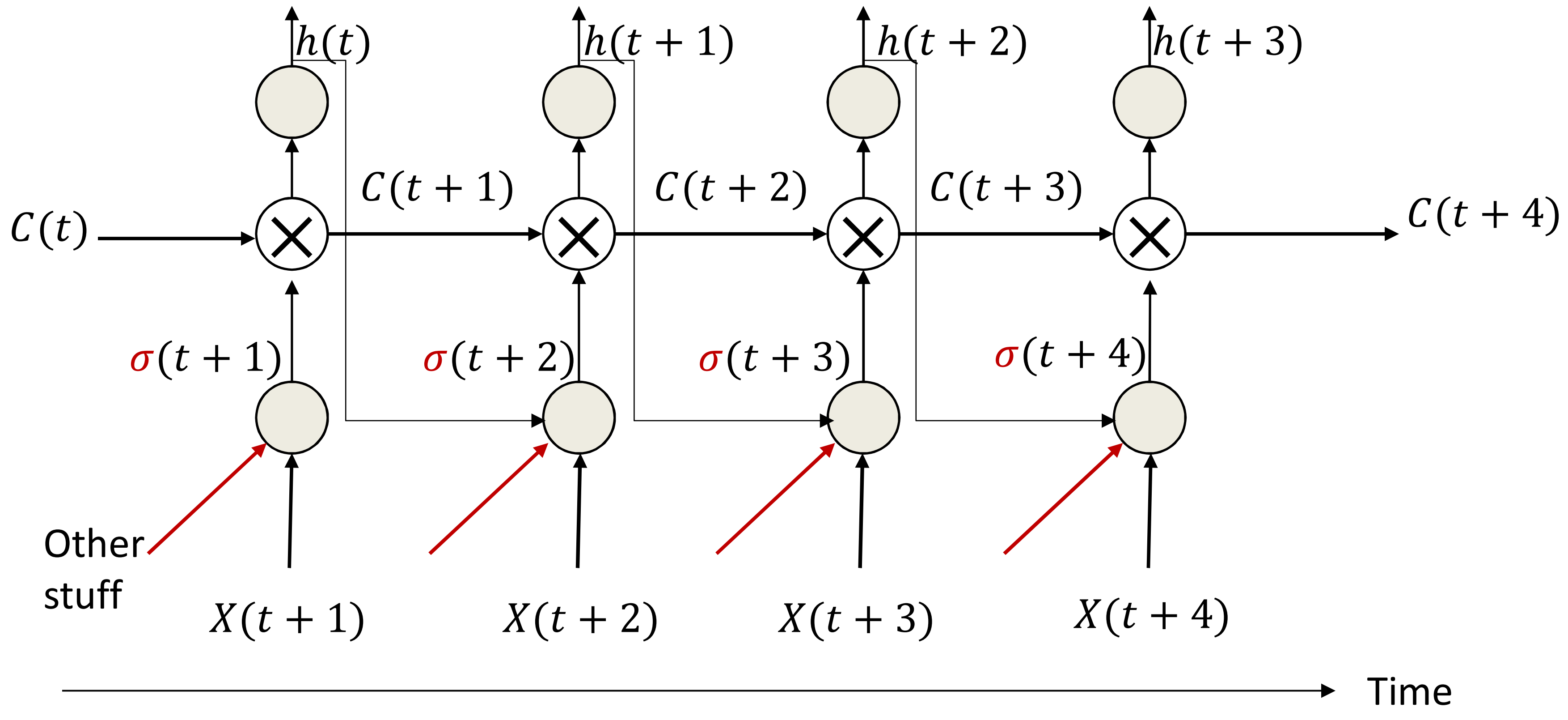
- Actual non-linear work is done by other portions of the network
  - Neurons that compute the workable state from the memory

Enter – the constant error carousel



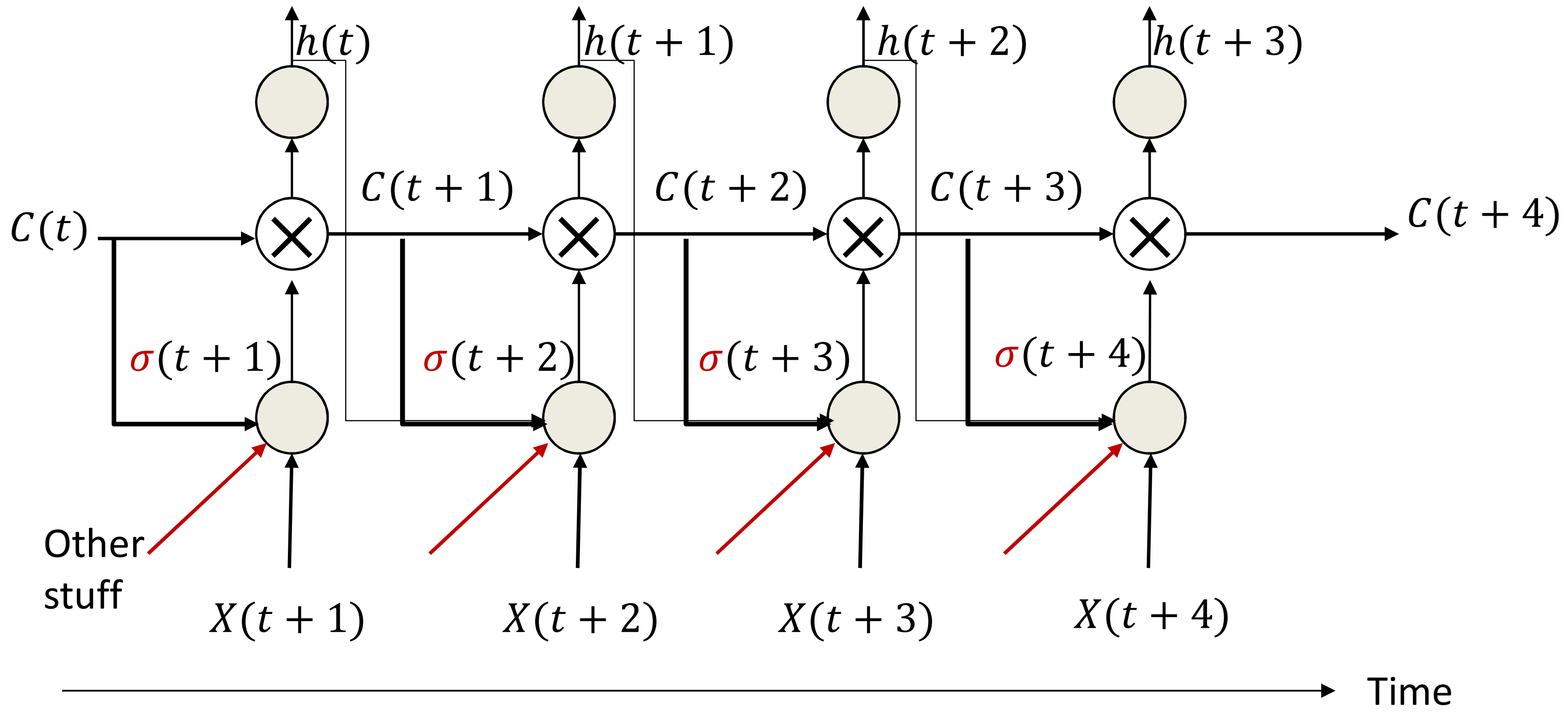
- The gate σ depends on current input, current hidden state…

Enter – the constant error carousel



- The gate σ depends on current input, current hidden state… and other stuff…

Enter – the constant error carousel



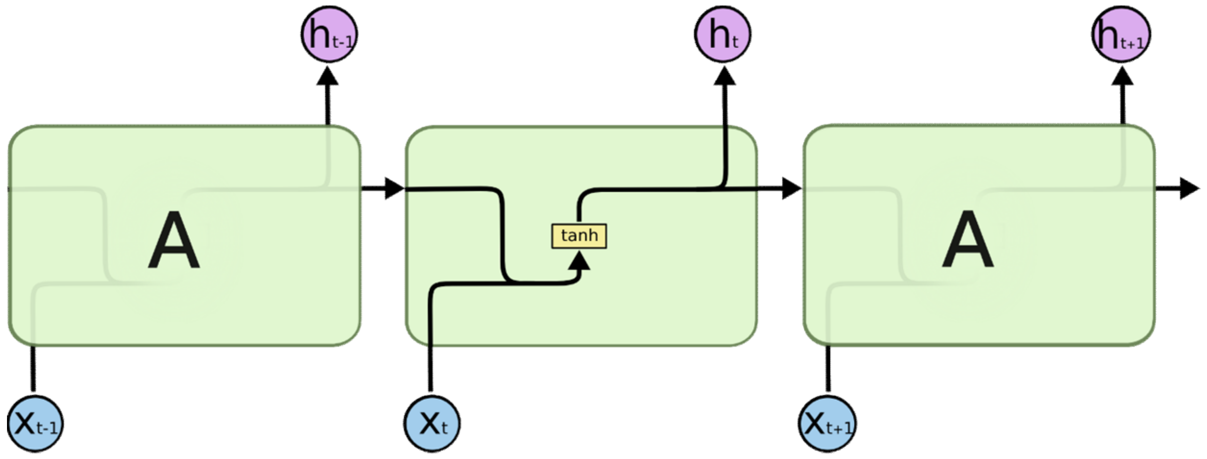
- The gate σ depends on current input, current hidden state… and other stuff…
- Including, obviously, what is currently in raw memory

Enter the *LSTM*

- *Long Short-Term Memory*
- Explicitly latch information to prevent decay / blowup

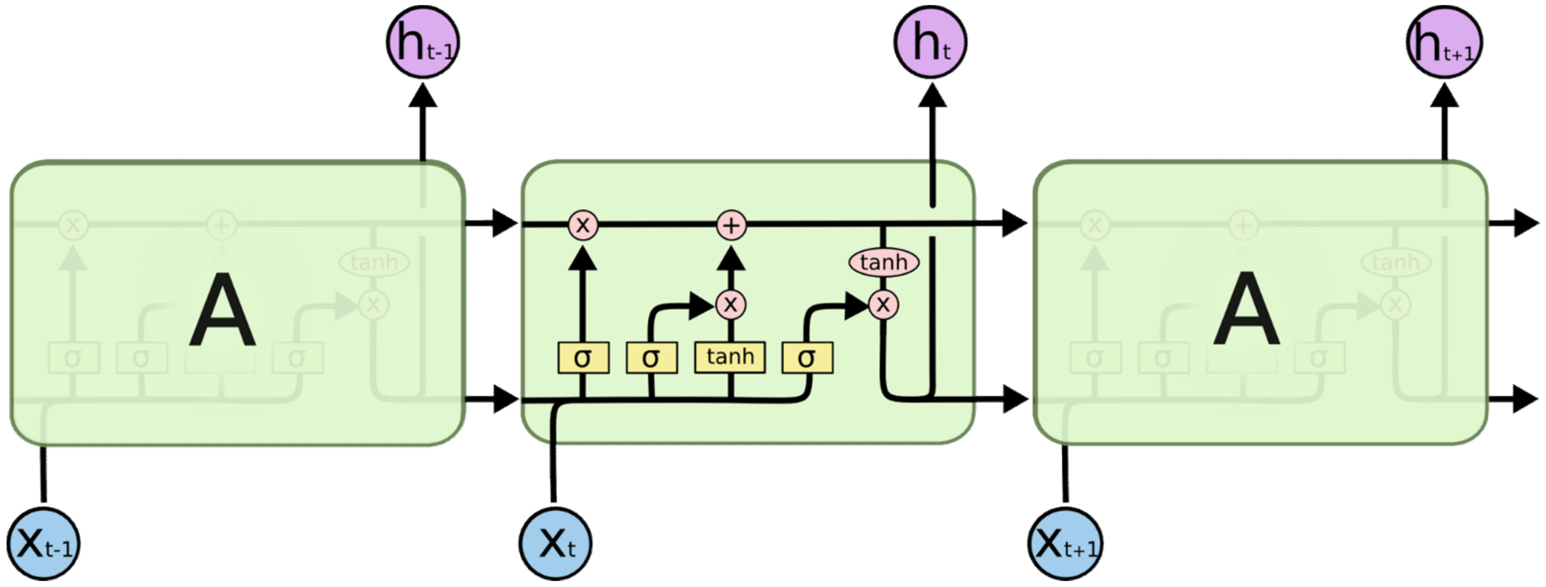
- Following notes borrow liberally from
- http://colah.github.io/posts/2015-08-Understanding-LSTMs/

Standard RNN



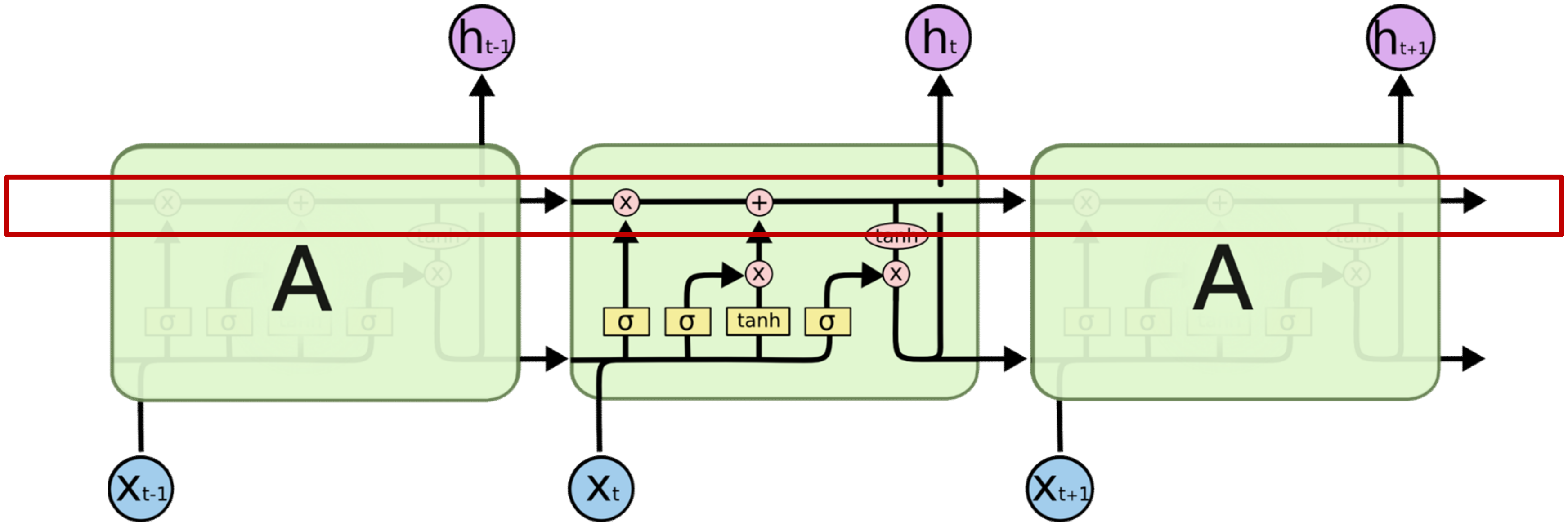
- Recurrent neurons receive past recurrent outputs and current input as inputs
- Processed through a tanh() activation function
  - As mentioned earlier, tanh() is the generally used activation for the hidden layer
- Current recurrent output passed to next higher layer and next time instant

Long Short-Term Memory



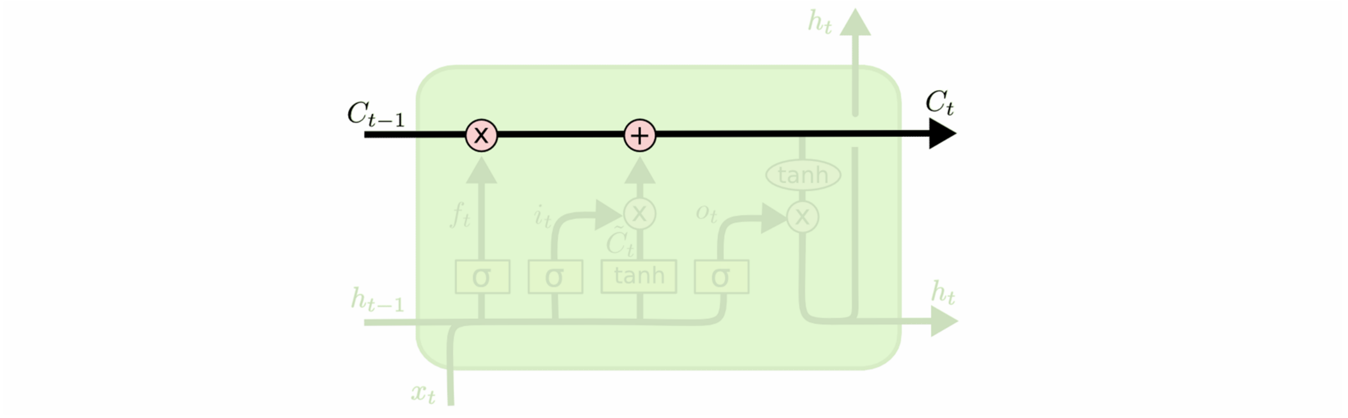
- The $\sigma()$ are *multiplicative gates* that decide if something is important or not
- Remember, every line actually represents a *vector*

LSTM: Constant Error Carousel



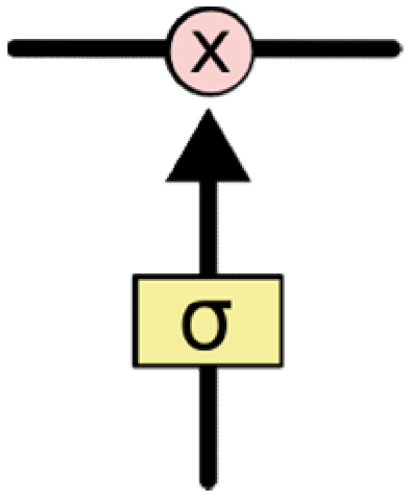
- Key component: a *remembered cell state*

LSTM: CEC



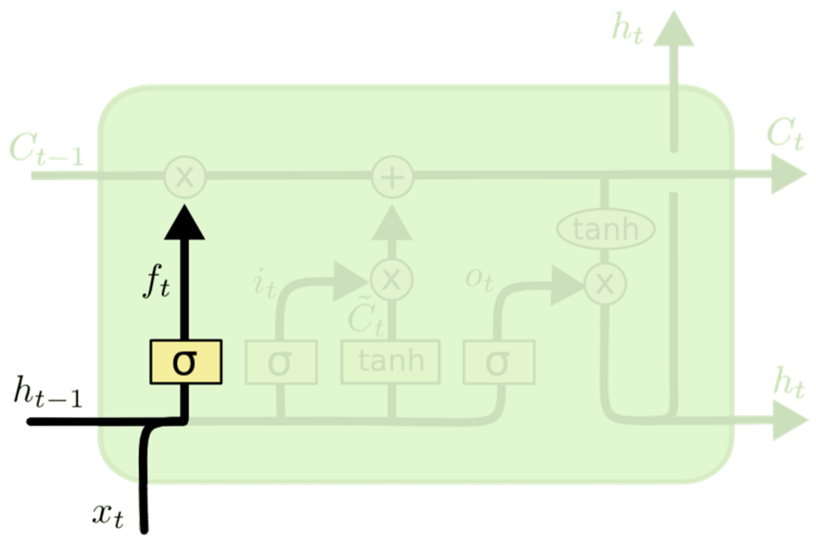
- C_t is the linear history carried by the *constant-error carousel*
- Carries information through, only affected by a gate
  – And *addition of history,* which too is gated..

LSTM: Gates



- Gates are simple sigmoidal units with outputs in the range (0,1)
- Controls how much of the information is to be let through

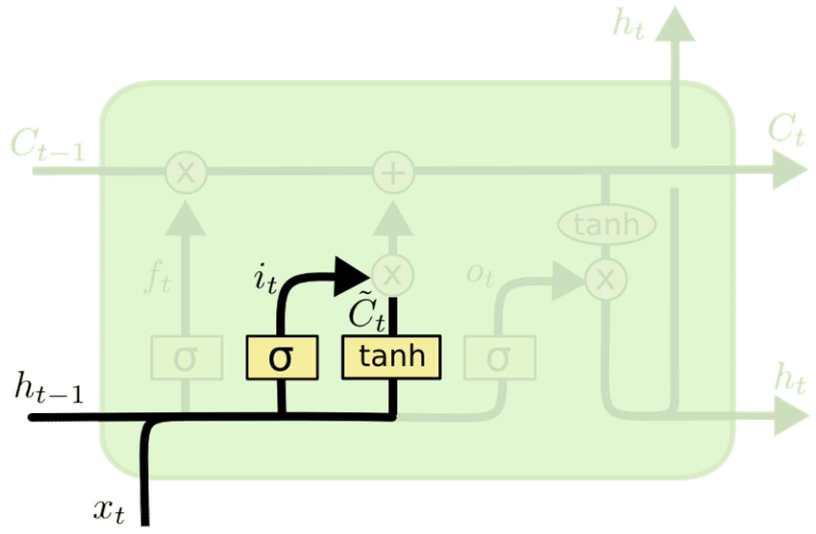
LSTM: Forget gate



$$f_t = \sigma\left(W_f \cdot [h_{t-1}, x_t] \ + \ b_f\right)$$

- The first gate determines whether to carry over the history or to forget it
  - More precisely, how much of the history to carry over
  - Also called the “forget” gate
  - Note, we’re actually distinguishing between the cell memory C and the state h that is coming over time! They’re related though

LSTM: Input gate

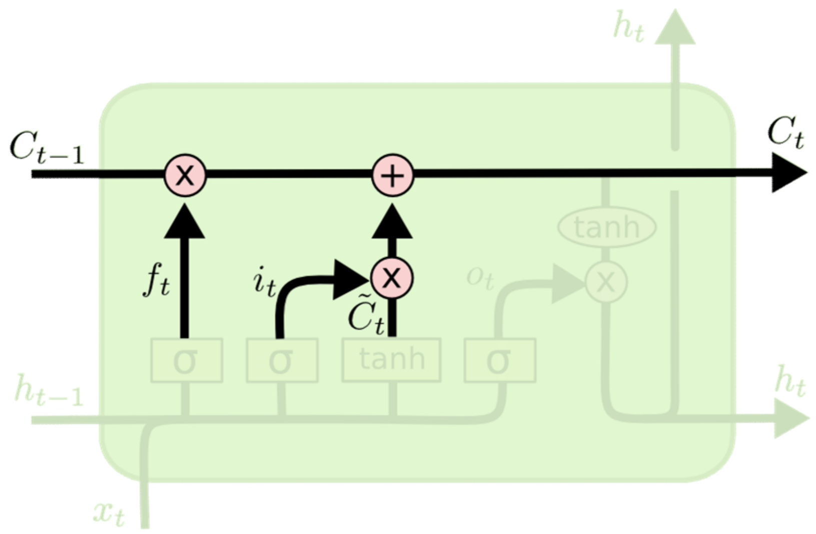


$$i_t = \sigma\left(W_i \cdot [h_{t-1}, x_t] \ + \ b_i\right)$$

$$\tilde{C}_t = \tanh(W_C \cdot [h_{t-1}, x_t] \ + \ b_C)$$

- The second input has two parts
  - A perceptron layer that determines if there's something new and interesting in the input
  - A gate that decides if its worth remembering

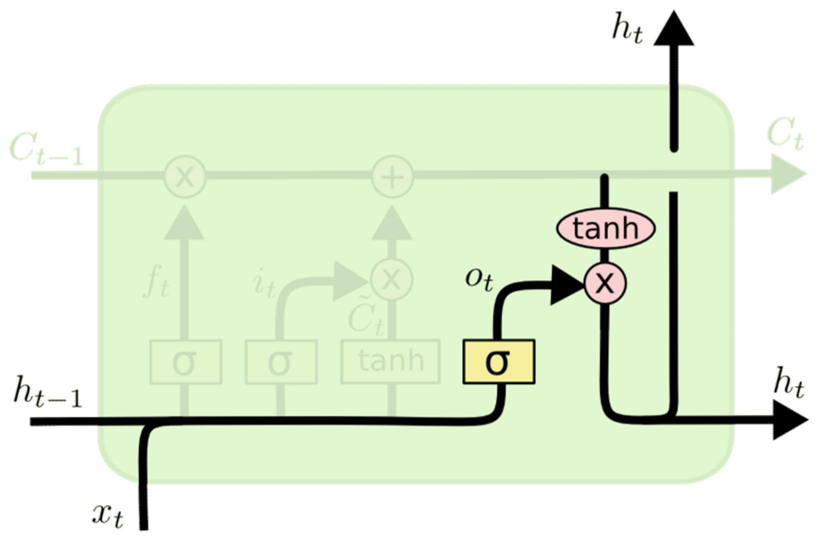
LSTM: Memory cell update



$$C_t = f_t * C_{t-1} + i_t * \tilde{C}_t$$

- The second input has two parts
  - A perceptron layer that determines if there's something interesting in the input
  - A gate that decides if its worth remembering
  - **If so its added to the current memory cell**

LSTM: Output and Output gate

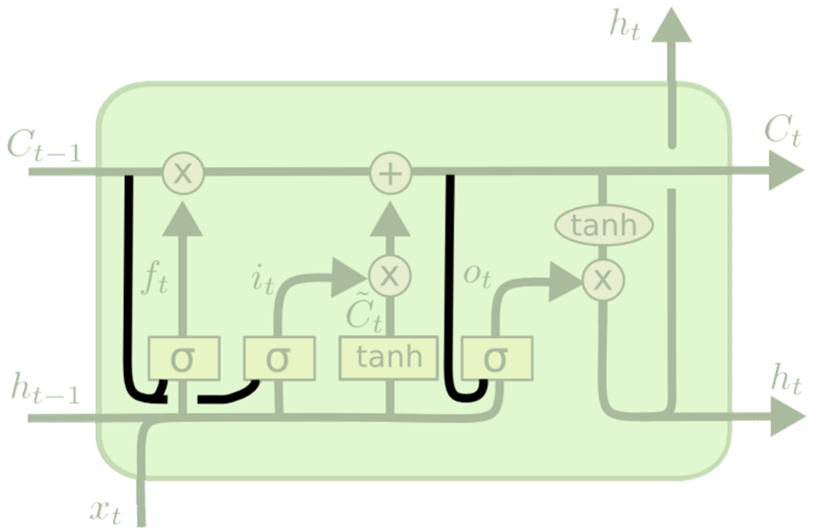


$$o_t = \sigma\left(W_o\ [h_{t-1}, x_t]\ +\ b_o\right)$$

$$h_t = o_t * \tanh\left(C_t\right)$$

- The *output* of the cell
  - Simply compress it with tanh to make it lie between 1 and -1
    - Note that this compression no longer affects our ability to *carry* memory forward
  - Controlled by an *output* gate
    - To decide if the memory contents are worth reporting at *this* time

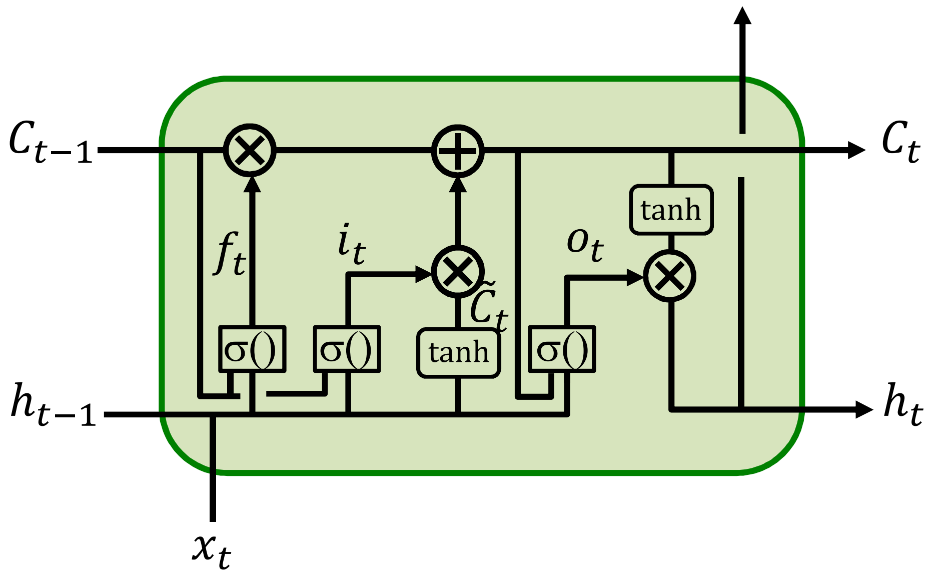
LSTM: The "Peephole" Connection



$$f_t = \sigma\left(W_f \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \; + \; b_f\right)$$
$$i_t = \sigma\left(W_i \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \; + \; b_i\right)$$
$$o_t = \sigma\left(W_o \cdot [\boldsymbol{C_t}, h_{t-1}, x_t] \; + \; b_o\right)$$

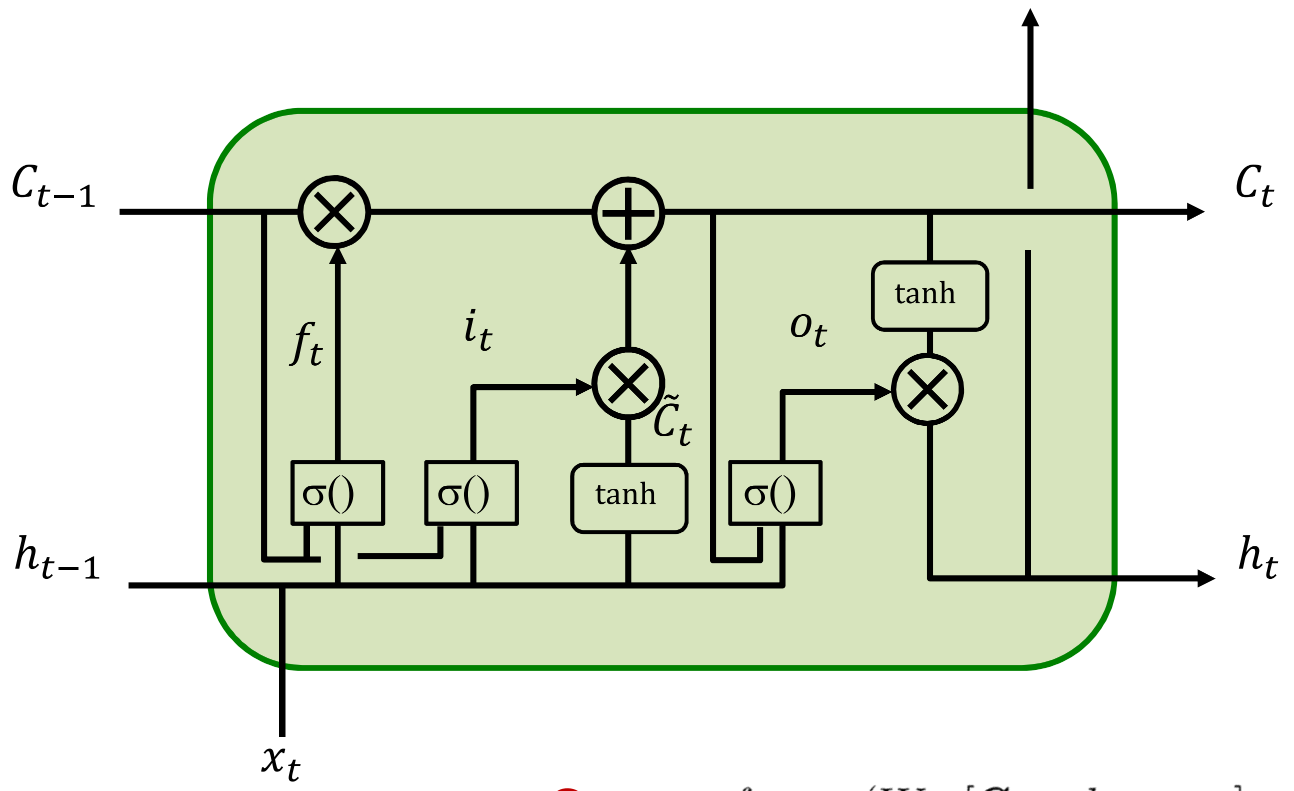
- The raw memory is informative by itself and can also be input
  - Note, we're using both C and h

The complete LSTM unit



- With input, output, and forget gates and the peephole connection..

Backpropagation rules: Forward



- Forward rules:

Gates

$$f_t = \sigma\left(W_f \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \ + \ b_f\right)$$
$$i_t = \sigma\left(W_i \cdot [\boldsymbol{C_{t-1}}, h_{t-1}, x_t] \ + \ b_i\right)$$
$$o_t = \sigma\left(W_o \cdot [\boldsymbol{C_t}, h_{t-1}, x_t] \ + \ b_o\right)$$

Variables

$$\tilde{C}_t = \tanh(W_C \cdot [h_{t-1}, x_t] \ + \ b_C)$$
$$C_t = f_t * C_{t-1} + i_t * \tilde{C}_t$$
$$h_t = o_t * \tanh\left(C_t\right)$$

Notes on the pseudocode

`Class LSTM_cell`

- We will assume an object-oriented program
- Each LSTM unit is assumed to be an “LSTM cell”
- There’s a new copy of the LSTM cell at each time, at each layer
- LSTM cells retain local variables that are not relevant to the computation outside the cell
  - These are static and retain their value once computed, unless overwritten

LSTM cell (single unit) Definitions

```
# Input:
#    C : current value of CEC
#    h : Current hidden state value (“output” of cell)
#    x:  Current input
# [W,b]: The set of all model parameters for the cell
#      These include all weights and biases
# Output
#    C : Next value of CEC
#    h : Next value of h
# In the function:  sigmoid(x) = 1/(1+exp(-x))
#                   performed component-wise

# Static local variables to the cell
static local $z_f$, $z_i$, $z_c$, $z_o$, f, i, o, $C_i$
function [C,h] = LSTM_cell.forward(C,h,x,[W,b])
    code on next slide
```

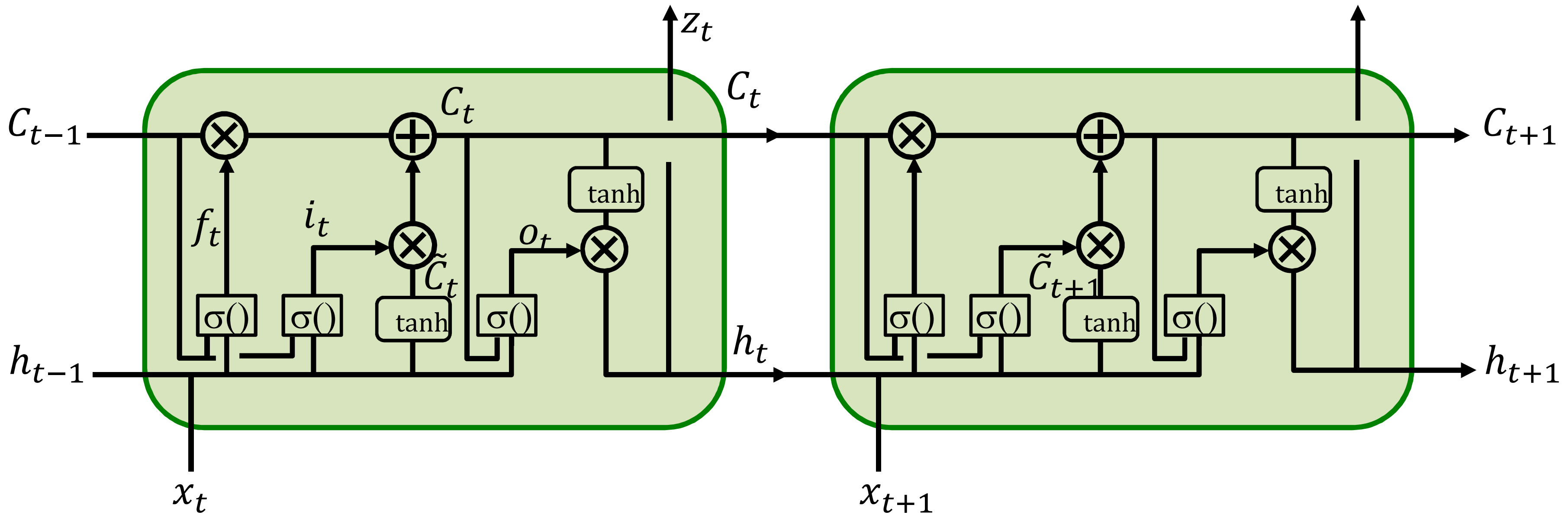
LSTM cell forward

```
# Continuing from previous slide
# Note: [W,h] is a set of parameters, whose individual elements are
#       shown in red within the code.  These are passed in

# Static local variables which aren't required outside this cell
static local $z_f$, $z_i$, $z_c$, $z_o$, f, i, o, $C_i$
function [$C_o$, $h_o$] = LSTM_cell.forward(C,h,x, [W,h])
    $z_f = W_{fc}C + W_{fh}h + W_{fx}x + b_f$
    f = sigmoid($z_f$) # forget gate

    $z_i = W_{ic}C + W_{ih}h + W_{ix}x + b_i$
    i = sigmoid($z_i$) # input gate

    $z_c = W_{cc}C + W_{ch}h + W_{cx}x + b_c$
    $C_i$ = tanh($z_c$)  # Detecting input pattern

    $C_o = f \circ C + i \circ C_i$ # "∘" is component-wise multiply

    $z_o = W_{oc}C_o + W_{oh}h + W_{ox}x + b_o$
    o = sigmoid($z_o$) # output gate

    $h_o = o \circ \tanh(C)$ # "∘" is component-wise multiply

    return $C_o$,$h_o$
```

LSTM network forward

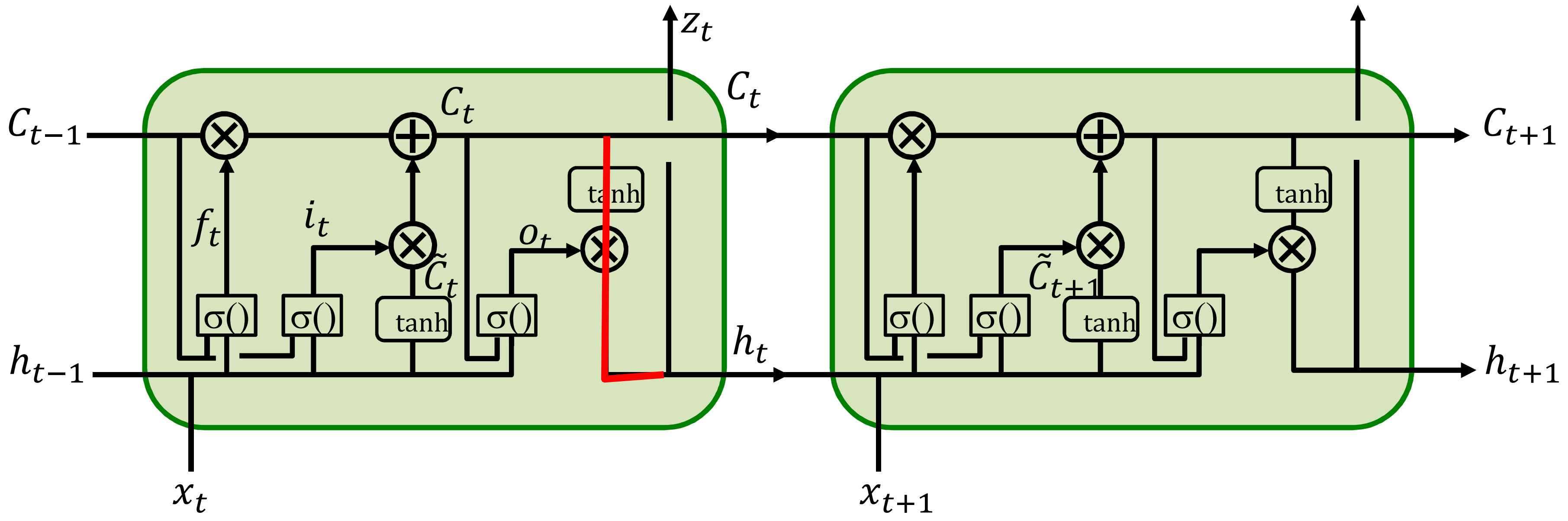
```
# Assuming h(-1,*) is known and C(-1,*)=0
# Assuming L hidden-state layers and an output layer
# Note: LSTM_cell is an indexed class with functions
# [W{l},b{l}] are the entire set of weights and biases
#                  for the lth hidden layer
# $W_o$ and $b_o$ are output layer weights and biases

for t = 0:T-1  # Including both ends of the index
    h(t,0) = x(t) # Vectors. Initialize h(0) to input
    for l = 1:L  # hidden layers operate at time t
        [C(t,l),h(t,l)] = LSTM_cell(t,l).forward(…
              …C(t-1,l),h(t-1,l),h(t,l-1)[W{l},b{l}])
    $z_o(t)$ = $W_o$h(t,L) + $b_o$
    Y(t) = softmax( $z_o(t)$ )
```

Backpropagation rules: Backward



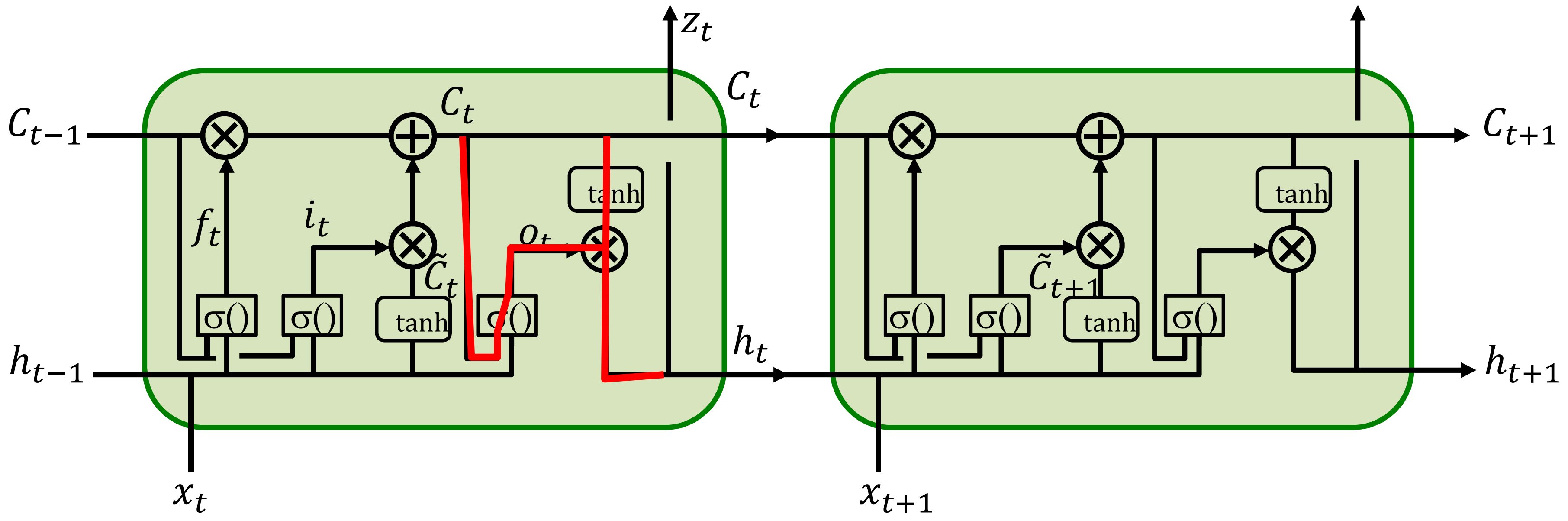
$\nabla_{C_t} Div =$

Backpropagation rules: Backward



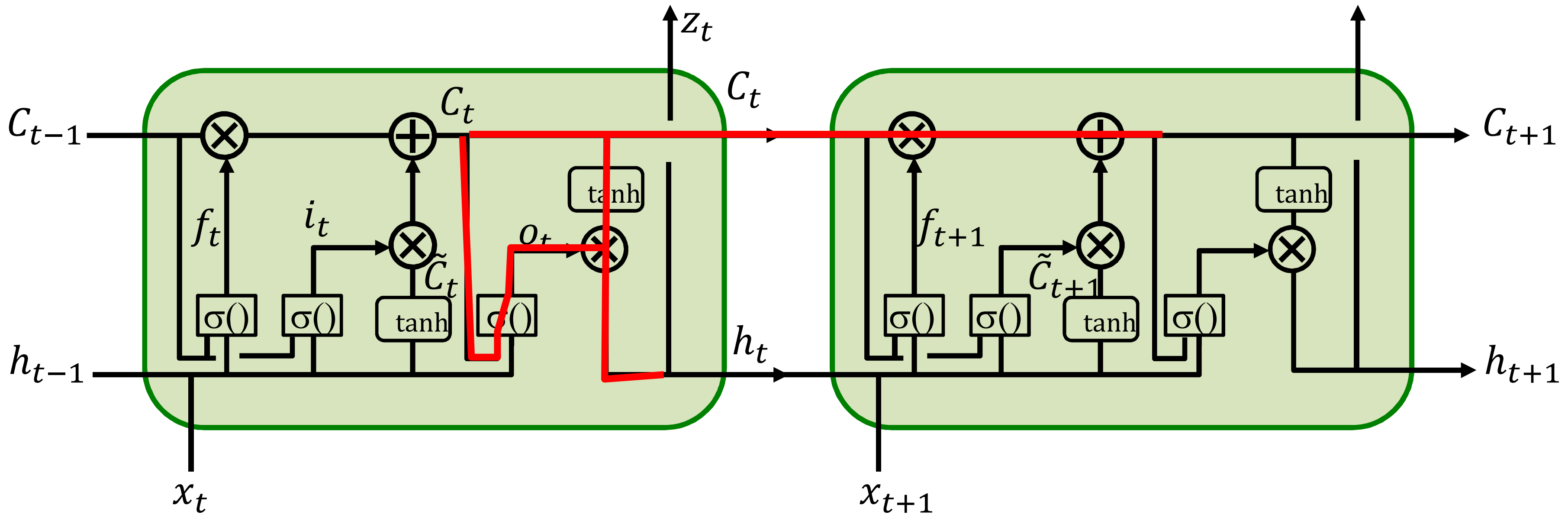
$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ o_t \circ tanh'(.)$$

Backpropagation rules: Backward



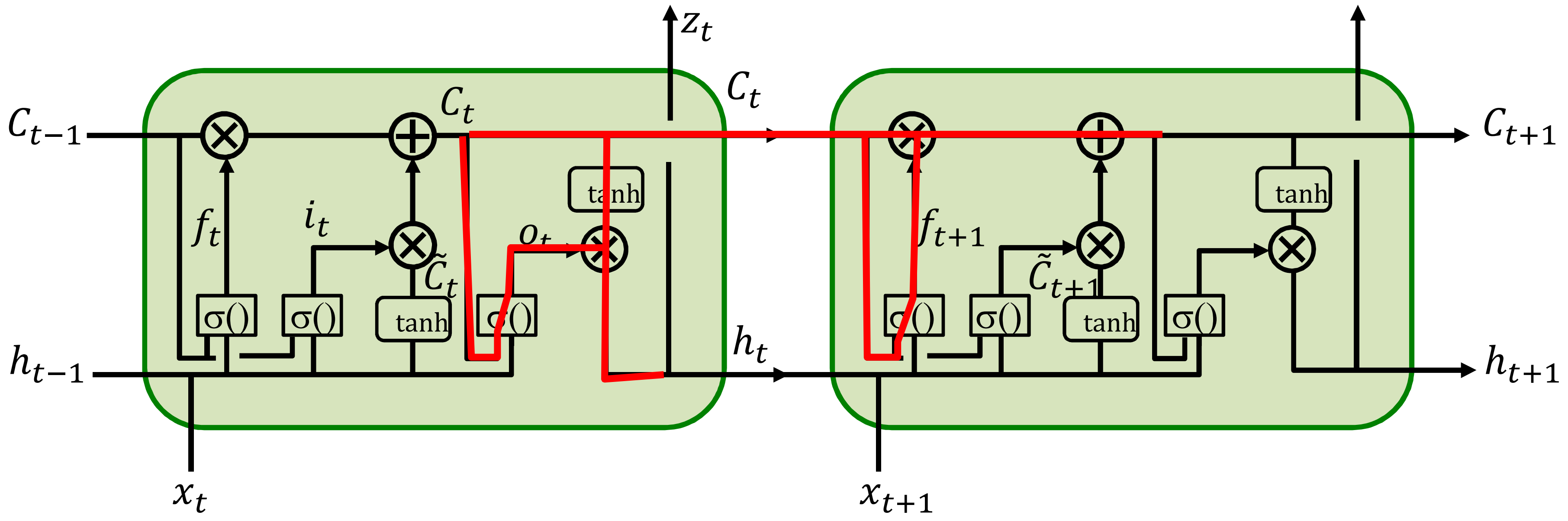
$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.) W_{Co})$$

Backpropagation rules: Backward



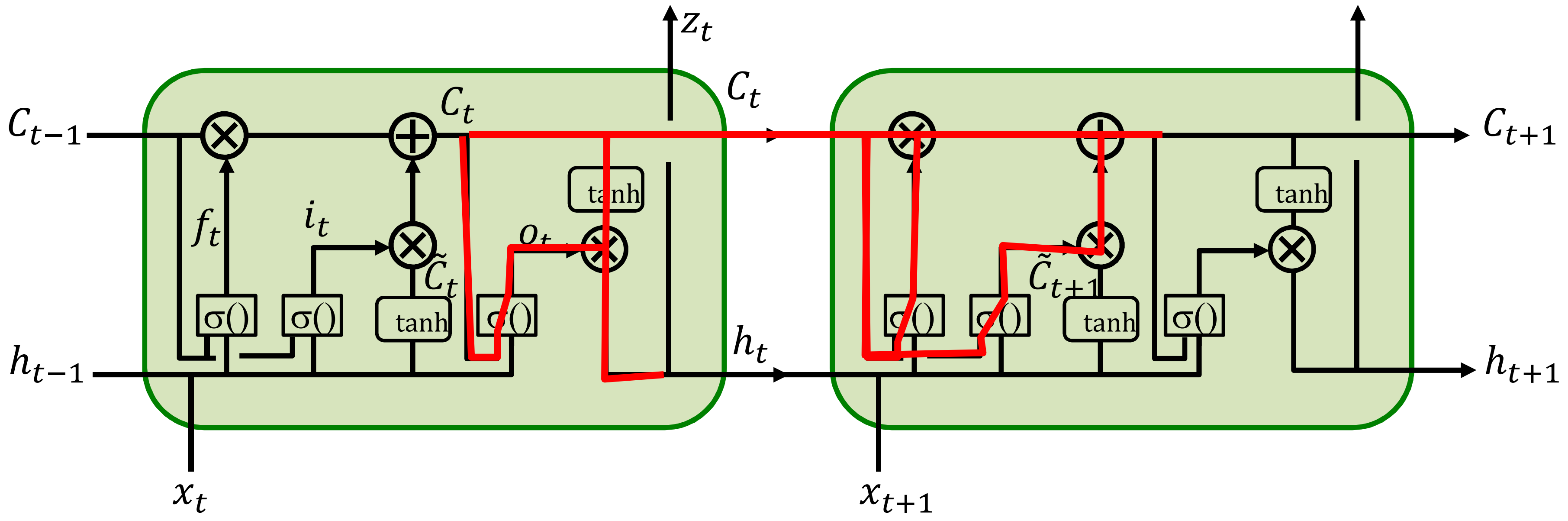
$$\nabla_{C_t}Div = \nabla_{h_t}Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{C_{t+1}}Div \circ f_{t+1} +$$

Backpropagation rules: Backward



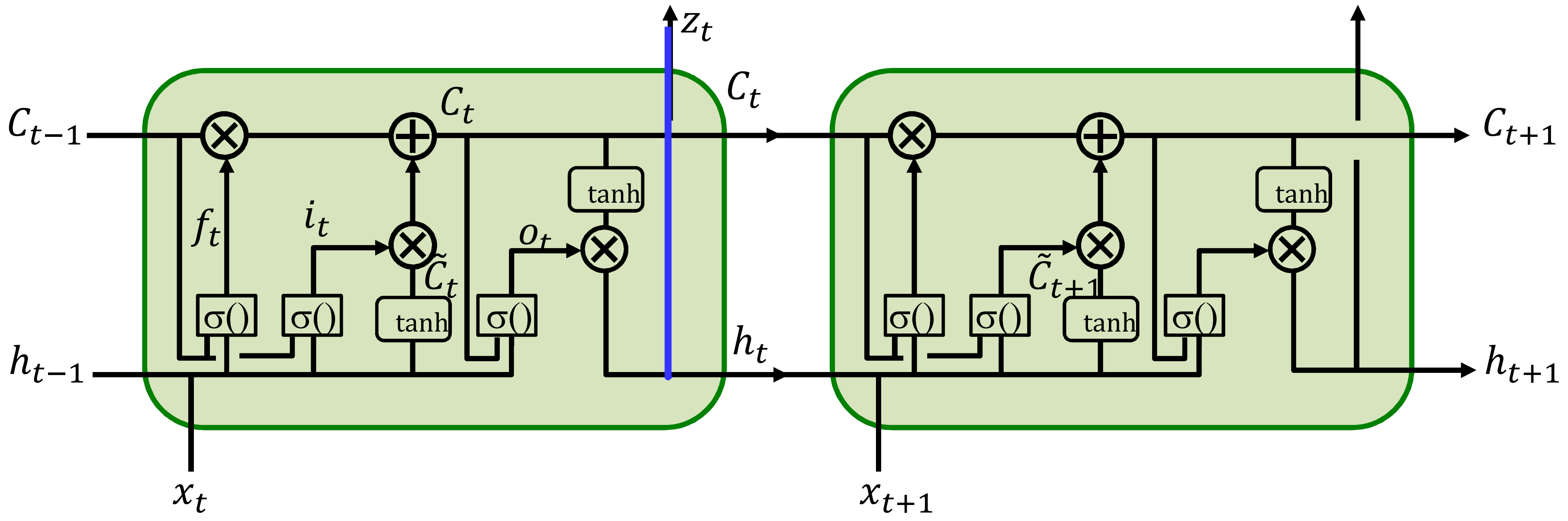
$$\nabla_{C_t}Div = \nabla_{h_t}Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.)W_{Co}) + \nabla_{C_{t+1}}Div \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf}\right)$$

Backpropagation rules: Backward



$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ \left(o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.) W_{Co}\right) +$$
$$\nabla_{C_{t+1}} Div \circ \left(f_{t+1} + C_t \circ \sigma'(.) W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.) W_{Ci}\right)$$

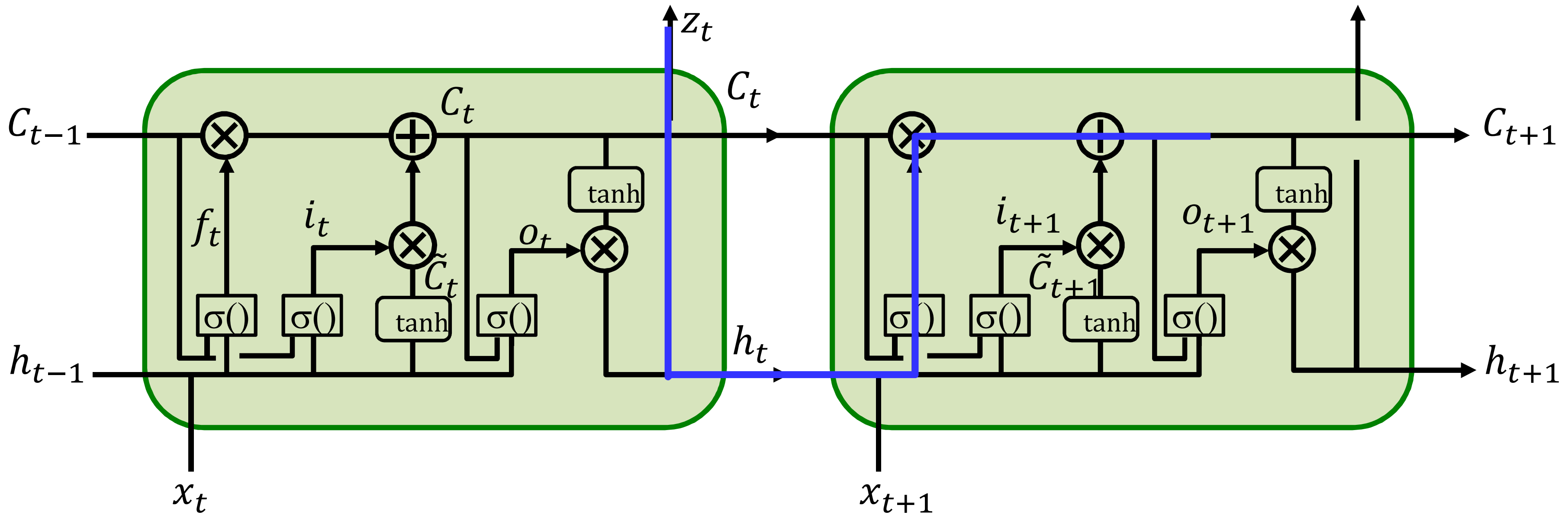
Backpropagation rules: Backward



$$\nabla_{C_t}Div = \nabla_{h_t}Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{C_{t+1}}Div \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci}\right)$$

$$\nabla_{h_t}Div = \nabla_{z_t}Div \nabla_{h_t}z_t$$

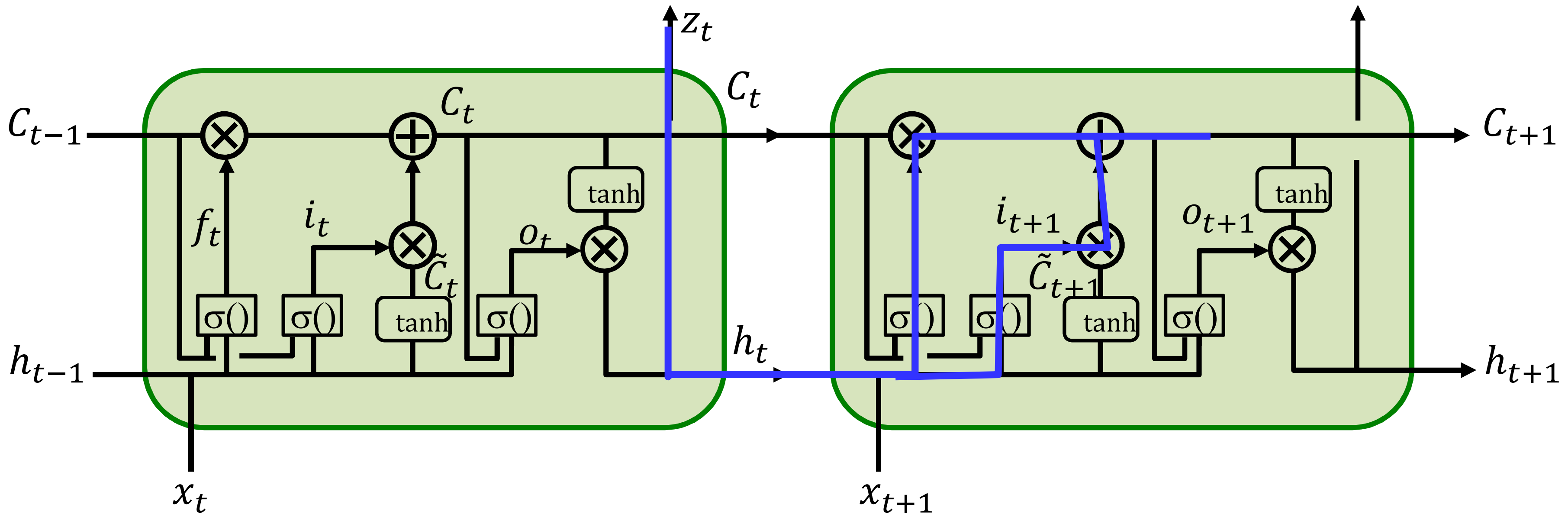
Backpropagation rules: Backward



$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.) W_{Co}) + \nabla_{C_{t+1}} Div \circ \left(f_{t+1} + C_t \circ \sigma'(.) W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.) W_{Ci}\right)$$

$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t + \nabla_{C_{t+1}} Div \circ C_t \circ \sigma'(.) W_{hf}$$

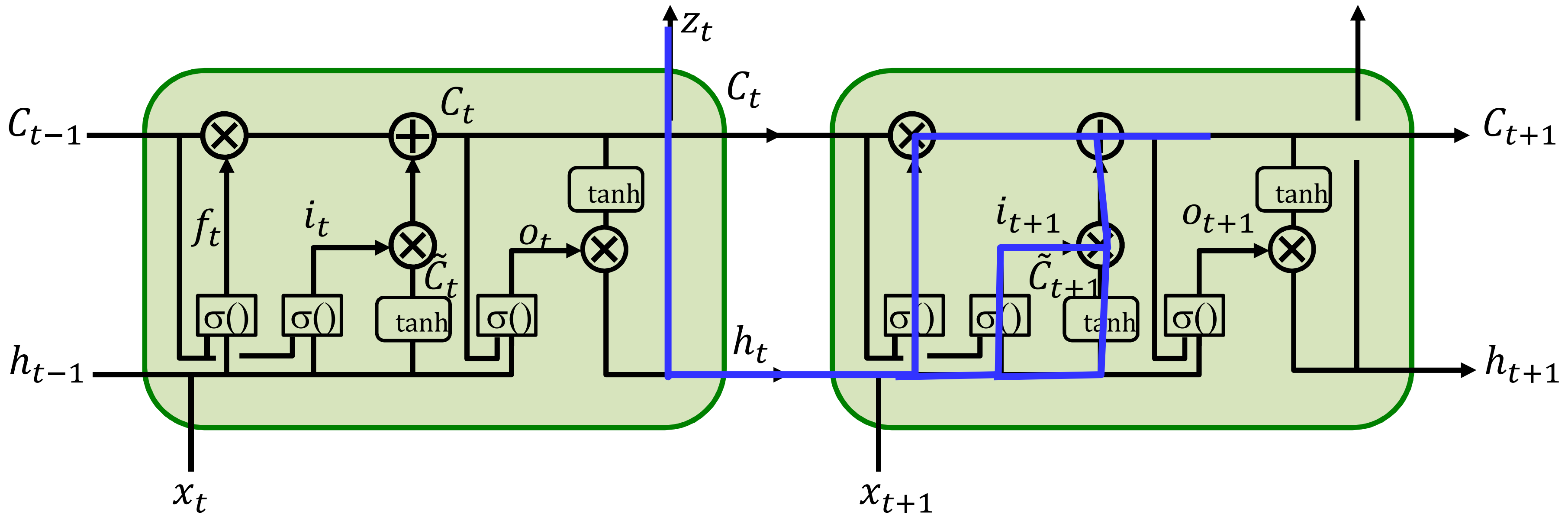
Backpropagation rules: Backward



$$\nabla_{C_t}Div = \nabla_{h_t}Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.)W_{Co}) + \nabla_{C_{t+1}}Div \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci}\right)$$

$$\nabla_{h_t}Div = \nabla_{z_t}Div\nabla_{h_t}z_t + \nabla_{C_{t+1}}Div \circ \left(C_t \circ \sigma'(.)W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{hi}\right)$$

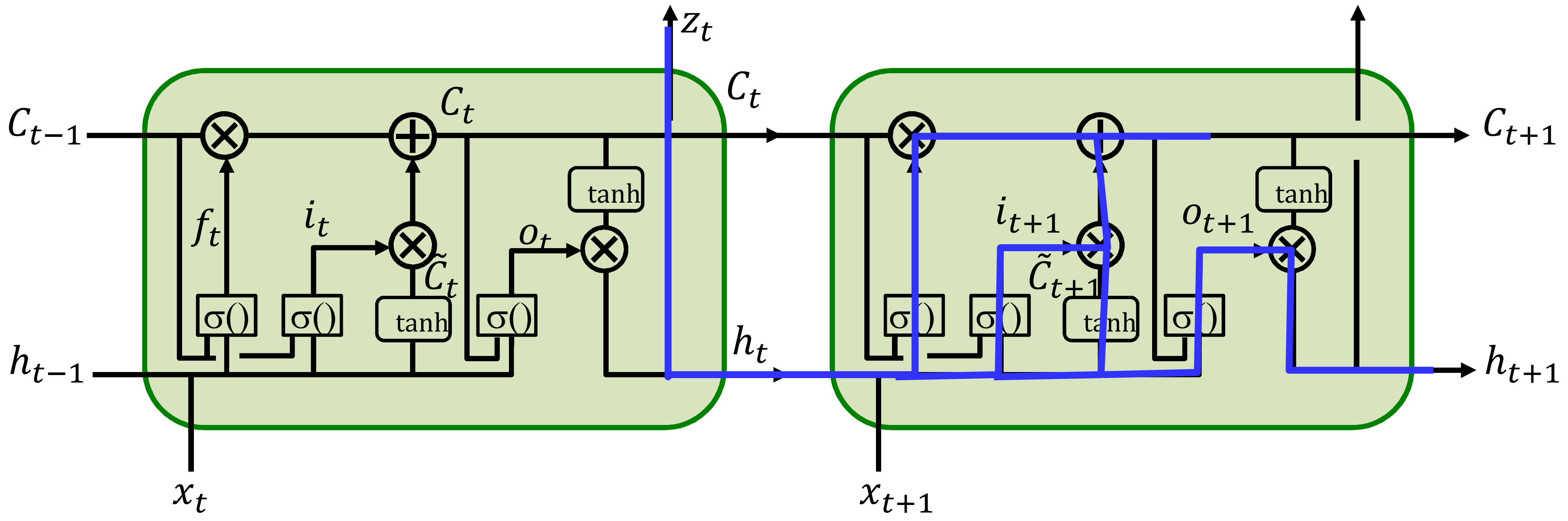
Backpropagation rules: Backward



$$\nabla_{C_t}Div = \nabla_{h_t}Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.)W_{Co}) +$$
$$\nabla_{C_{t+1}}Div \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci}\right)$$

$$\nabla_{h_t}Div = \nabla_{z_t}Div\nabla_{h_t}z_t + \nabla_{C_{t+1}}Div \circ \left(C_t \circ \sigma'(.)W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{hi}\right) +$$
$$\nabla_{C_{t+1}}Div \circ i_{t+1} \circ tanh'(.)W_{hi}$$

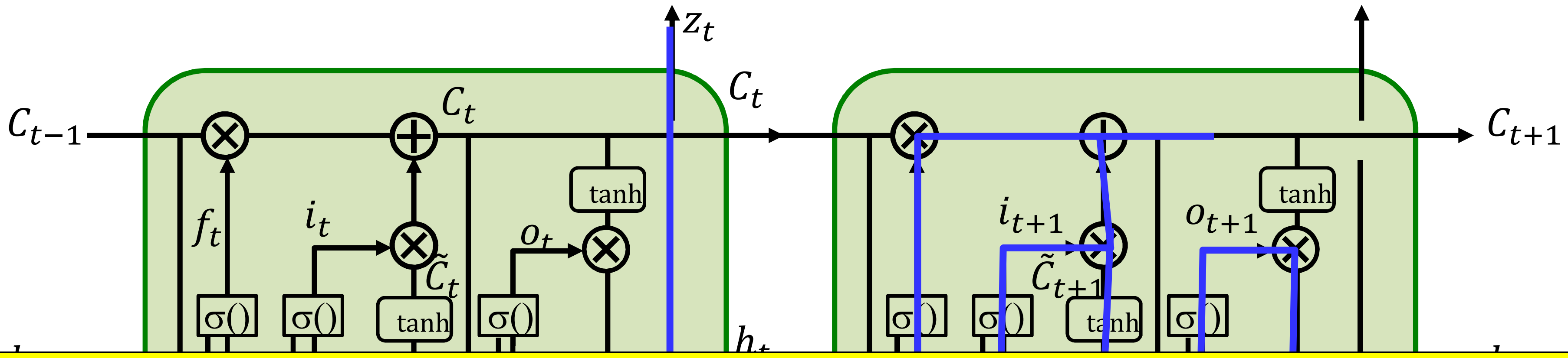
Backpropagation rules: Backward



$$\nabla_{C_t} Div = \nabla_{h_t} Div \circ (o_t \circ tanh'(.) + tanh(.) \circ \sigma'(.) W_{Co}) + \nabla_{C_{t+1}} Div \circ \left(f_{t+1} + C_t \circ \sigma'(.) W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.) W_{Ci}\right)$$

$$\nabla_{h_t} Div = \nabla_{z_t} Div \nabla_{h_t} z_t + \nabla_{C_{t+1}} Div \circ \left(C_t \circ \sigma'(.) W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.) W_{hi}\right) + \nabla_{C_{t+1}} Div \circ o_{t+1} \circ tanh'(.) W_{hi} + \nabla_{h_{t+1}} Div \circ tanh(.) \circ \sigma'(.) W_{ho}$$

Backpropagation rules: Backward



Not explicitly deriving the derivatives w.r.t weights; Left as an exercise

$$\nabla_{C_t}Div = \nabla_{h_t}Div \circ (o_t \circ tanh'(.)W_{Ch} + tanh(.) \circ \sigma'(.)W_{Co}) + \\ \nabla_{h_t}C_{t+1} \circ \left(f_{t+1} + C_t \circ \sigma'(.)W_{Cf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{Ci}\right)$$

$$\nabla_{h_t}Div = \nabla_{z_t}Div\nabla_{h_t}z_t + \nabla_{C_{t+1}}Div \circ \left(C_t \circ \sigma'(.)W_{hf} + \tilde{C}_{t+1} \circ \sigma'(.)W_{hi}\right) + \\ \nabla_{C_{t+1}}Div \circ o_{t+1} \circ tanh'(.)W_{hi} + \nabla_{h_{t+1}}Div \circ tanh(.) \circ \sigma'(.)W_{ho}$$

Notes on the backward pseudocode

`Class LSTM_cell`

- We first provide backward computation *within a cell*
- For the backward code, we will assume the static variables computed during the forward are still available
- The following slides first show the forward code for reference
- Subsequently we will give you the backward, and explicitly indicate *which* of the forward equations each backward equation refers to
  - *The backward code for a cell is long (but simple) and extends over multiple slides*

LSTM cell forward (for reference)

```
# Continuing from previous slide
# Note: [W,h] is a set of parameters, whose individual elements are
#       shown in red within the code.  These are passed in

# Static local variables which aren't required outside this cell
static local $z_f$, $z_i$, $z_c$, $z_o$, f, i, o, $C_i$
function [$C_o$, $h_o$] = LSTM_cell.forward(C,h,x, [W,h])
     $z_f$ = $W_{fc}C$ + $W_{fh}h$ + $W_{fx}x$ + $b_f$
     f = sigmoid($z_f$) # forget gate

     $z_i$ = $W_{ic}C$ + $W_{ih}h$ + $W_{ix}x$ + $b_i$
     i = sigmoid($z_i$) # input gate

     $z_c$ = $W_{cc}C$ + $W_{ch}h$ + $W_{cx}x$ + $b_c$
     $C_i$ = tanh($z_c$)  # Detecting input pattern

     $C_o$ = $f \circ C + i \circ C_i$ # "∘" is component-wise multiply

     $z_o$ = $W_{oc}C_o$ + $W_{oh}h$ + $W_{ox}x$ + $b_o$
     o = sigmoid($z_o$) # output gate

     $h_o$ = $o \circ$tanh(C) # "∘" is component-wise multiply

     return $C_o$,$h_o$
```

LSTM cell backward

```
# Static local variables carried over from forward
static local $z_f$, $z_i$, $z_c$, $z_o$, f, i, o, $C_i$
function [dC,dh,dx,d[W, b]]=LSTM_cell.backward(dC$_o$, dh$_o$, C, h, C$_o$, h$_o$, [W,b])
    # First invert $h_o = o \circ tanh(C)$
    do = dh$_o$ ∘ tanh(C$_o$)$^T$
    d tanhC$_o$ = dh$_o$ ∘ o
    dC$_o$ += dtanhC$_o$ ∘ (1-tanh$^2$(C$_o$))$^T$ #(1-tanh$^2$) is the derivative of tanh

    # Next invert o = sigmoid(z$_o$)
    dz$_o$ = do ∘ sigmoid(z$_o$)$^T$ ∘ (1-sigmoid(z$_o$))$^T$ # do x derivative of sigmoid(z$_o$)

    # Next invert $z_o = W_{oc}C_o + W_{oh}h + W_{ox}x + b_o$
    dC$_o$ += dz$_o$W$_{oc}$  #  Note - this is a regular matrix multiply
    dh = dz$_o$ W$_{oh}$
    dx = dz$_o$ W$_{ox}$

    dW$_{oc}$ = C$_o$dz$_o$  # Note - this multiplies a column vector by a row vector
    dW$_{oh}$ = h dz$_o$
    dW$_{ox}$ = x dz$_o$
    db$_o$ = dz$_o$

    # Next invert $C_o = f \circ C + i \circ C_i$
    dC = dC$_o$ ∘ f
    dC$_i$ = dC$_o$ ∘ i
    di = dC$_o$ ∘ C$_i$
    df = dC$_o$ ∘ C
```

LSTM cell backward (continued)

```
# Next invert $C_i$ = tanh($z_c$)
$dz_c = dC_i \circ (1-\tanh^2(z_c))^T$

# Next invert $z_c = W_{cc}C + W_{ch}h + W_{cx}x + b_c$
$dC \mathrel{+}= dz_c W_{cc}$
$dh \mathrel{+}= dz_c\ W_{ch}$
$dx \mathrel{+}= dz_c\ W_{cx}$

$dW_{cc} = C\ dz_c$
$dW_{ch} = h\ dz_c$
$dW_{cx} = x\ dz_c$
$db_c = dz_c$

# Next invert i = sigmoid($z_i$)
$dz_i = di \circ \mathrm{sigmoid}(z_i)^T \circ (1-\mathrm{sigmoid}(z_i))^T$

# Next invert $z_i = W_{ic}C + W_{ih}h + W_{ix}x + b_i$
$dC \mathrel{+}= dz_i\ W_{ic}$
$dh \mathrel{+}= dz_i\ W_{ih}$
$dx \mathrel{+}= dz_i\ W_{ix}$

$dW_{ic} = C\ dz_i$
$dW_{ih} = h\ dz_i$
$dW_{ix} = x\ dz_i$
$db_i = dz_i$
```

LSTM cell backward (continued)

```
# Next invert f = sigmoid($z_f$)
$dz_f$ = df ∘ sigmoid($z_f$)$^T$ ∘ (1-sigmoid($z_f$))$^T$

# Finally invert $z_f$ = $W_{fc}$C + $W_{fh}$h + $W_{fx}$x + $b_f$
dC += $dz_f$ $W_{fc}$
dh += $dz_f$ $W_{fh}$
dx += $dz_f$ $W_{fx}$

$dW_{fc}$ = C $dz_f$
$dW_{fh}$ = h $dz_f$
$dW_{fx}$ = x $dz_f$
$db_f$ = $dz_f$

return dC, dh, dx, d[W, b]
# d[W,b] is shorthand for the complete set
  of weight and bias derivatives
```

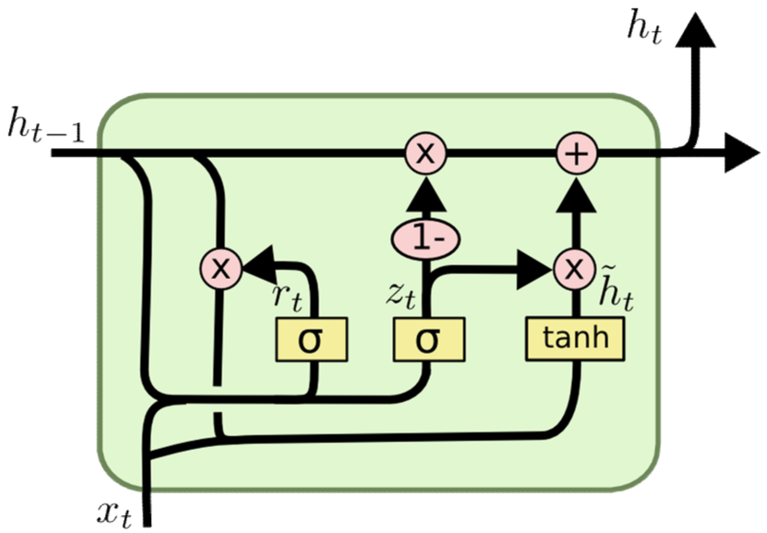
LSTM network forward (for reference)

```
# Assuming h(-1,*) is known and C(-1,*)=0
# Assuming L hidden-state layers and an output layer
# Note: LSTM_cell is an indexed class with functions
# [W{l},b{l}] are the entire set of weights and biases
#              for the lth hidden layer
# Wo and bo are output layer weights and biases

for t = 0:T-1  # Including both ends of the index
    h(t,0) = x(t) # Vectors. Initialize h(0) to input
    for l = 1:L  # hidden layers operate at time t
        [C(t,l),h(t,l)] = LSTM_cell(t,l).forward(…
              …C(t-1,l),h(t-1,l),h(t,l-1)[W{l},b{l}])
    zo(t) = Woh(t,L) + bo
    Y(t) = softmax( zo(t) )
```

LSTM network backward

```
# Assuming h(-1,*) is known and C(-1,*)=0
# Assuming L hidden-state layers and an output layer
# Note: LSTM_cell is an indexed class with functions
# [W{l},b{l}] are the entire set of weights and biases
#              for the lth hidden layer
# W_o and b_o are output layer weights and biases
# Y is the output of the network
# Assuming dW_o and db_o and d[W{l} b{l}] (for all l) are
#          all initialized to 0 at the start of the computation

for t = T-1:0  # Including both ends of the index
    dz_o  = dY(t) ∘ sigmoid(z_o(t))^T ∘(1- sigmoid(z_o(t)))^T
    dW_o += h(t,L) dz_o(t)
    dh(t,L)  = dz_o(t)W_o
    db_o += dz_o(t)

    for l = L-1:0
        [dC(t,l),dh(t,l),dx(t,l),d[W, b]] = …
             … LSTM_cell(t,l).backward(…
             … dC(t+1,l), dh(t+1,l)+dx(t,l+1), C(t,l), h(t,l), …
             … C(t,l), h(t,l),[W(l),b(l)])
        d[W{l} b{l}] += d[W,b]
```

Gated Recurrent Units: Lets simplify the LSTM



$$z_t = \sigma\left(W_z \cdot [h_{t-1}, x_t]\right)$$

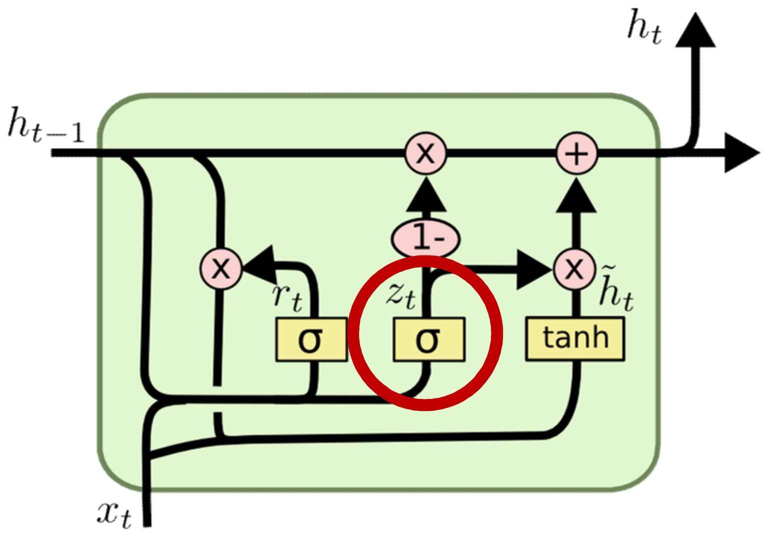
$$r_t = \sigma\left(W_r \cdot [h_{t-1}, x_t]\right)$$

$$\tilde{h}_t = \tanh\left(W \cdot [r_t * h_{t-1}, x_t]\right)$$

$$h_t = (1 - z_t) * h_{t-1} + z_t * \tilde{h}_t$$

- Simplified LSTM which addresses some of your concerns of *why*

Gated Recurrent Units: Lets simplify the LSTM



$$z_t = \sigma\left(W_z \cdot [h_{t-1}, x_t]\right)$$

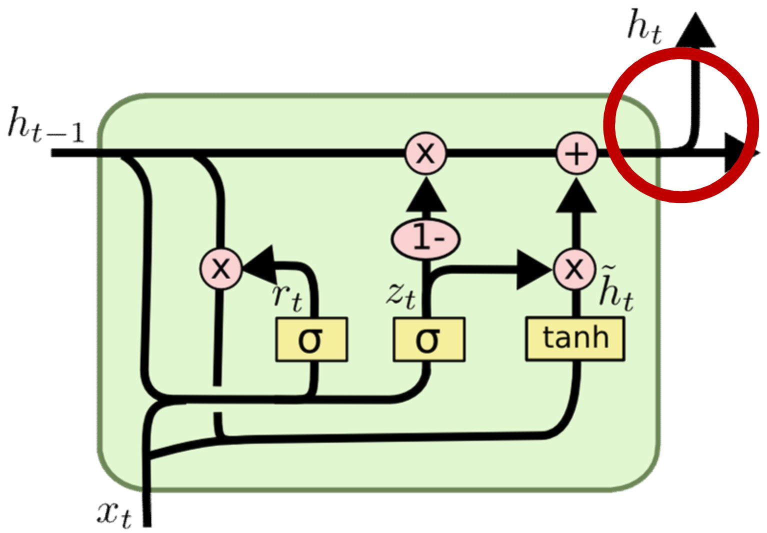
$$r_t = \sigma\left(W_r \cdot [h_{t-1}, x_t]\right)$$

$$\tilde{h}_t = \tanh\left(W \cdot [r_t * h_{t-1}, x_t]\right)$$

$$h_t = (1 - z_t) * h_{t-1} + z_t * \tilde{h}_t$$

- Combine forget and input gates
  - In new input is to be remembered, then this means old memory is to be forgotten
    - Why compute twice?

Gated Recurrent Units: Lets simplify the LSTM



$$z_t = \sigma\left(W_z \cdot [h_{t-1}, x_t]\right)$$

$$r_t = \sigma\left(W_r \cdot [h_{t-1}, x_t]\right)$$

$$\tilde{h}_t = \tanh\left(W \cdot [r_t * h_{t-1}, x_t]\right)$$

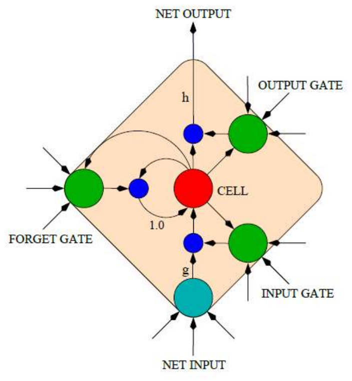
$$h_t = (1 - z_t) * h_{t-1} + z_t * \tilde{h}_t$$

- Don’t bother to separately maintain compressed and regular memories
  - Pointless computation!
  - Redundant representation

LSTM Equations

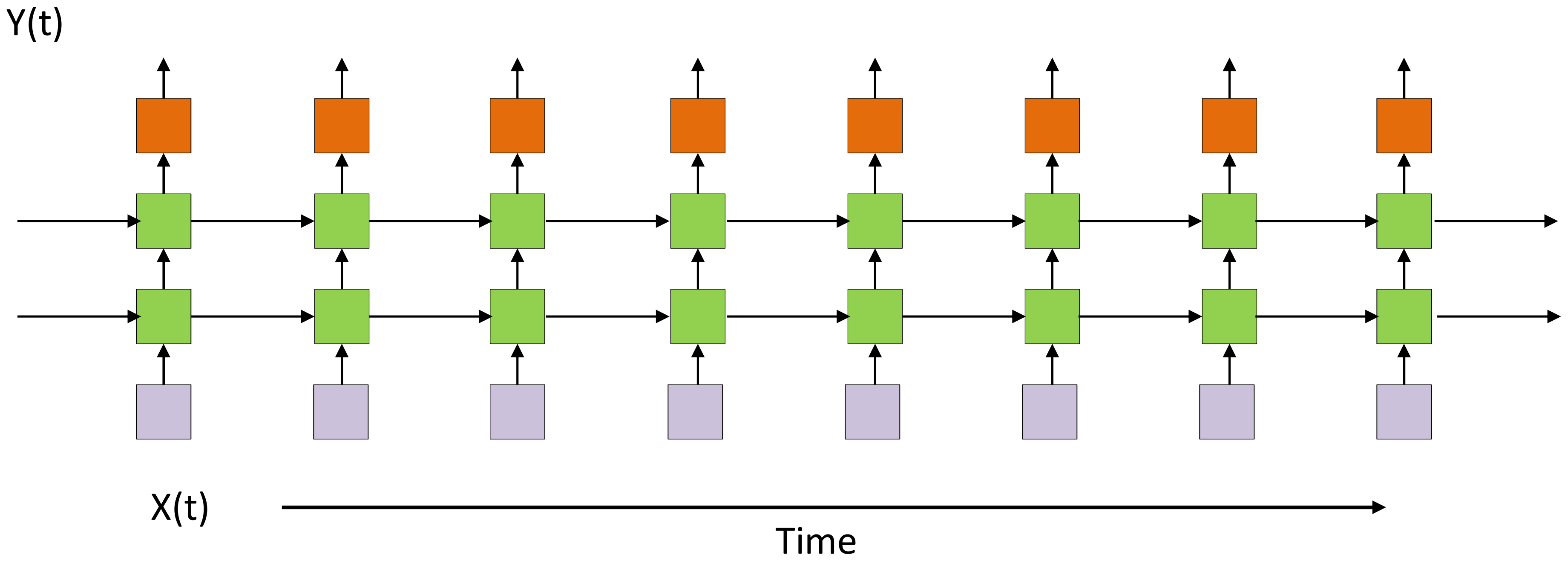
- i: input gate, how much of the new information will be let through the memory cell.
- f: forget gate, responsible for information should be thrown away from memory cell.
- o: output gate, how much of the information will be passed to expose to the next time step.
- g: self-recurrent which is equal to standard RNN
- c_t: internal memory of the memory cell
- s_t: hidden state
- y: final output

- $i = \sigma\left(x_t U^i + s_{t-1} W^i\right)$
- $f = \sigma\left(x_t U^f + s_{t-1} W^f\right)$
- $o = \sigma(x_t U^o + s_{t-1} W^o)$
- $g = \tanh(x_t U^g + s_{t-1} W^g)$
- $c_t = c_{t-1} \circ f + g \circ i$
- $s_t = \tanh(c_t) \circ o$
- $y = softmax(V s_t)$



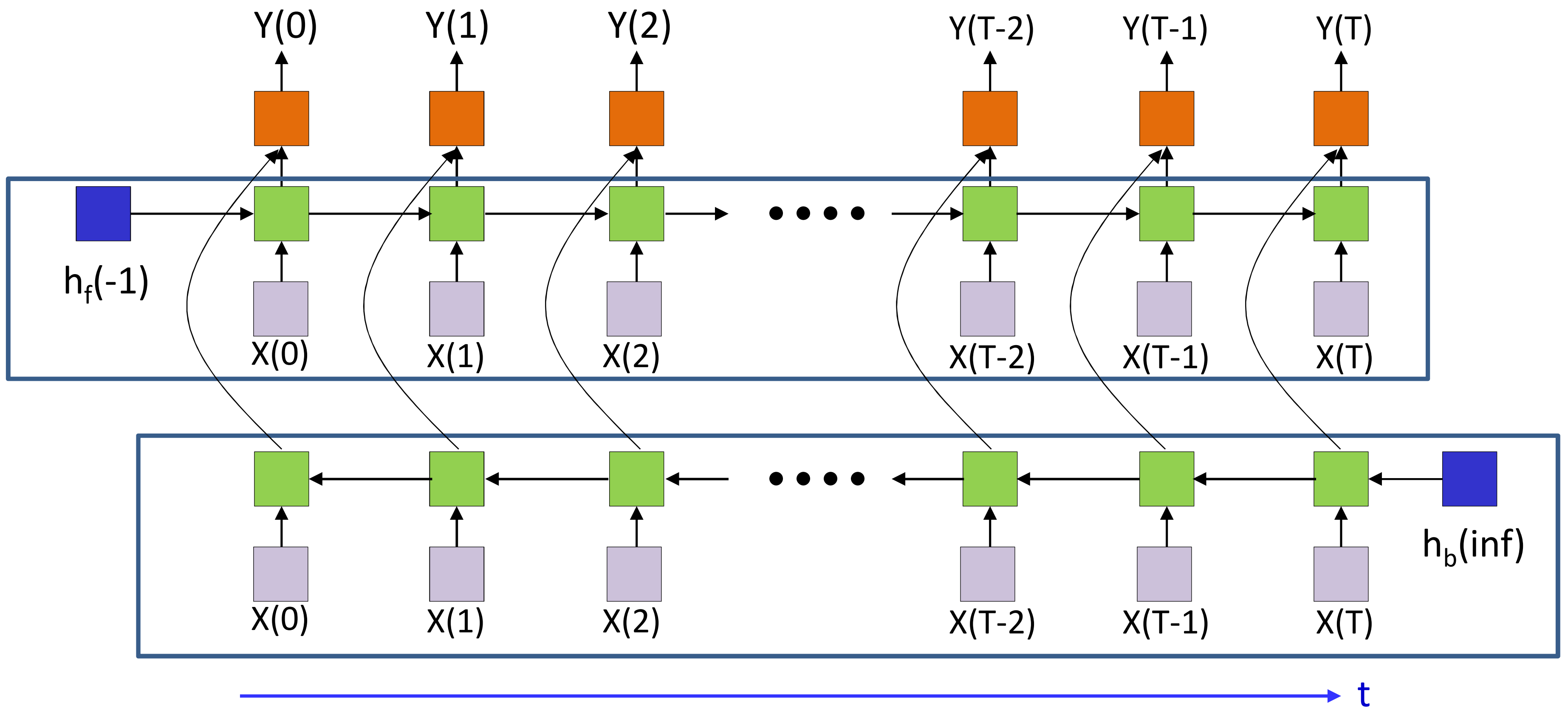
LSTM Memory Cell

LSTM architectures example



- Each green box is now an entire LSTM or GRU unit
- Also keep in mind each box is an *array* of units

Bidirectional LSTM



- Like the BRNN, but now the hidden nodes are LSTM units.
- Can have multiple layers of LSTM units in either direction
  - Its also possible to have MLP feed-forward layers between the hidden layers..
- The output nodes (orange boxes) may be complete MLPs

Story so far

- Recurrent networks are poor at memorization
  - Memory can explode or vanish depending on the weights and activation
- They also suffer from the vanishing gradient problem during training
  - Error at any time cannot affect parameter updates in the too-distant past
  - E.g. seeing a “close bracket” cannot affect its ability to predict an “open bracket” if it happened too long ago in the input

- LSTMs are an alternative formalism where memory is made more directly dependent on the input, rather than network parameters/structure
  - Through a “Constant Error Carousel” memory structure with no weights or activations, but instead direct switching and “increment/decrement” from pattern recognizers
  - Do not suffer from a vanishing gradient problem but ***do suffer from*** ***exploding*** **gradient issue**

Significant issues

- The Divergence
- How to use these nets..
- This and more in next couple of classes..