# Component Analysis Methods for Signal Processing

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# Component Analysis for PR

- Computer Vision & Image Processing
  - Structure from motion.
  - Spectral graph methods for segmentation.
  - Appearance and shape models.
  - Fundamental matrix estimation and calibration.
  - Compression.
  - Classification.
  - Dimensionality reduction and visualization.
- Signal Processing
  - Spectral estimation, system identification (e.g. Kalman filter), sensor array processing (e.g. cocktail problem, eco cancellation), blind source separation, …
- Computer Graphics
  - Compression (BRDF), synthesis,…
- Speech, bioinformatics, combinatorial problems.

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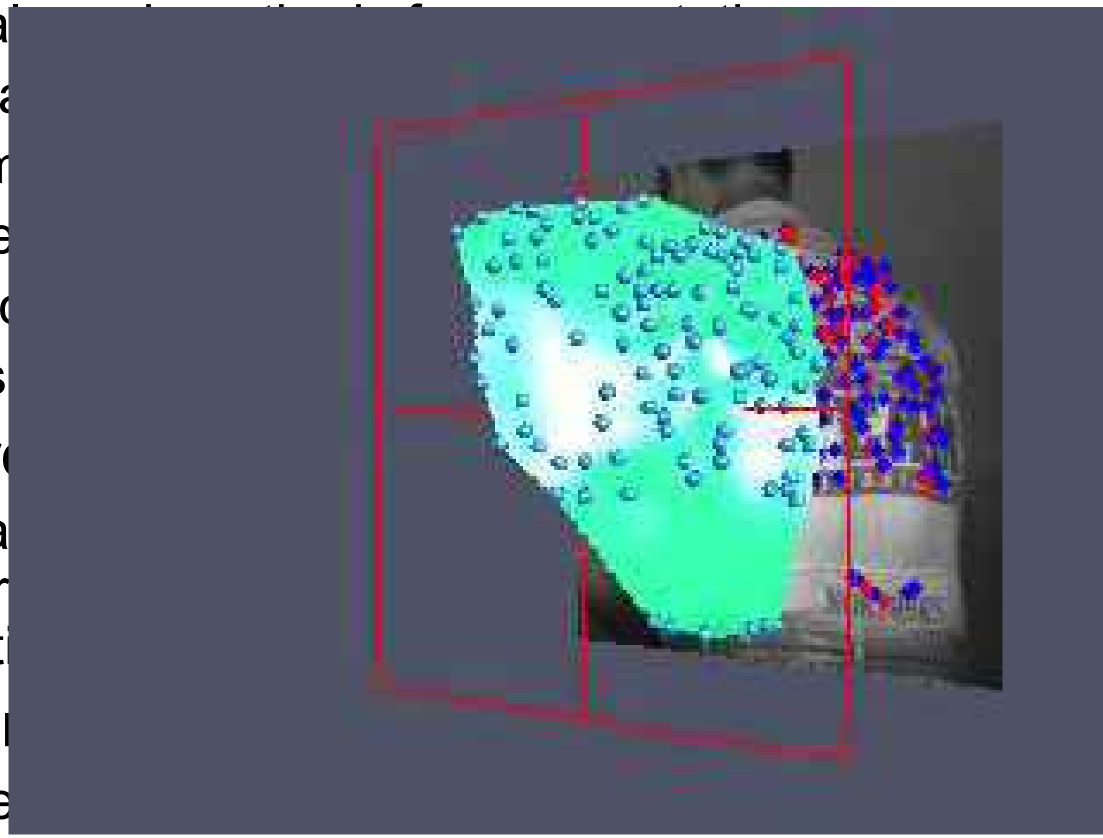

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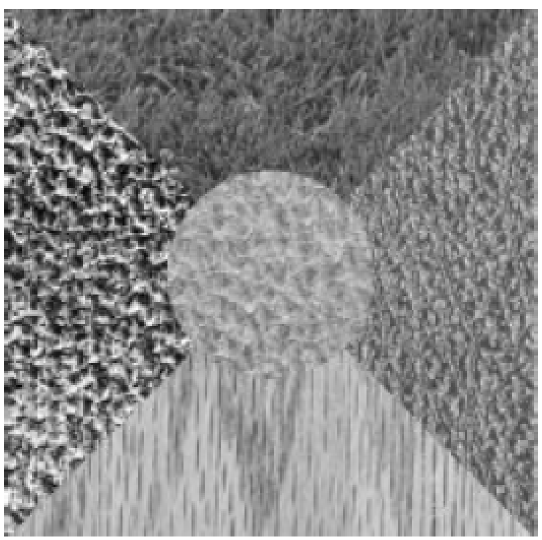

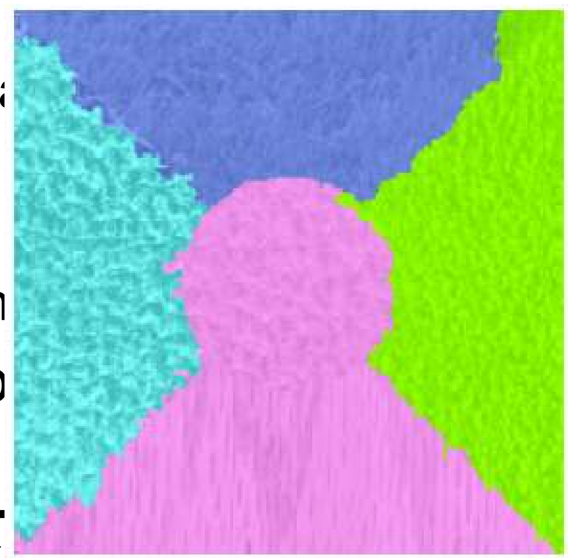

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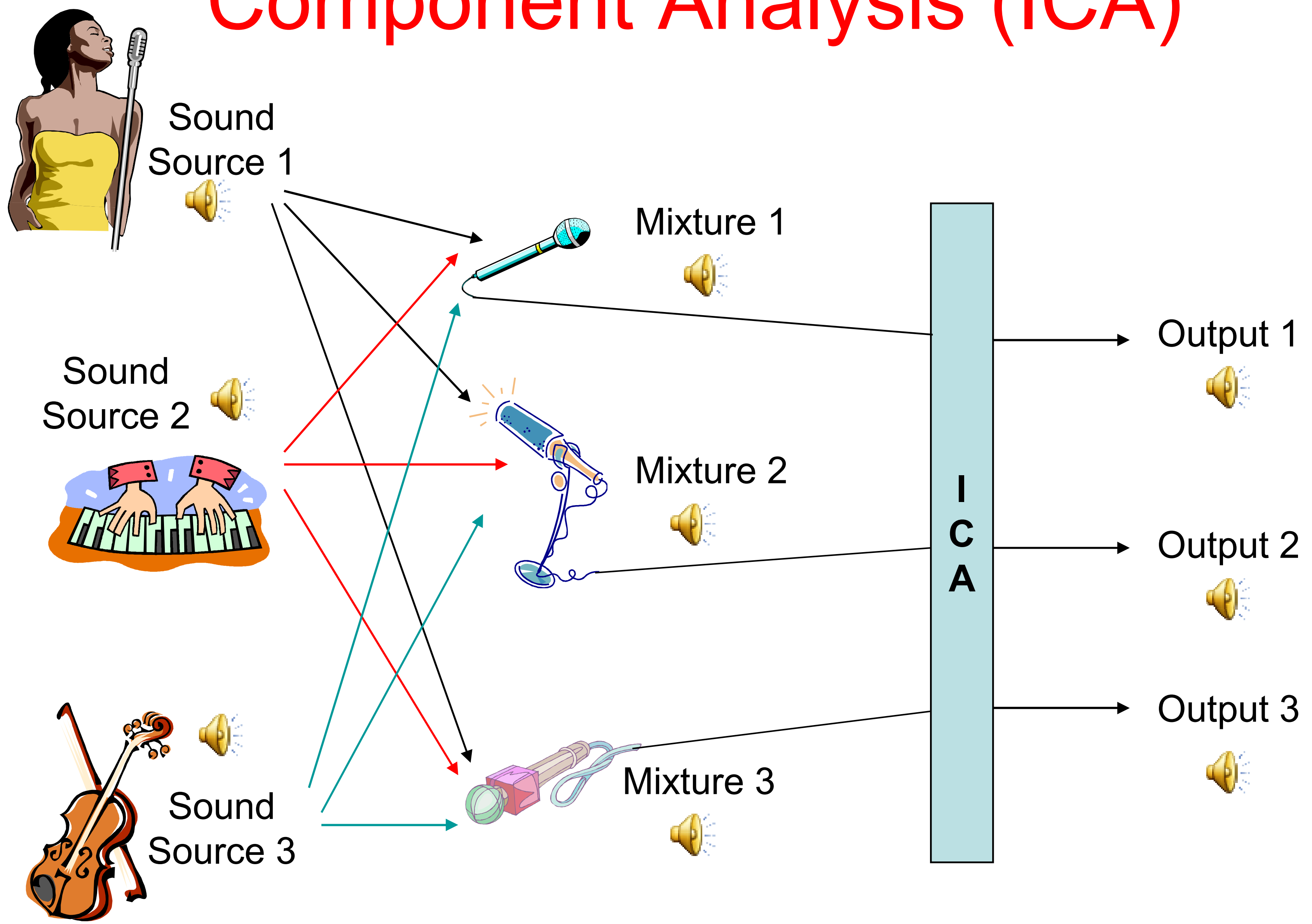
# Component Analysis (ICA)



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# Why Component Analysis for PR?

- Learn from high dimensional data and few samples.
  - Useful for dimensionality reduction.

- Easy to incorporate
  - Robustness to noise, missing data, outliers (de la Torre & Black, 2003a)
  - Invariance to geometric transformations (de la Torre & Black, 2003b; de la Torre & Nguyen,2007)
  - Non-linearities (Kernel methods) (Scholkopf & Smola,2002; Shawe-Taylor & Cristianini,2004)
  - Probabilistic (latent variable models) (Everitt,1984)
  - Multi-factorial (tensors) (Paatero & Tapper, 1994 ;O'Leary & Peleg,1983; Vasilescu & Terzopoulos,2002; Vasilescu & Terzopoulos,2003)
  - Exponential family PCA (Gordon,2002; Collins et al. 01)

- Efficient methods $O( d \quad n< <n^2 )$

  d: features

  $n< <n^2$: samples

# Are CA Methods Popular/Useful/Used?

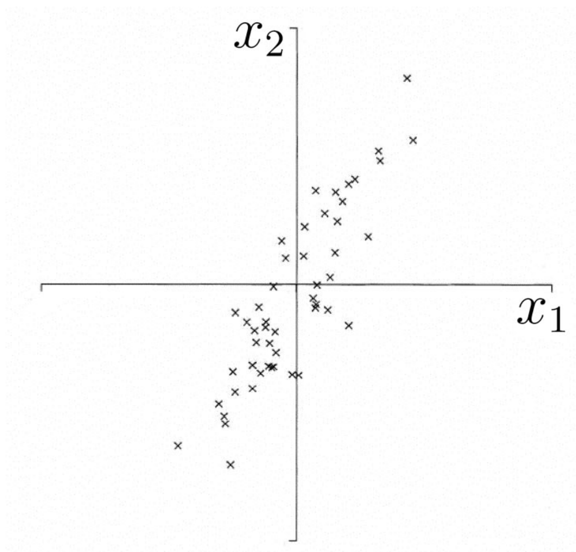
- About 20% of CVPR-06 papers use CA.
- Google:
  - Results **1** - **10** of about **1,870,000** for **"principal component analysis"**.
  - Results **1** - **10** of about **506,000** for **"independent component analysis"**.
  - Results **1** - **10** of about **273,000** for **"linear discriminant analysis"**.
  - Results **1** - **10** of about **46,100** for **"negative matrix factorization"**.
  - Results **1** - **10** of about **491,000** for **"kernel methods"**.
- Still work to do
  - Results **1** - **10** of about **65,300,000** for **"Britney Spears"**.

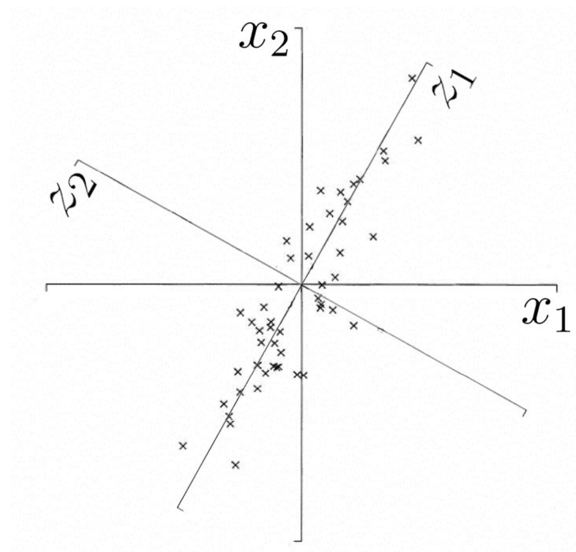
# Outline

- Introduction
- Generative models
  - Principal Component Analysis (PCA)
  - Non-negative Matrix Factorization (NMF)
  - Independent Component Analysis (ICA)
  - Multidimensional Scaling (MDS)
- Discriminative models
  - Linear Discriminant Analysis (LDA).
  - Oriented Component Analysis (OCA).
  - Canonical Correlation Analysis (CCA).
- Standard extensions of linear models
  - Kernel methods.
  - Latent variable models.
  - Tensor factorization

# Principal Component Analysis (PCA)

(Pearson, 1901; Hotelling, 1933;Mardia et al., 1979; Jolliffe, 1986; Diamantaras, 1996)



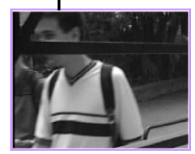

- PCA finds the directions of maximum variation of the data based on linear correlation.
- PCA decorrelates the original variables.

# PCA

d = pixels

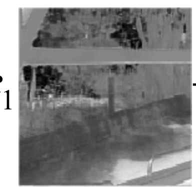
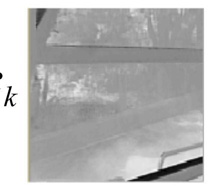
n = images

$$\mathbf{D} = \left[\mathbf{d}_1 \;\; \mathbf{d}_2 \;\; \ldots \;\; \mathbf{d}_n\right] \approx \mathbf{B}\mathbf{C} + \boldsymbol{\mu}\mathbf{1}_n^T$$



$$\mathbf{D} \in \Re^{d\times n} \quad \mathbf{B} \in \Re^{d\times k}$$

$$\mathbf{C} \in \Re^{k\times n} \quad \boldsymbol{\mu} \in \Re^{d\times 1}$$

 $\approx \mu$  $+ c_1$  $+ c_2$  $+ \ldots\ldots + c_k$ 

- Assuming 0 mean data, the basis **B** that preserve the maximum variation of the signal is given by the eigenvectors of **DD**$^T$.

$$\underbrace{\mathbf{D}\mathbf{D}^T}_{d \times d}\mathbf{B} = \mathbf{B}\boldsymbol{\Lambda}$$

# Snap-shot Method & SVD

- If d>>n (e.g. images 100*100 vs. 300 samples) no $\mathbf{DD}^T$.
- $\mathbf{DD}^T$ and $\mathbf{D}^T\mathbf{D}$ have the same eigenvalues (energy) and related eigenvectors (by **D**).
- **B** is a linear combination of the data! (Sirovich, 1987)

$$\mathbf{DD}^T\mathbf{B} = \mathbf{B}\boldsymbol{\Lambda} \quad \mathbf{B} = \mathbf{D}\boldsymbol{\alpha} \quad \mathbf{D}^T\mathbf{DD}^T\mathbf{D}\boldsymbol{\alpha} = \mathbf{D}^T\mathbf{D}\boldsymbol{\alpha}\boldsymbol{\Lambda}$$

- [**α**,**L**]=eig($\mathbf{D}^T\mathbf{D}$) **B**=**D** **α**(diag(diag(**L**))) $^{-0.5}$

- SVD factorizes the data matrix **D** as:

(Beltrami, 1873; Schmidt, 1907; Golub & Loan, 1989)

$$\mathbf{DD}^T = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T$$

$$\mathbf{D}^T\mathbf{D} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T$$

| PCA | SVD |
| --- | --- |
| $\mathbf{D} = \mathbf{BC}$ | $\mathbf{D} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ |
| $\mathbf{B} \in \Re^{d \times k}$ $\quad \mathbf{C} \in \Re^{k \times n}$ | $\mathbf{U} \in \Re^{d \times k}$ $\quad \boldsymbol{\Sigma} \in \Re^{k \times n}$ $\quad \mathbf{V} \in \Re^{n \times n}$ |
| $\mathbf{B}^T\mathbf{B} = \mathbf{I}$ $\quad \mathbf{CC}^T = \boldsymbol{\Lambda}$ | $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ $\quad \mathbf{V}^T\mathbf{V} = \mathbf{I}$ $\quad \boldsymbol{\Sigma}$ *diagonal* |

# Error Function for PCA

- PCA minimizes the following **CONVEX** function.

(Eckardt & Young, 1936; Gabriel & Zamir, 1979; Baldi & Hornik, 1989; Shum et al., 1995; de la Torre & Black, 2003a)

$$E_1(\mathbf{B},\mathbf{C}) = \sum_{i=1}^{n} \left\| \mathbf{d}_i - \mathbf{B}\mathbf{c}_i \right\|_2^2 = \left\| \mathbf{D} - \mathbf{B}\mathbf{C} \right\|_F$$

- Not unique solution: $\mathbf{B}\mathbf{R}\mathbf{R}^{-1}\mathbf{C} = \mathbf{B}\mathbf{C}$ $\qquad \mathbf{R} \in \Re^{k \times k}$
- To obtain same PCA solution **R** has to satisfy:

$$\hat{\mathbf{B}} = \mathbf{B}\mathbf{R} \qquad \hat{\mathbf{C}} = \mathbf{R}^{-1}\mathbf{C}$$

$$\hat{\mathbf{B}}^T\hat{\mathbf{B}} = \mathbf{I} \qquad \hat{\mathbf{C}}\hat{\mathbf{C}}^T = \Lambda$$

- **R** is computed as a generalized k×k eigenvalue problem.

(de la Torre, 2006)

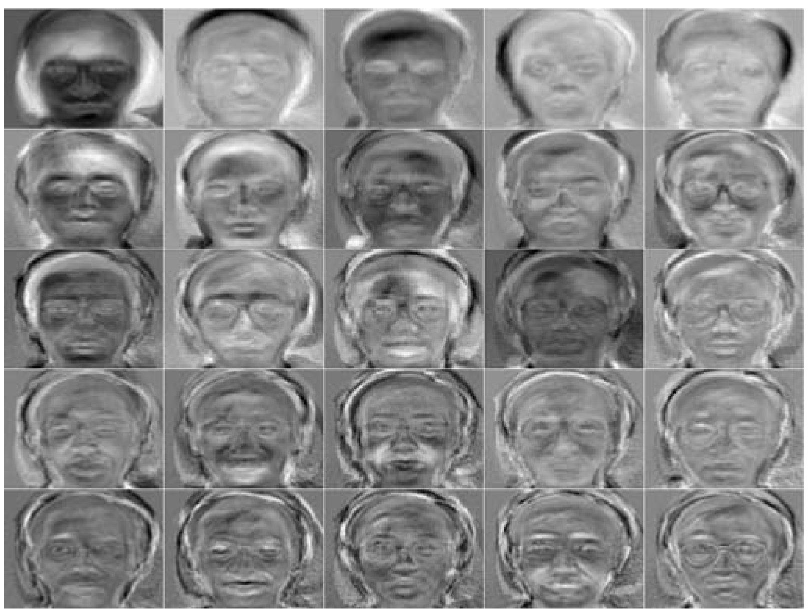
$$\left(\mathbf{C}\mathbf{C}^T\right)^{-1}\mathbf{R} = \mathbf{B}^T\mathbf{B}\mathbf{R}\Lambda^{-1}$$

# PCA/SVD in Computer Vision

- PCA/SVD has been applied to:
  - Recognition (eigenfaces:Turk & Pentland, 1991; Sirovich & Kirby, 1987; Leonardis & Bischof, 2000; Gong et al., 2000; McKenna et al., 1997a)
  - Parameterized motion models (Yacoob & Black, 1999; Black et al., 2000; Black, 1999; Black & Jepson, 1998)
  - Appearance/shape models (Cootes & Taylor, 2001; Cootes et al., 1998; Pentland et al., 1994; Jones & Poggio, 1998; Casia & Sclaroff, 1999; Black & Jepson, 1998; Blanz & Vetter, 1999; Cootes et al., 1995; McKenna et al., 1997; de la Torre et al., 1998b; de la Torre et al., 1998b)
  - Dynamic appearance models (Soatto et al., 2001; Rao, 1997; Orriols & Binefa, 2001; Gong et al., 2000)
  - Structure from Motion (Tomasi & Kanade, 1992; Bregler et al., 2000; Sturm & Triggs, 1996; Brand, 2001)
  - Illumination based reconstruction (Hayakawa, 1994)
  - Visual servoing (Murase & Nayar, 1995; Murase & Nayar, 1994)
  - Visual correspondence (Zhang et al., 1995; Jones & Malik, 1992)
  - Camera motion estimation (Hartley, 1992; Hartley & Zisserman, 2000)
  - Image watermarking (Liu & Tan, 2000)
  - Signal processing (Moonen & de Moor, 1995)
  - Neural approaches (Oja, 1982; Sanger, 1989; Xu, 1993)
  - Bilinear models (Tenenbaum & Freeman, 2000; Marimont & Wandell, 1992)
  - Direct extensions (Welling et al., 2003; Penev & Atick, 1996)

# “Intercorrelations among variables are the bane of the multivariate researcher’s struggle for meaning”

**Cooley and Lohnes, 1971**



# Part-based Representation

- The firing rates of neurons are never negative.
- Independent representations.

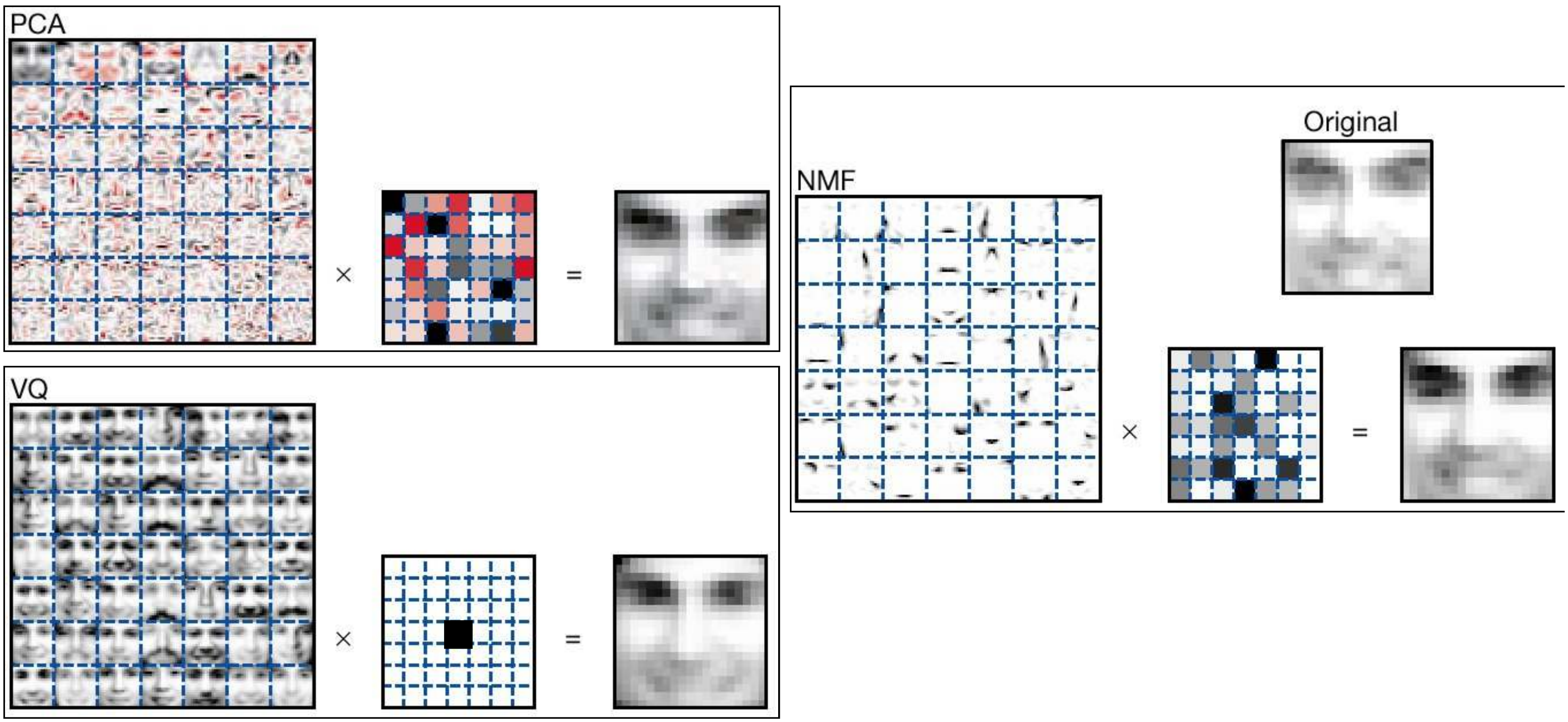
NMF & ICA

# Non-negative Matrix Factorization

- Positive factorization.

$$E(\mathbf{B},\mathbf{C}) = \|\mathbf{D}-\mathbf{BC}\|_F \quad \mathbf{B},\mathbf{C} \geq 0$$

- Leads to part-based representation.



# Nonnegative Factorization

(Lee & Seung, 1999;Lee & Seung, 2000)

$$\min_{\mathbf{B}\geq 0,\mathbf{C}\geq 0} F = \sum_{ij} \left| d_{ij} - (\mathbf{BC})_{ij} \right|^2$$

Inference:

$$\mathbf{C}_{ij} \leftarrow \mathbf{C}_{ij} \frac{(\mathbf{B}^{\mathrm{T}}\mathbf{D})_{ij}}{(\mathbf{B}^{\mathrm{T}}\mathbf{BV})_{ij}}$$

Derivatives:

$$\frac{\partial F}{\partial \mathbf{C}_{ij}} = (\mathbf{B}^{\mathrm{T}}\mathbf{BC})_{ij} - (\mathbf{B}^{\mathrm{T}}\mathbf{C})_{ij}$$

Learning:

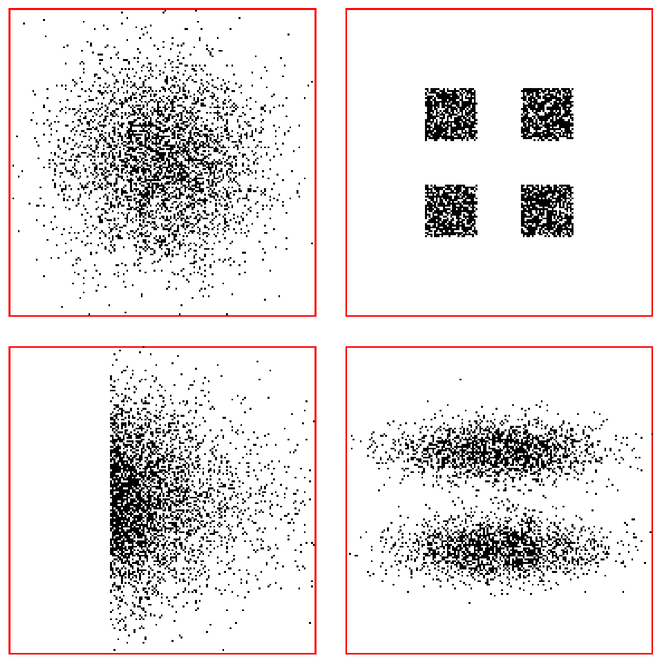
$$\mathbf{B}_{ij} \leftarrow \mathbf{B}_{ij} \frac{(\mathbf{DC}^{T})_{ij}}{(\mathbf{BC}\mathbf{C}^{T})_{ij}}$$

$$\frac{\partial F}{\partial \mathbf{B}_{ij}} = (\mathbf{BCC}^{T})_{ij} - (\mathbf{DC}^{T})_{ij}$$

- Multiplicative algorithm can be interpreted as diagonally rescaled gradient descent.

# Independent Component Analysis

- We need more than second order statistics to represent the signal.

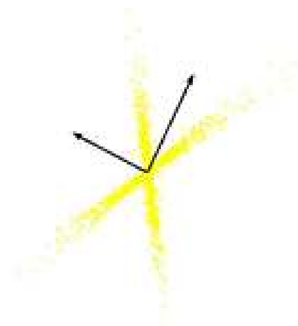


# ICA

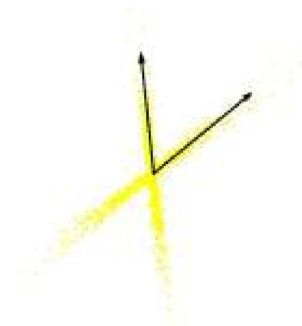
(Hyvrinen et al., 2001)

$$\mathbf{D} = \mathbf{BC} \quad \mathbf{C} \approx \mathbf{S} = \mathbf{WD} \quad \mathbf{W} \approx \mathbf{B}^{-1}$$

- Look for $s_i$ that are independent.
- PCA finds uncorrelated variables, the independent components have non Gaussian distributions.
- Uncorrelated $E(s_i s_j) = E(s_i)E(s_j)$
- Independent $E(g(s_i)f(s_j)) = E(g(s_i))E(f(s_j))$ for any non-linear f,g

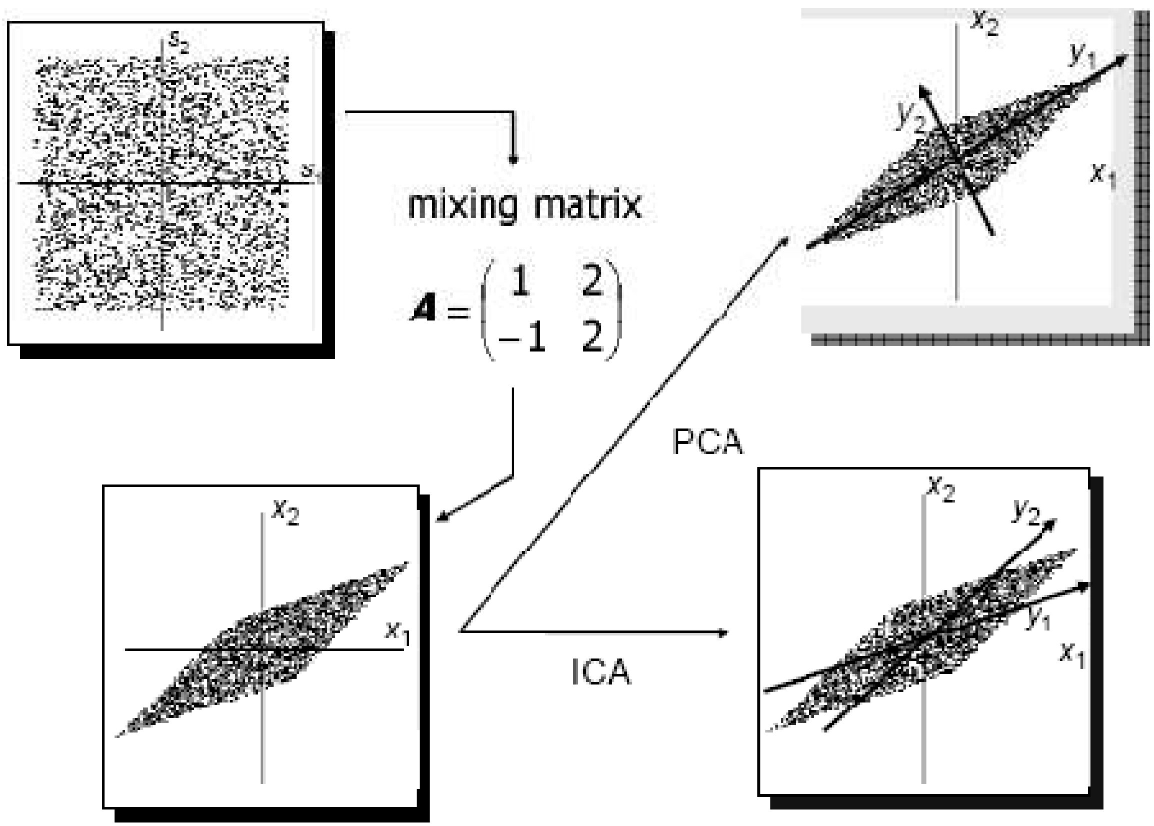


PCA



ICA

# ICA vs PCA



$s_2$
$s_1$
mixing matrix
$A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$
$x_2$
$x_1$
PCA
$x_2$
$y_1$
$y_2$
$x_1$
ICA
$x_2$
$y_2$
$y_1$
$x_1$

# Many optimization criteria

- Minimize high order moments: e.g. kurtosis

  kurt($\boldsymbol{W}$) = $E\{s^4\} - 3(E\{s^2\})^2$
- Many other information criteria.
- Also an error function: (Olhausen & Field, 1996)

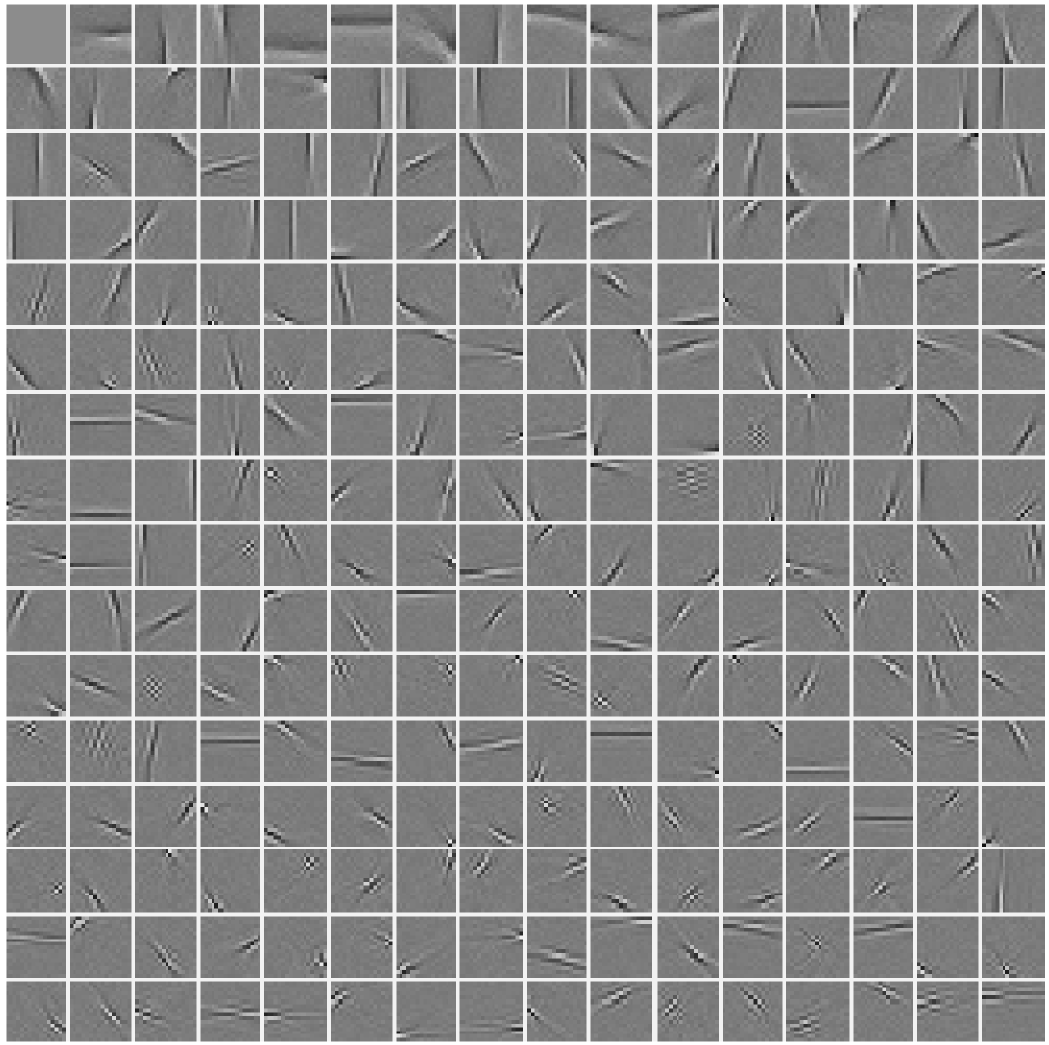
$$\sum_{i=1}^{n} \left\| \mathbf{d}_i - \mathbf{B}\mathbf{c}_i \right\| + \sum_{i=1}^{n} S(\mathbf{c}_i)$$

Sparseness (e.g. S=| |)

- Other sparse PCA.

  (Chennubhotla & Jepson, 2001b; Zou et al., 2005; dAspremont et al., 2004;)

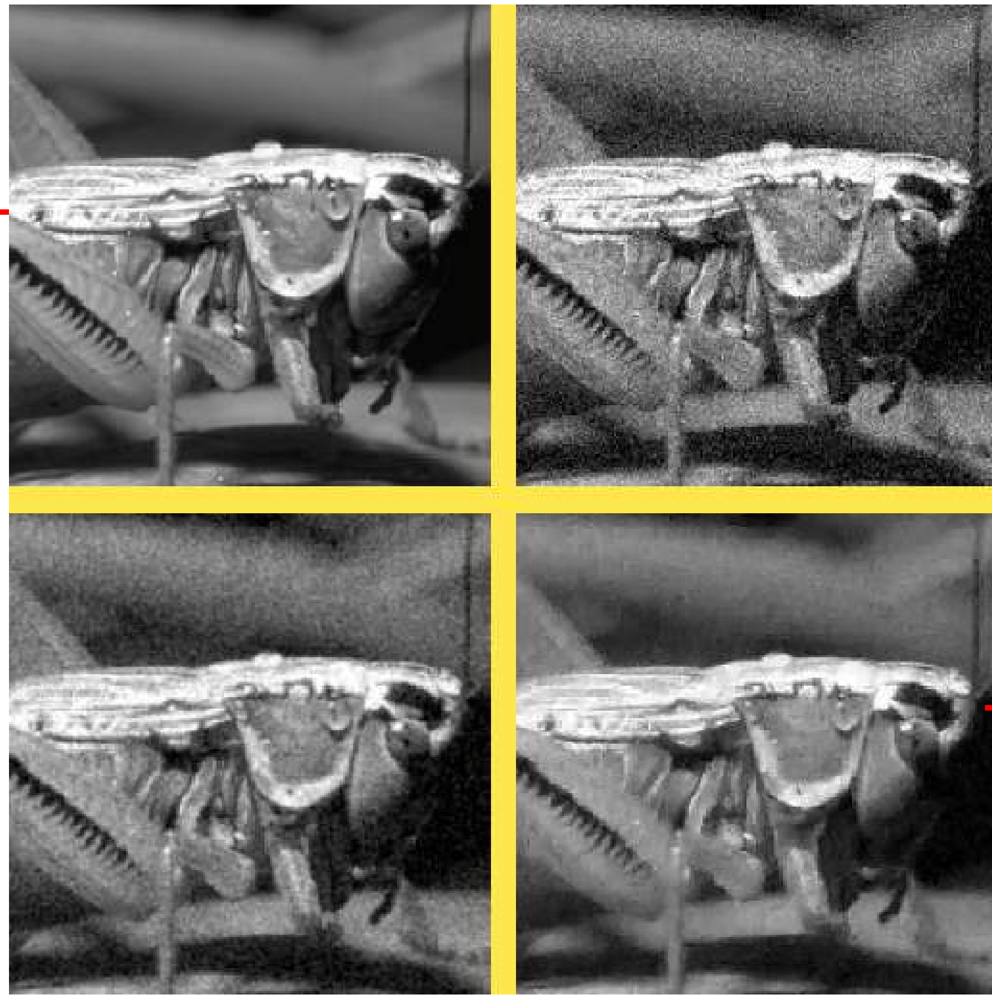
# Basis of natural images



# Denoising

Original image

Noisy Image (30% noise)



Denoise (Wiener filter)

ICA

# Multidimensional Scaling (MDS)

- MDS takes a matrix of pair-wise distances and finds an embedding that preserves the interpoint distances.

## An example: map of the US

| | Chicago | Raleigh | Boston | Seattle | S.F. | Austin | Orlando |
|---|---|---|---|---|---|---|---|
| Chicago | 0 | | | | | | |
| Raleigh | 641 | 0 | | | | | |
| Boston | 851 | 608 | 0 | | | | |
| Seattle | 1733 | 2363 | 2488 | 0 | | | |
| S.F. | 1855 | 2406 | 2696 | 684 | 0 | | |
| Austin | 972 | 1167 | 1691 | 1764 | 1495 | 0 | |
| Orlando | 994 | 520 | 1105 | 2565 | 2458 | 1015 | 0 |

# MDS(II)

Optimize w.r.t $y_i$

$$\sum_i \sum_j (\delta_{ij} - d_{ij})^2$$

Observed distance between points i and j in p-space

Distance between the points in two-dimensional space

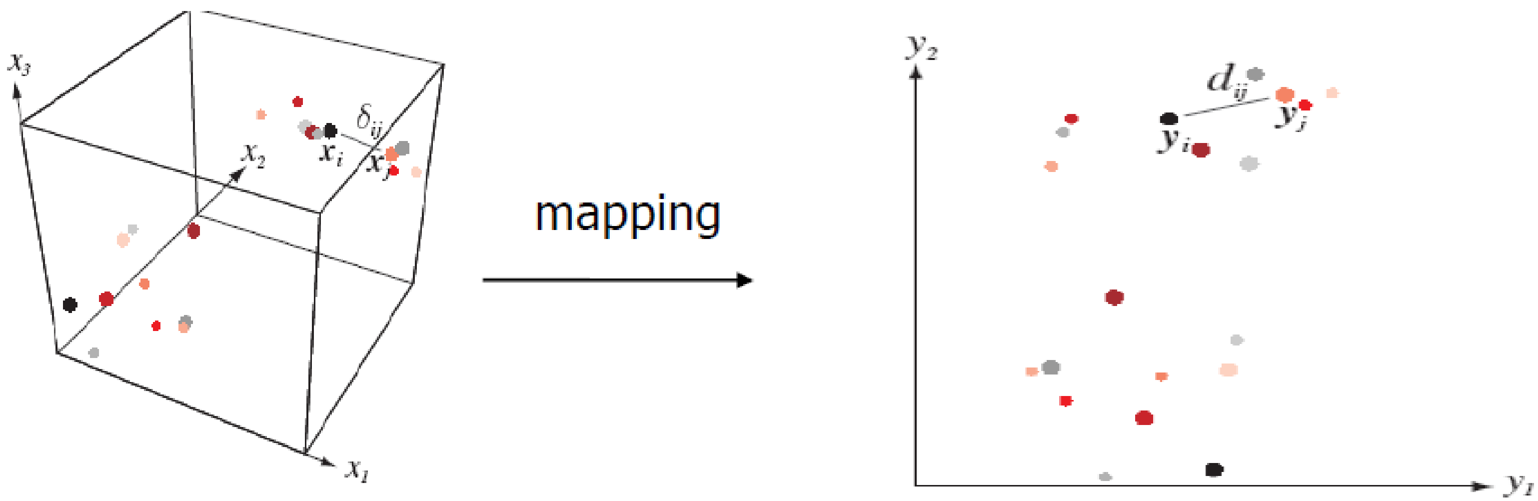
Criterion is invariant wrt rotations and translations.
However it is not invariant to scaling
Better criterion is

$$\frac{\sum_i \sum_j (\delta_{ij} - d_{ij})^2}{\sum_{i,j} d_{ij}^{\ 2}}$$

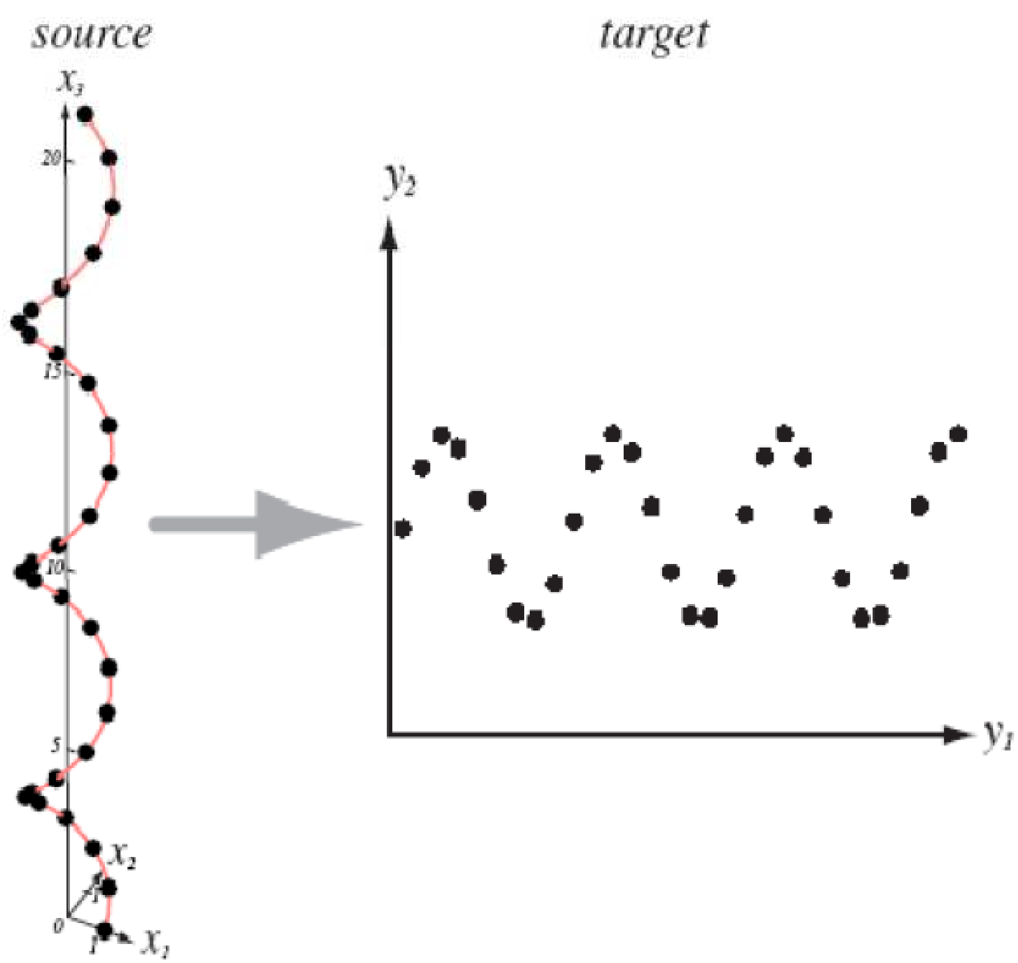
or

$$\sqrt{\frac{\sum_i \sum_j (\delta_{ij} - d_{ij})^2}{\sum_{i,j} d_{ij}^{\ 2}}}$$

Called stress



# MDS (III)



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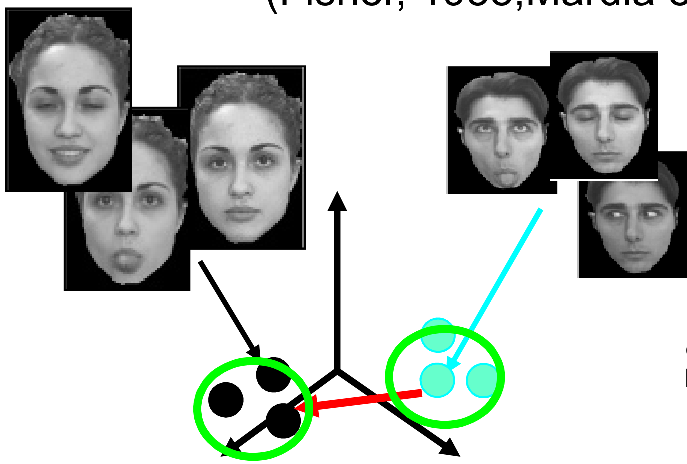
# Linear Discriminant Analysis (LDA)

(Fisher, 1938;Mardia et al., 1979; Bishop, 1995)



$$\mathbf{S}_b = \sum_{i=1}^{C}\sum_{j=1}^{C}(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)(\boldsymbol{\mu}_i - \boldsymbol{\mu}_j)^T$$

$$\mathbf{S}_t = \mathbf{D}\mathbf{D}^T = \sum_{i=1}^{n}\mathbf{d}_i\mathbf{d}_i^T$$

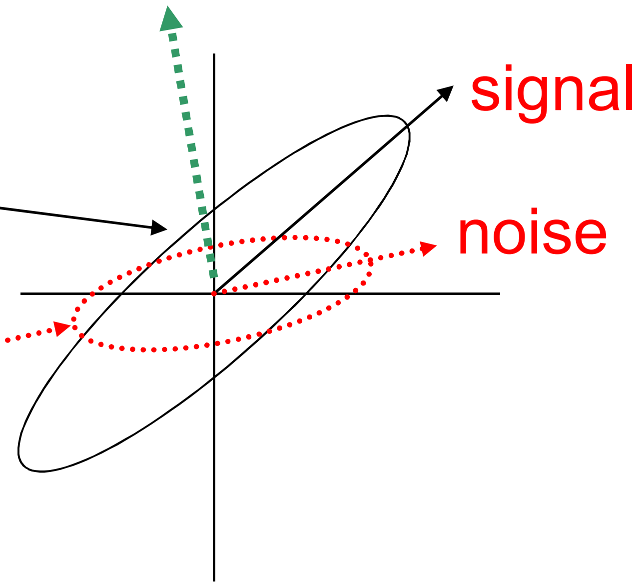
$$J(\mathbf{B}) = \frac{|\mathbf{B}^T\mathbf{S}_\mathbf{b}\mathbf{B}|}{|\mathbf{B}^T\mathbf{S}_t\mathbf{B}|} \qquad \mathbf{S}_\mathbf{b}\mathbf{B} = \mathbf{S}_t\mathbf{B}\boldsymbol{\Lambda}$$

$$\mathbf{S}_w = \sum_{j=1}^{c}\sum_{i=1}^{C_i}(\mathbf{d}_i - \boldsymbol{\mu}_j)(\mathbf{d}_i - \boldsymbol{\mu}_j)^T$$

- Optimal linear dimensionality reduction if classes are Gaussian with equal covariance matrix.

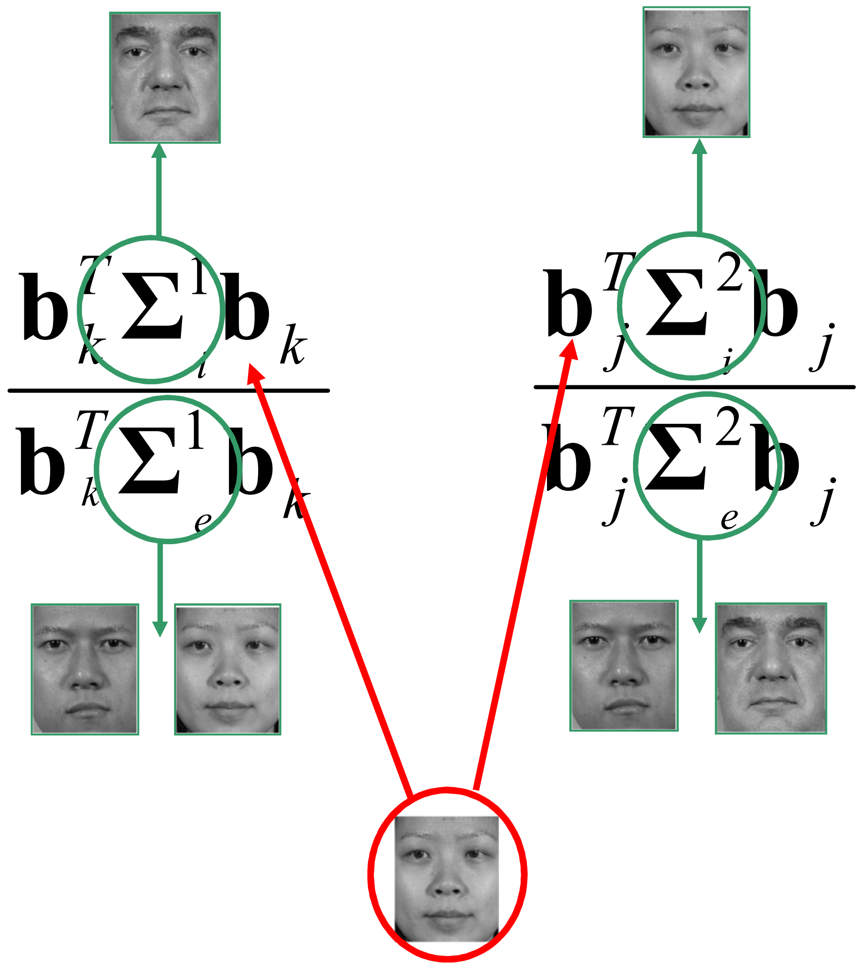
# Oriented Component Analysis (OCA)

$$\frac{\mathbf{b}_{OCA}^{T}\,\boldsymbol{\Sigma}_{signal}\,\mathbf{b}_{OCA}}{\mathbf{b}_{OCA}^{T}\,\boldsymbol{\Sigma}_{noise}\,\mathbf{b}_{OCA}}$$



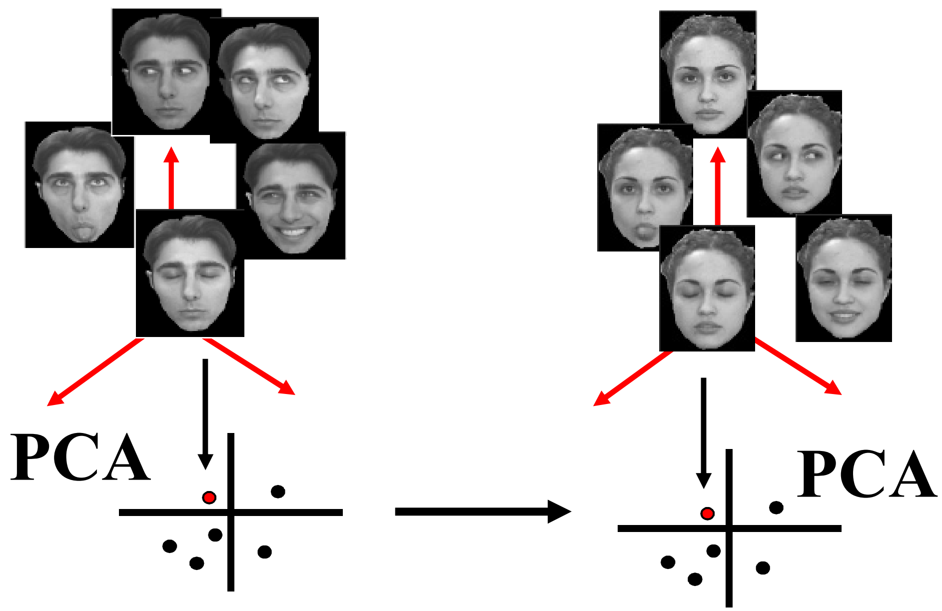

- Generalized eigenvalue problem: $\boldsymbol{\Sigma}_i \mathbf{b}_k = \boldsymbol{\Sigma}_e \mathbf{b}_k \lambda$
- $\mathbf{b}_{oca}$ is steered by the distribution of noise.

# OCA for face recognition



# Canonical Correlation Analysis (CCA)

- PCA independently and general mapping




- Signals dependent signals with small energy can be lost.

# Canonical Correlation Analysis (CCA)

(Mardia et al., 1979; Borga)

- Learn relations between multiple data sets? (e.g. find features in one set related to another data set)
- Given two sets $\mathbf{X} \in \Re^{d_1 \times n}$ *and* $\mathbf{Y} \in \Re^{d_2 \times n}$, CCA finds the pair of directions $\mathbf{w_x}$ and $\mathbf{w_y}$ that maximize the correlation between the projections (assume zero mean data)

$$\rho = \frac{\mathbf{w}_x^T \mathbf{X}^T \mathbf{Y} \mathbf{w}_y}{\sqrt{\mathbf{w}_x^T \mathbf{X}^T \mathbf{X} \mathbf{w}_x^T \mathbf{w}_y^T \mathbf{Y}^T \mathbf{Y} \mathbf{w}_y^T}}$$

- Several ways of optimizing it:

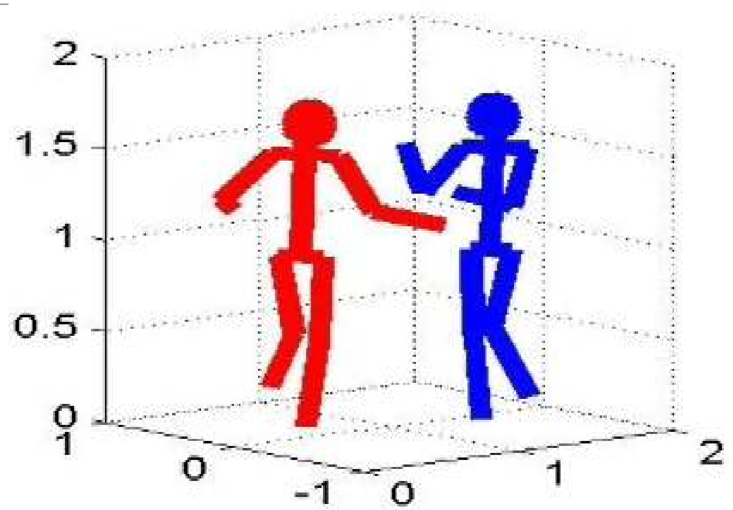
$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{X}^T\mathbf{Y} \\ \mathbf{X}^T\mathbf{Y} & \mathbf{0} \end{bmatrix} \in \Re^{(d_1+d_2)\times(d_1+d_2)}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{X}^T\mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^T\mathbf{Y} \end{bmatrix} \in \Re^{(d_1+d_2)\times(d_1+d_2)} \quad \mathbf{w} = \begin{bmatrix} \mathbf{w}_x \\ \mathbf{w}_y \end{bmatrix}$$

- An stationary point of r is the solution to CCA.

$$\mathbf{Aw} = \lambda \mathbf{Bw}$$

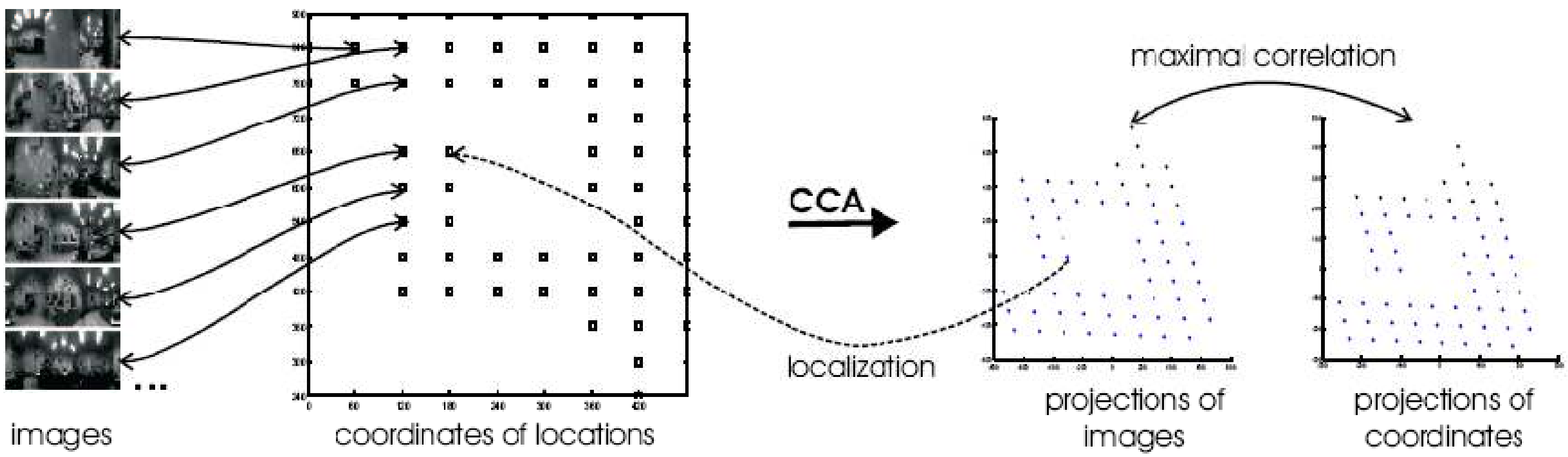
# Dynamic Coupled Component Analysis





# Robot localization with Canonical Correlation Analysis

(Skocaj & Leonardis, 2000)



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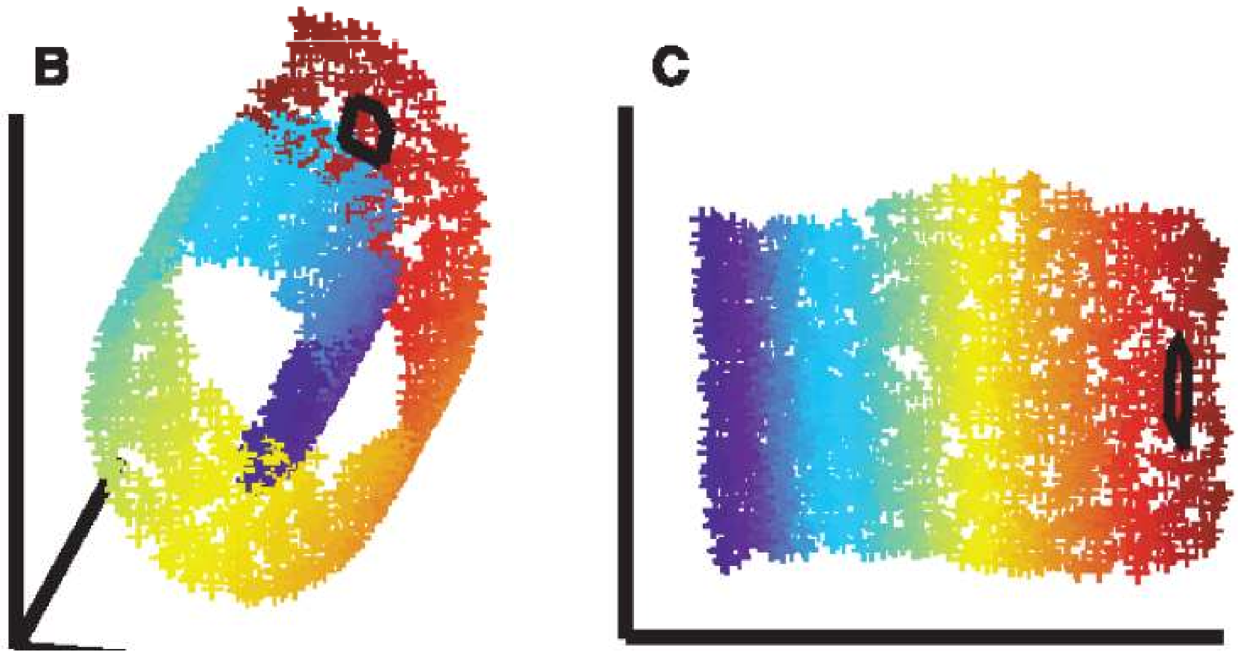
  - Independent Component Analysis (ICA)
  - Multidimensional Scaling (MDS)
- Discriminative models
  - Linear Discriminant Analysis (LDA).
  - Oriented Component Analysis (OCA).
  - Canonical Correlation Analysis (CCA).
- Standard extensions of linear models
  - Kernel methods.
  - Latent variable models.
  - Tensor factorization

# Kernel Methods

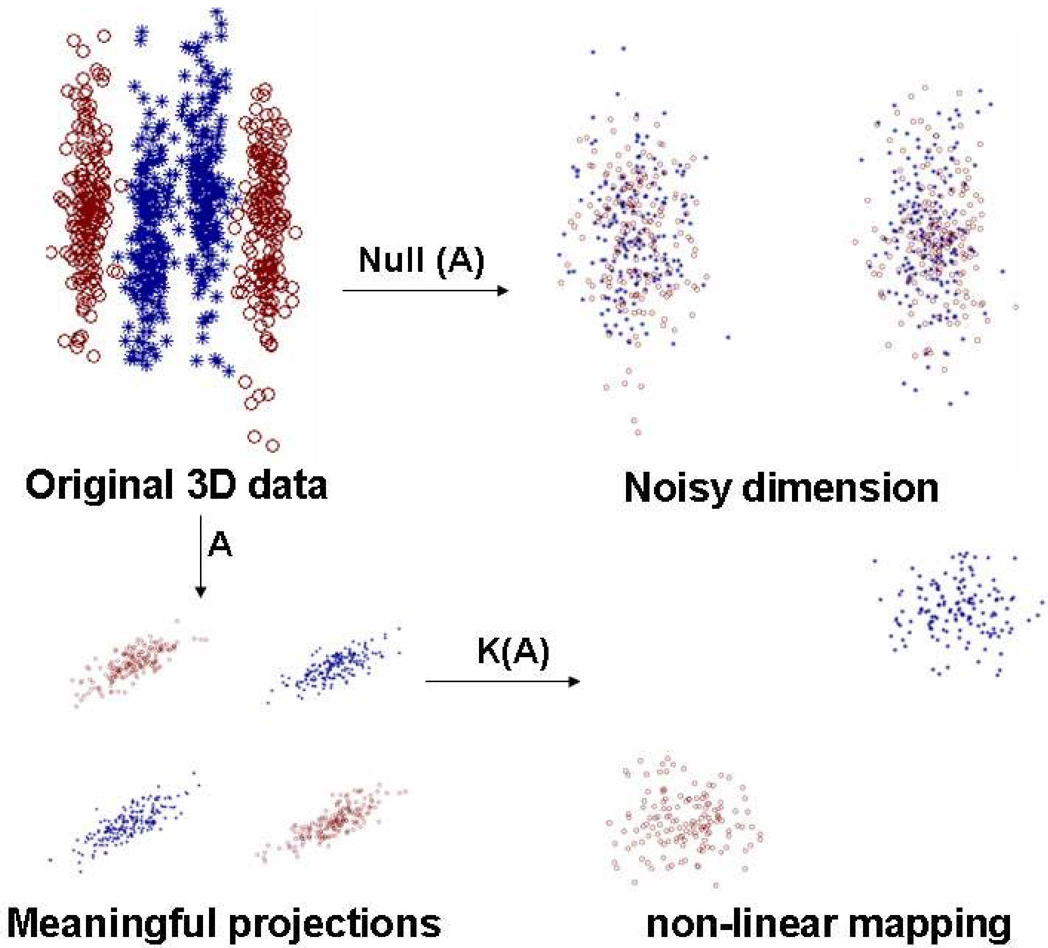
# Linear methods fail

- When data points sit on a non-linear manifold
  - We won't find a good **linear** mapping from the data points to a plane, because there isn't any
  - In the end, linear methods do nothing more than rotate/translate/scale data

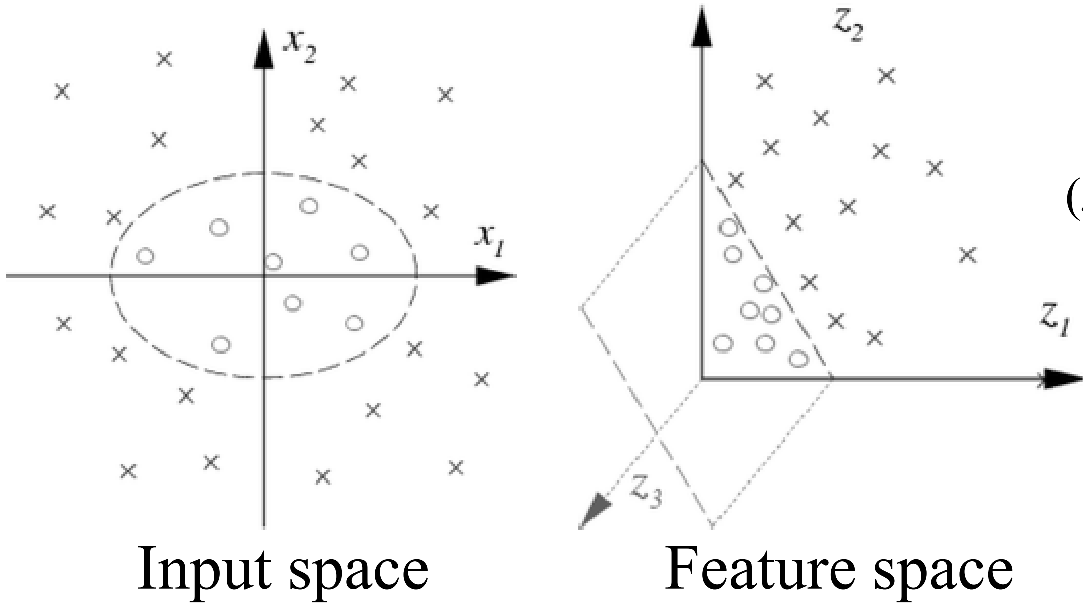


# Linear methods fail

- Learning a non-linear representation for classification



# Kernel Methods for Classification




$$(x_1, x_2) \rightarrow (x_1, x_2, x_1^2 + x_2^2) = (z_1, z_2, z_3)$$

- The kernel defines an implicit mapping (usually high dimensional and non-linear) from input to feature space, so the data becomes linearly separable.
- Computation in the feature space can be costly because it is (usually) high dimensional
  - The feature space is typically infinite-dimensional!

# Kernel Methods

- Suppose $\phi(.)$ is given as follows

$$\phi(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = (1, \sqrt{2}x_1, \sqrt{2}x_2, x_1^2, x_2^2, \sqrt{2}x_1x_2)$$

- An inner product in the feature space is

$$\langle \phi(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}), \phi(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}) \rangle = (1 + x_1y_1 + x_2y_2)^2$$

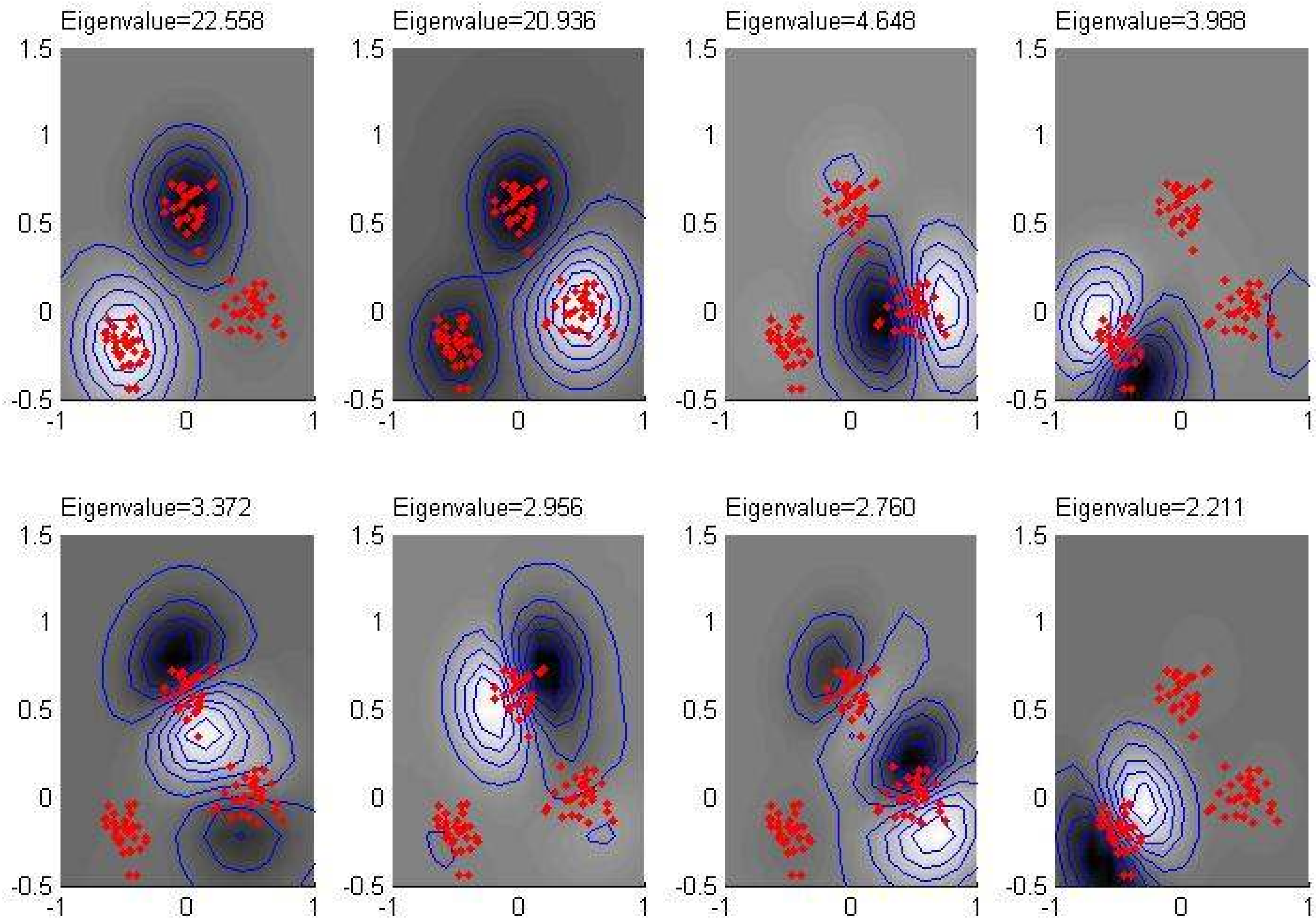
- So, if we define the kernel function as follows, there is no need to carry out $\phi(.)$ explicitly

$$K(\mathbf{x}, \mathbf{y}) = (1 + x_1y_1 + x_2y_2)^2$$

- This use of kernel function to avoid carrying out $\phi(.)$ explicitly is known as the kernel trick. In any linear algorithm that can be expressed by inner products can be made nonlinear by going to the feature space

# Kernel PCA

(Scholkopf et al., 1998)



# Kernel PCA

(Scholkopf et al., 1998)

- Eigenvectors of the cov. Matrix in feature space.

$$\overline{\mathbf{C}} = \frac{1}{n}\sum_{i=1}^{n}\Phi(\mathbf{d}_i)\Phi(\mathbf{d}_i)^{\mathrm{T}} \qquad \overline{\mathbf{C}}\,\mathbf{b}_1 = \mathbf{b}_1\lambda$$

- Eigenvectors lie in the span of data in feature space.

$$\mathbf{b}_1 = \sum_{i=1}^{n}\alpha_i\Phi(\mathbf{d}_i)$$

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\Phi(\mathbf{d}_i)K(\mathbf{d}_i,\mathbf{d}_j) = [\sum_{i=1}^{n}\alpha_i\Phi(\mathbf{d}_i)]\lambda$$

$$\mathbf{K}\boldsymbol{\alpha} = \boldsymbol{\alpha}\lambda$$

# Latent Variable Models

# Factor Analysis

- A Gaussian distribution on the coefficients and noise is added to PCA➔ Factor Analysis. (Mardia et al., 1979)

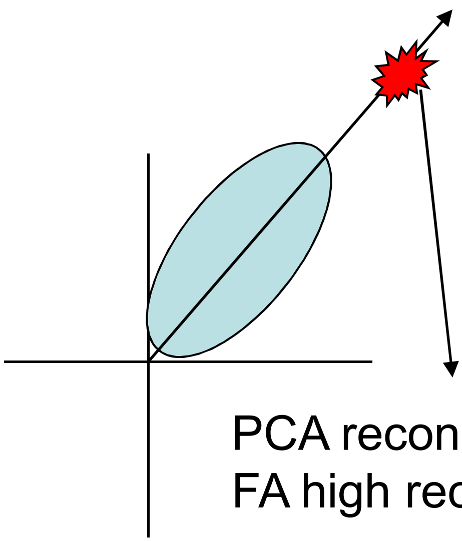
$$\mathbf{d} = \boldsymbol{\mu} + \mathbf{B}\mathbf{c} + \boldsymbol{\eta}$$

$$p(\mathbf{c}) = N(\mathbf{c} \mid \mathbf{0}, \mathbf{I}_k) \qquad p(\mathbf{d} \mid \mathbf{c}, \mathbf{B}) = N(\mathbf{d} \mid \boldsymbol{\mu} + \mathbf{B}\mathbf{c}, \Psi)$$

$$p(\boldsymbol{\eta}) = N(\mathbf{c} \mid \mathbf{0}, \Psi) \qquad \Psi = diag(\eta_1, \eta_2, \ldots, \eta_d)$$

$$\mathrm{cov}(\mathbf{d}) = E((\mathbf{d} - \boldsymbol{\mu})(\mathbf{d} - \boldsymbol{\mu})^T) = \mathbf{B}\mathbf{B}^T + \Psi$$

- Inference (Roweis & Ghahramani, 1999;Tipping & Bishop, 1999a)



$p(\mathbf{c}, \mathbf{d})$ Jointly Gaussian

$$p(\mathbf{c} \mid \mathbf{d}) = N(\mathbf{c} \mid \mathbf{m}, \mathbf{V})$$

$$\mathbf{m} = \mathbf{B}^T (\mathbf{B}\mathbf{B}^T + \Psi)^{-1} (\mathbf{d} - \boldsymbol{\mu})$$

$$\mathbf{V} = (\mathbf{I} + \mathbf{B}^T \Psi^{-1} \mathbf{B})^{-1}$$

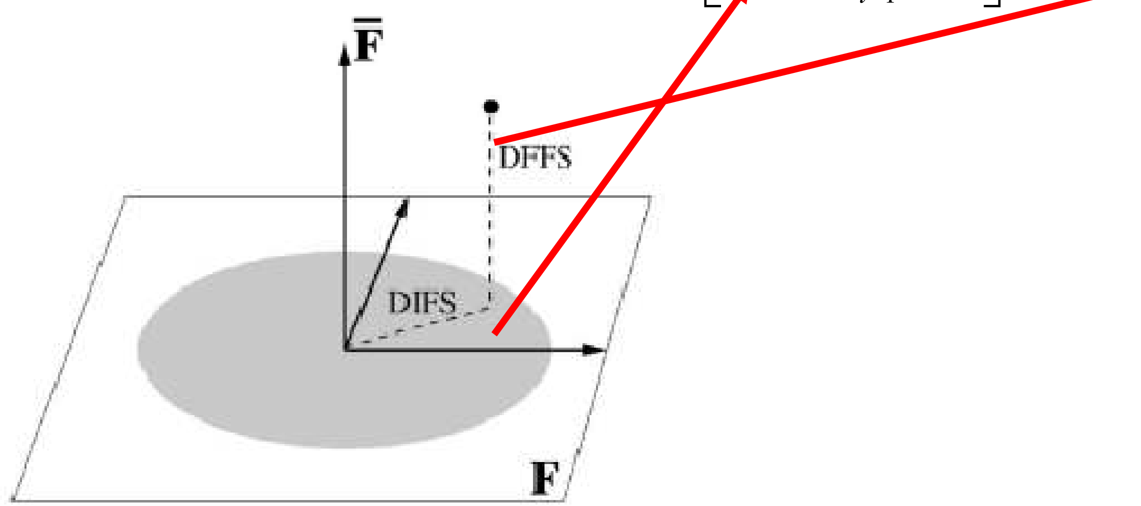
PCA reconstruction low error.
FA high reconstruction error (low likelihood).

# Ppca

- If $\Psi = E(\boldsymbol{\eta}\boldsymbol{\eta}^T) = \varepsilon \mathbf{I}_d$ PPCA.
- If $\varepsilon \to 0$ is equivalent to PCA. $\varepsilon \to 0 \quad \mathbf{B}^T(\mathbf{B}\mathbf{B}^T + \Psi)^{-1} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$
- Probabilistic visual learning (Moghaddam & Pentland, 1997;)

$$p(\mathbf{d}) = \int p(\mathbf{d}\,|\,\mathbf{c})\,p(\mathbf{c})\,d\mathbf{c} = \frac{e^{-\frac{1}{2}(\mathbf{d}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{d}-\boldsymbol{\mu})}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}} = \frac{e^{-\frac{1}{2}(\mathbf{d}-\boldsymbol{\mu})^T (\mathbf{B}\mathbf{B}^\mathbf{T}+\varepsilon\mathbf{I})^{-1}(\mathbf{d}-\boldsymbol{\mu})}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}} = \left[\frac{e^{-\frac{1}{2}\sum_{i=1}^{k}\frac{c_i^2}{\lambda_i}}}{(2\pi)^{\frac{d}{2}}\prod_{i=1}^{k}\lambda_i^{1/2}}\right]\left[\frac{e^{-\frac{\varepsilon^2(\mathbf{d})}{2\rho}}}{(2\pi\rho)^{\frac{(d-k)}{2}}}\right]$$

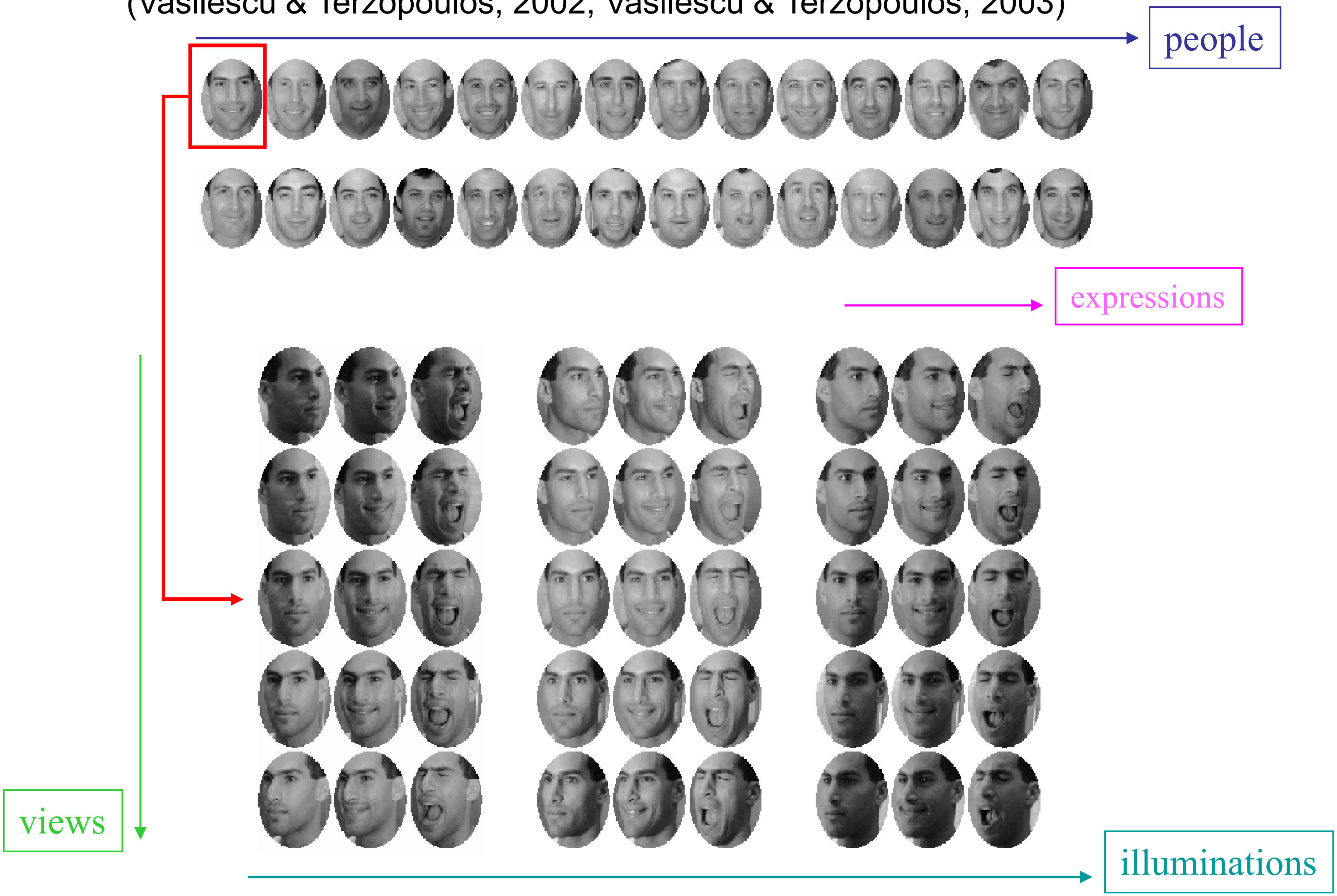
$$\mathbf{c}_i = \mathbf{B}^T\mathbf{d}_i$$



# Tensor Factorization

# Tensor faces

(Vasilescu & Terzopoulos, 2002; Vasilescu & Terzopoulos, 2003)



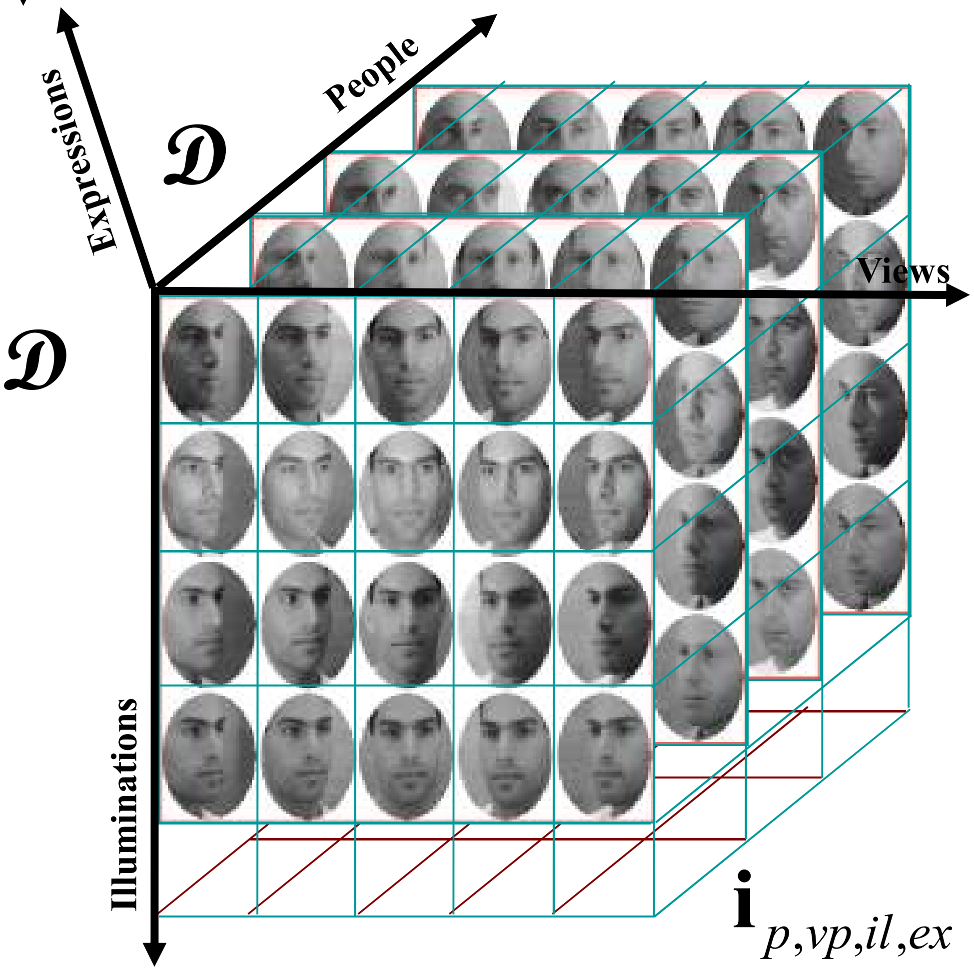
# Eigenfaces

- Facial images (identity change)

- Eigenfaces bases vectors capture the variability in facial appearance (do not decouple pose, illumination, …)

# Data Organization

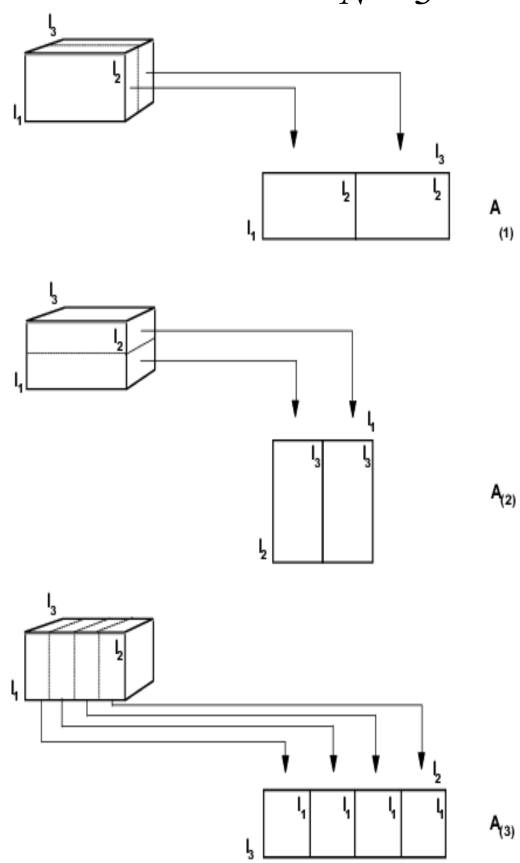
- Linear/PCA: **Data Matrix**
  - $\mathbb{R}^{\text{pixels x images}}$
  - a matrix of image vectors
- Multilinear: *Data Tensor* $\mathcal{D}$
  - $\mathbb{R}^{\text{people x views x illums x express x pixels}}$
  - N-dimensional matrix
  - 28 people, 45 images/person
  - 5 views, 3 illuminations, 3 expressions per person

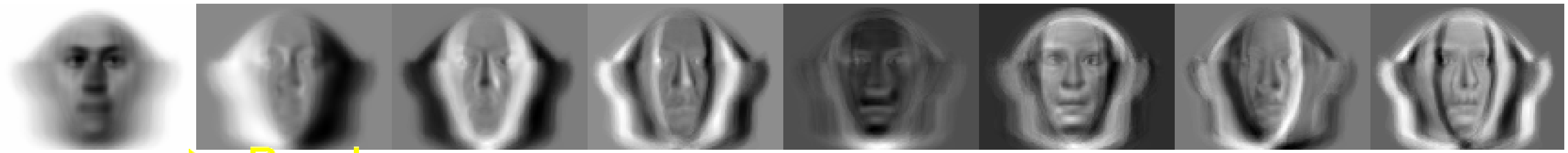
# N-Mode SVD Algorithm

$$\mathcal{D} = \mathcal{Z} \times_1 \mathbf{U}_{\text{people}} \times_2 \mathbf{U}_{\text{views}} \times_3 \mathbf{U}_{\text{illums.}} \times_4 \mathbf{U}_{\text{express.}} \times_5 \mathbf{U}_{\text{pixels}}$$

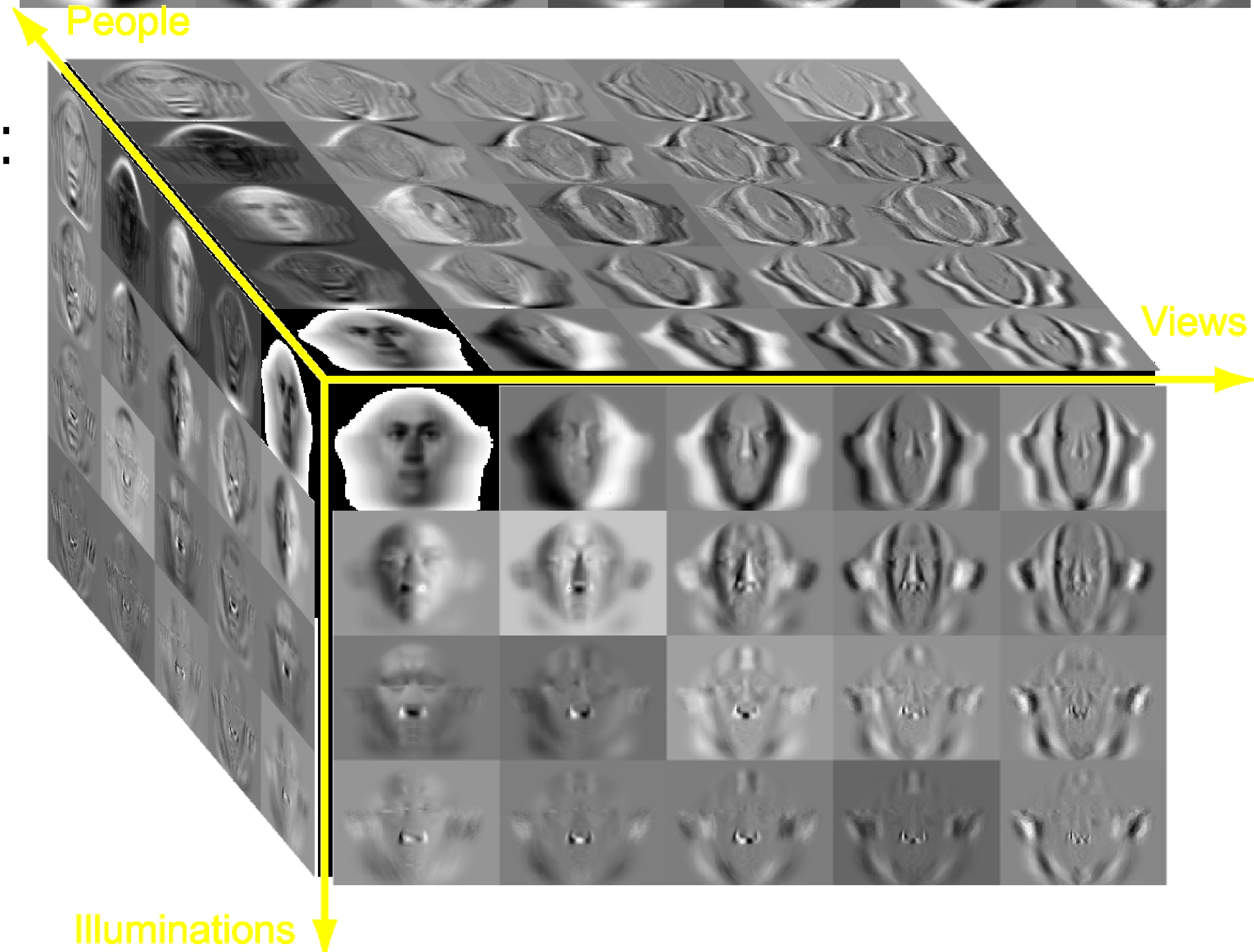
$N = 3$



PCA:



TensorFaces:



# Strategic Data Compression = Perceptual Quality

- TensorFaces data reduction in illumination space primarily degrades illumination effects (cast shadows, highlights)
- PCA has *lower mean square error* but *higher perceptual error*

**Original**

176 basis vectors

**TensorFaces**

6 illum + 11 people param.

66 basis vectors

**TensorFaces**

Mean Sq. Err. = **409.15**

3 illum + 11 people param.

33 basis vectors

**PCA**

Mean Sq. Err. = **85.75**

33 parameters

33 basis vectors

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